HOMEWORK 6 FOR 18.745, FALL 2012
DUE FRIDAY, OCTOBER 19 BY 3PM.

The base field is algebraically closed of characteristic zero and the Lie algebras are finite dimensional unless stated otherwise.

(1) Which of the following Lie algebras admit a nondegenrate invariant symmetric bilinear form? Which ones have an outer derivation?
  (a) A non-abelian two-dimensional Lie algebra.
  (b) The Lie algebra with basis x, y, z such that $[x, y] = y$, $[x, z] = -z$, $[y, z] = 0$.
  (c) The Lie algebra with basis x, y, z such that $[x, y] = z$, $[x, z] = 0$, $[y, z] = 0$.

(2) Recall the isomorphism of $\mathfrak{sl}_2$ modules $V_n \otimes V_m = V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}$, $(n \geq m)$. Consider the operator $C' := e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$ acting on the left hand side of this isomorphism. Describe its action on the right hand side.

  [Hint: For representations V, W of $\mathfrak{sl}_2$ the action of C on $V \otimes W$ does not coincide with the tensor product of its actions on V and W. Write down the formula for the difference of the two operators.]

(3) A Lie algebra is called *reductive* if its radical coincides with its center.[1] Show that $\mathfrak{g}$ is reductive iff its adjoint representation is completely reducible (i.e. is a direct sum of irreducible submodules).

(4) Let k be an arbitrary field of characteristic different from 2, and $A = k[x, y]$ be the ring of polynomials in two variables. It is easy to see that A has a Poisson algebra structure such that $\{x, y\} = 1$. [You are not required to write down a proof of this. See homework 3 for the definition of a Poisson algebra.]
  (a) Show that the subspace A_2 of homogeneous quadratic polynomials forms a Lie subalgebra which is isomorphic to $\mathfrak{sl}_2$.
  (b) Consider A as a module over $A_2 \cong \mathfrak{sl}_2$. Assuming again that the base field is algebraically closed of characteristic zero, show that it is a direct sum of finite dimensional irreducible submodules, so that each irreducible $\mathfrak{sl}_2$ module is isomorphic to exactly one of the summands.
  (c) Assume now that $char(k) = p > 2$. Show that the space of homogeneous polynomials of degree p is an $\mathfrak{sl}_2$ submodule which is not completely reducible. More precisely, find a two-dimensional submodule with trivial action and show that there are no nonzero quotient modules with a trivial action.

(5) (Optional) Recall the Jacobson-Morozov filtration on a finite dimensional vector space V equipped with a nilpotent endomorphism e. Show that it is given by the following formula:

$$V_{\geq i} = \sum_{a,b,b-a \geq i-1} (Ker(e^a) \cap Im(e^b)).$$

[1] Lie algebra $\mathfrak{gl}_n$ is the basic example of a reductive Lie algebra.

(6) (Weil representation)

(a) (Optional) Show that the space of polynomials in n variables carries an action of $\mathfrak{sl}_2$, such that $e : P \mapsto \frac{1}{2}(\sum x_i^2)P$, $f : P \mapsto -\frac{1}{2}\sum \frac{\partial^2 P}{\partial x_i^2}$.

(b) (Very optional) Let $k = \mathbb{C}$. Recall that the space of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ embeds into the space of distributions $\mathcal{D}$, which is the topological dual of the space $\mathcal{S}$ of smooth rapidly decreasing complex valued functions on $\mathbb{R}^n$ (Schwarz space). Show that the above action extends to an action of $\mathfrak{sl}_2$ on $\mathcal{D}$, and the operator $\exp(ie)\exp(if)\exp(ie)$ acts there by Fourier transform.