MATH 18.726, SPRING 2015 - PROBLEM SET # 7

(1) The *Grassmann variety* $Gr(r,n)$ (which we will consider as a scheme over a field k) is characterized as follows: for a scheme X over k we have a canonical (functorial) identification between $Hom(X, Gr(r,n))$ and the set of pairs $(\mathcal{E}, p)$, where $\mathcal{E}$ is a locally free coherent sheaf of rank r on X and p is a surjective map $\mathcal{O}_X^{\oplus n} \to \mathcal{E}$.
   (a) Assuming existence of $Gr(r,n)$ use the r-th exterior power of p to construct a morphism $Gr(r,n) \to \mathbb{P}^N$ where $N = \binom{n}{r} - 1$.
   (b) Assuming furthermore that the map you constructed is a closed embedding (it is called *Plücker embedding*) construct a map from the blow up of diagonal in $(\mathbb{P}^n)^2$ to $Gr(2, n+1)$ sending a pair of distinct points to the line passing through them.

(2) Let π be the map from the blow up of a point on $\mathbb{A}^n$ to $\mathbb{A}^n$. Describe the coherent sheaf $\pi^*(J)$, where J is the ideal sheaf of the point.

(3) Prove that the deformation to normal cone to a closed scheme Z in an affine scheme X can be also described as the complement to the strict transform of $X \times \{0\}$ in the blow up of $X \times \mathbb{A}^1$ along $Z \times \{0\}$.

(4) Let X be a smooth degree 4 surface in $\mathbb{P}^3$. [This is an example of a $K3$ *surface*]. Compute the dimensions of $H^1(\mathcal{O}_X)$, $H^2(\mathcal{O}_X)$ and $H^1(\Omega^1_X)$.

   [In view of strong Lefschetz theorem and Serre duality this provides full information on dimension of cohomology of $X(\mathbb{C})$

(5) Problem 8.6 in Harthshorne II.8.