HOMEWORK 2 FOR 18.745, FALL 2012
DUE FRIDAY, SEPTEMBER 21 BY 3PM.

(1) Show that all three dimensional non-abelian nilpotent Lie algebras over a given field are isomorphic.

(2) (after problem 9 in Bourbaki, exercises for §I.4) Let $\mathfrak{g}$ be the semi-direct product of the one dimensional algebra by the three dimensional abelian algebra where a generator of the one dimensional algebra acts on the three dimensional space by a regular nilpotent endomorphism (i.e. by one Jordan block).

   (a) Show that $\mathfrak{g}$ is nilpotent and find the dimensions of the terms in its upper and lower central series and in the derived series.

   (b) (Optional) Show that every nonabelian four dimensional nilpotent Lie algebra is either isomorphic to $\mathfrak{g}$ or the product of the three dimensional Heisenberg Lie algebra by the one dimensional algebra.

(3) (a) Let k be a field of characteristic $p > 0$. Give an example of a solvavle Lie algebra $\mathfrak{g}$ over k whose commutator is not nilpotent. Your Lie algebra should have dimension $p + 3$ and the terms of the derived series should satisfy:

$$\dim(\mathfrak{g}^{(1)}) = p + 1,\ \dim(\mathfrak{g}^{(2)}) = p,\ \mathfrak{g}^{(3)} = 0.$$

   (b) Let k be a field of characteristic zero. Give an example of an infinite dimensional solvable Lie algebra $\mathfrak{g}$ over k whose commutator is not nilpotent. The terms of the derived series should satisfy:

$$\dim(\mathfrak{g}/\mathfrak{g}^{(1)}) = 2,\ \dim(\mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}) = 1,\ \mathfrak{g}^{(3)} = 0.$$

   [Hint: use Heisenberg representation of the three dimensional Heisenberg Lie algebra].

(4) (a) Show that inner derivations of a Lie algebra form an ideal in the Lie algebra of all its derivations.
   The quotient of the Lie algebra of derivations by that ideal is called the Lie algebra of *outer derivations*.

   (b) Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal in a Lie algebra $\mathfrak{g}$. Let

$$\mathfrak{c} = \{x \in \mathfrak{g} \mid [x, y] = 0\ \forall y \in \mathfrak{h}\}$$

   be the *centralizer* of $\mathfrak{h}$.
   Check that the adjoint action induces a homomorphism $\mathfrak{g}/\mathfrak{h} \to Out(\mathfrak{h})$ where $Out(\mathfrak{h})$ denotes the Lie algebra of outer derivations. Let $\mathfrak{k}$ denote its kernel.
   Show that $\mathfrak{c}$ is a central extension of $\mathfrak{k}$ by the center of $\mathfrak{h}$.