**HOMEWORK 4 FOR 18.745, FALL 2012**
**DUE FRIDAY, OCTOBER 5 BY 3PM.**

(1) Let $V$ be an $n$-dimensional vector space. Show that the representation $V \otimes \Lambda^{n-1} V$ of $\mathfrak{gl}(V)$ is isomorphic to the tensor product of the adjoint representation by a one dimensional representation and describe that one dimensional representation.

(2) Classify ideals in the Lie algebra of (nonstrictly) upper triangular matrices $\mathfrak{b} \subset \mathfrak{sl}_n$ which are contained in strictly upper triangular matrices.

(3) Let $\mathfrak{g}$ be a Lie algebra with an invariant symmetric bilinear form. Let $\mathfrak{k} \subset \mathfrak{g}$ be the kernel of the form, i.e. $\mathfrak{k} = \{x \in \mathfrak{g} | \ (x, y) = 0 \forall y \in \mathfrak{g}\}$.
   (a) Show that $\mathfrak{k}$ is an ideal in $\mathfrak{g}$.
   (b) Fix integers $d_1, \ldots, d_m$, $\sum d_i = n$, and let $\mathfrak{g} \subset \mathfrak{sl}_n$ be the algebra of (non-strictly) block upper triangular matrices with corresponding sizes of blocks. Lie algebra $\mathfrak{g}$ is equipped with an invariant symmetric bilinear form $(x, y) = Tr(xy)$. Describe $\mathfrak{k}$.

(4) Let $V = k^{2m}$ be a symplectic vector space and set $W = V^{\oplus n}$, the sympectic form on $V$ induces one on $W$. Consider the homomorphism $\mathfrak{sp}(V) \to \mathfrak{sp}(W)$ sending a $2m \times 2m$ matrix $A$ to the block diagonal matrix where each of $n$ diagonal $2m \times 2m$ blocks equals $A$. Show that the centralizer of the image of $\mathfrak{sp}(V)$ in $\mathfrak{sp}(W)$ is isomorphic to $\mathfrak{so}(n)$.

   [Hint: if $V_1$ is equipped with a skew-symmetric bilinear form and $V_2$ is equipped with a symmetric bilinear form, then $V_1 \otimes V_2$ carries a natural skew-symmetric form and we get a homomorphism $\mathfrak{sp}(V_1) \oplus \mathfrak{so}(V_2) \to sp(V_1 \otimes V_2)$.