HOMEWORK 10 FOR 18.745, FALL 2012
DUE FRIDAY, NOVEMBER 16 BY 3PM.

In problems below $\Sigma \subset E$ is a (reduced) root system, the *root lattice* P is the subgroup in E generated by Σ, $\Sigma^{\check{}} \subset E^*$ is the dual roots system, $\Sigma^{\check{}} = \{\check{\alpha} \mid \alpha \in \Sigma\}$, where $\langle \check{\alpha}, \lambda \rangle = 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}$, and the *weight lattice* $\Lambda = \{\lambda \in \Lambda \mid \langle \check{\alpha}, \lambda \rangle \in \mathbb{Z}\}$ for $\check{\alpha} \in \Sigma^{\check{}}$.

(1) Suppose that $\lambda \in P$ is a vector of minimal nonzero length. Show that $\lambda \in \Sigma$.

(2) (Λ/P and minuscule weights)[1] In this problem the root system is assumed to be irreducible.

A nonzero weight $\lambda \in \Lambda$ is called *minuscule* if $\langle \check{\alpha}, \lambda \rangle$ is either 0 or 1 for every $\alpha \in \Sigma^+$.

A weight is called *fundamental* if $\langle \check{\alpha}, \lambda \rangle = 1$ for one simple roots and vanishes for the other simple roots.

(a) Check that every nonzero coset P/Λ contains a minuscule weight.
[Hint: If λ is (nonstrictly) dominant and not miniscule there exists a positive coroot $\check{\alpha}$ of maximal length and such that $\langle \check{\alpha}, \lambda \rangle \geq 2$. Pick such $\check{\alpha}$ with minimal height (i.e. with minimal sum of coefficients in the decomposition of $\check{\alpha}$ as a combination of simple coroots). Then show that for the corresponding root α the weight $\lambda - \alpha$ is again dominant. Thus minuscule weights are minimal elements with respect to the standard partial order in their P-cosets intersected with the set of dominant weights.]

(b) Check that every minuscule weight is fundamental.

(c) Let ω be a fundamental weight and α be a simple root s.t. $\langle \check{\alpha}, \lambda \rangle = 1$. Show that ω is minuscule iff the following condition holds. Let $\gamma \in \check{\Sigma}$ be the highest root. Write down γ as a linear combination of $\check{\alpha}_i$ for α_i is simple. Then the coefficient of $\check{\alpha}$ equals one.

(d) (Optional) Let A be the set of elements in E satisfying: $\langle \check{\alpha}, \lambda \rangle \in [0, 1]$ for all $\alpha \in \Sigma^+$. Consider the group of maps from E to E generated by W and translations by elements of P; it is called the affine Weyl group, denote it by $\hat{W}$. Check that every element in E is in the $\hat{W}$ orbit of a unique element of A. Deduce that every nonzero coset P/Λ contains a unique minuscule weight.

(e) Let Σ be the root system of Lie algebra $\mathfrak{sl}_n$. Describe minuscule weights and confirm that they form a basis of E.
[One can show that the last statement fails for any simple Lie algebra other than $\mathfrak{sl}_n$.]

[1]One can show that if Σ is the root system of a semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and G is the group of automorphisms of $\mathfrak{g}$ generated by $\exp(ad(x))$ (say) for nilpotent x, then the finite abelian group Λ/P is dual to the fundamental group $\pi_1(G)$. Thus the problem shows that $|pi_1(G)| - 1$ is the number of minuscule weights.