HOMEWORK 3 FOR 18.745, FALL 2012
DUE FRIDAY, SEPTEMBER 28 BY 3PM.

(1) Recall the central series $\mathfrak{g}^1 = \mathfrak{g}$, $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$. Suppose that the Lie algebra $\mathfrak{g}$ is generated by two elements and the 6th term of its central series $\mathfrak{g}^6$ vanishes. What is the maximal possible dimension of $\mathfrak{g}$?

[You may use a calculator if you like.]

[Hint: Consider a homomorphism from the free Lie algebra with two generators to $\mathfrak{g}$, what can you say about its kernel?]

(2) Let $U(\mathfrak{g})_n$ be the terms of the standard filtration on the enveloping of a Lie algebra $\mathfrak{g}$ and $gr_n(U(\mathfrak{g})) = U(\mathfrak{g})_n/U(\mathfrak{g})_{n-1}$, so that:

$gr(U(\mathfrak{g})) := \oplus gr_n(U(\mathfrak{g})) \cong Sym(\mathfrak{g})$ by PBW Theorem.

(a) Show that the commutator in the associative algebra $U(\mathfrak{g})$ induces a well defined bilinear map $gr_n(U(\mathfrak{g})) \times gr_m(U(\mathfrak{g})) \to gr_{n+m-1}(U(\mathfrak{g}))$. This map is denoted by $(x, y) \mapsto \{x, y\}$.

(b) Show that $gr(U(\mathfrak{g}))$ equipped with the bracket $\{,\}$ is a Lie algebra.

(c) Check that for $x \in U(\mathfrak{g})$ the map $y \mapsto \{x, y\}$ is a derivation of the commutative associative algebra $gr(U(\mathfrak{g})) \cong Sym(\mathfrak{g})$.

[A commutative associative algebra equipped also with a Lie algebra structure satisfying the property stated in part (c) of the problem is called a *Poisson algebra*.]

(3) Let $\mathfrak{g}$ be the two dimensional nonabelian algebra (in particular, $\mathfrak{g}$ is solvable).

(a) Check that the Killing form for $\mathfrak{g}$ is nonzero.

(b) Assuming characteristic of the base field is zero, show that the center of the enveloping algebra $U(\mathfrak{g})$ is one dimensional.

(c) (Optional) Assume that characteristic of the base field is $p > 0$. Show that the center of $U(\mathfrak{g})$ is generated by y^p and $x^p - x$ where x, y is a basis such that $[x, y] = y$.