HOMEWORK 5 FOR 18.745, FALL 2012
DUE FRIDAY, OCTOBER 12 BY 3PM.

The base field is algebraically closed of characteristic zero unless stated otherwise.

(1) Let V, W be irreducible representations of $\mathfrak{sl}_2$, $\dim(V) = 3$, $\dim(W) = 10$. Describe the summands in the decomposition $V \otimes W \cong \oplus V_{n_i}$ into a sum of irreducible modules, and write down the images of the highest weight vectors in V_{n_i} under that isomorphism.

[The answer should be given as a linear combination of products of elements in the standard basis of V, W].

(2) Let k be a characteristic zero field (not necessarily algebraically closed).
   (a) Show that a simple Lie algebra of dimension three is isomorphic to $\mathfrak{so}(V)$ for some three dimensional vector space with a nondegenerate symmetric bilinear form.
   [Hint: use adjoint representation and the Killing form].
   (b) Check that for three dimensional spaces V, V' equipped with quadratic forms we have $\mathfrak{so}(V) \cong \mathfrak{so}(V')$ iff there exists an isomorphism $V \cong V'$ carrying the form on V to one proportional to the form on V'.
   [Hint: $\mathfrak{so}(V) \cong \Lambda^2 \cong V^*$ for a three dimensional space V].
   (c) (Optional) Check that there are exactly two isomorphism classes of three dimensional simple Lie algebras over $\mathbb{R}$: $\mathfrak{sl}_2(\mathbb{R})$ and $(\mathbb{R}^3, \times)$ where $\times$ denotes vector product as defined in elementary multivariable calculus.
   [You can use the standard properties of vector product stated in calculus textbooks.]

(3) Recall that up to an isomorphism and scaling there exist three distinct nondegenerate quadratic forms on $V = \mathbb{R}^4$: $x^2+y^2+z^2+t^2$, $x^2+y^2+z^2-t^2$ and $x^2+y^2-z^2-t^2$. Show that $\mathfrak{so}(V)$ is simple in exactly one of the three cases.

[Hint: you can use that $\mathfrak{so}(V\otimes\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$ contains exactly two nonzero proper ideals, so it remains to see when one of these two is defined over $\mathbb{R}$].

(4) (Optional) [Lefschetz $\mathfrak{sl}_2$] Let V be a symplectic vector space, let $\omega \in \Lambda^2(V^*)$ be the symplectic form. Let $\varpi \in \Lambda^2(V)$ be the corresponding form on V^*. Set $\Lambda V = \oplus_i \Lambda^i(V)$, $\Lambda V^* = \oplus_i \Lambda^i(V^*)$ and define operators $e : \Lambda V \to \Lambda V$, $e : x \mapsto \omega \wedge x$, $e' : \Lambda V^* \to \Lambda V^*$, $e' : y \mapsto \varpi \wedge y$. Using the standard isomorphism $\Lambda^i(V)^* \cong \Lambda^i(V^*)$ define $f : \Lambda V \to \Lambda V$ as the operator adjoint to e'. Check that e, f generate an action of $\mathfrak{sl}_2$ on ΛV. Describe the action of h and the summands in the decomposition of ΛV into a sum of irreducible modules.