1 The Cremona (or quadratic) transformation is a rational morphism $\mathbb{P}^2 \rightarrow \mathbb{P}^2$ given by $\phi : (t_0 : t_1 : t_2) \mapsto (t_0t_1 : t_0t_2 : t_1t_2)$.

(a) Show that $\phi$ is birational and find its inverse.

(b) Find maximal open subsets $U, V \subset \mathbb{P}^2$, such that $\phi$ induces an isomorphism $U \rightarrow V$.

2 (Optional problem) The Fermat cubic surface $X \subset \mathbb{P}^3$ is given by the equation $x^3 + y^3 + z^3 + w^3 = 0$.

(a) Check that the rational map sending $(x : y : z : w)$ with $w + y \neq 0$ or $x + z \neq 0$ to $(yz - wx : wy - wz + x^2 + w^2 - w(z + x) + y^2 - xy + wy + x^2 - wx + xz)$, is a birational isomorphism between $X$ and $\mathbb{P}^2$.

(b) Deduce that the Cremona group (defined in footnote 1) contains a subgroup $S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$.

3 Let $F_n \in \text{Coh}(A^1)$ be the cokernel of the map $\mathcal{O} \rightarrow \mathcal{O}(−n)$ (where $t$ is the coordinate on $A^1$). We have a surjective map $F_{n+1} \rightarrow F_n$. Show that the inverse limit $\varprojlim_{\text{Sh}} F_n$ in the category of sheaves of $\mathcal{O}$-modules is not isomorphic to the inverse limit $\varprojlim_{\text{QCoh}} F_n$ in the category of quasicoherent sheaves (though both limits exist). Moreover, check that $\varprojlim_{\text{Sh}} F_n$ is a non-quasicoherent sheaf supported at zero, while $\varprojlim_{\text{QCoh}} F_n$ has full support.

4 (a) Show that $\text{QCoh}(\mathbb{P}^1)$ contains no projective object $\mathcal{P}$ with a nonzero map $\mathcal{P} \rightarrow \mathcal{O}_{x_0}$, $x_0 \in \mathbb{P}^1$, where $\mathcal{O}_{x_0}$ is the sky-scraper at a point $x_0$, i.e. the direct image of $\mathcal{O}$ under the embedding $x_0 \rightarrow \mathbb{P}^1$.

[Hint: You can use a result to be proved in class that a quasicoherent sheaf is a union of its coherent subsheaves. If $\mathcal{P}$ is a projective object apply $\text{Hom}(\mathcal{P}, \mathcal{O})$ to the surjection $\mathcal{O}(−n) \rightarrow \mathcal{O}_{x_0}$. Get a map $\mathcal{P} \rightarrow \mathcal{O}(−n)$, which must be nonzero on a coherent subsheaf $\mathcal{F} \subset \mathcal{P}$ surjecting to $\mathcal{O}_{x_0}$. Now take $n \gg 0$ and get a contradiction].

(b) (Optional problem) Show that $j_*(\mathcal{O})/\mathcal{O}$ is an injective object in $\text{QCoh}(\mathbb{P}^1)$, where $j : A^1 \rightarrow \mathbb{P}^1$ is the embedding.

5 For a subvariety $X \subset \mathbb{P}^n$ not contained in a linear subspace, the $k$-th secant variety $S_k(X)$ is the closure of the union of all $k$-planes in $\mathbb{P}^n$ spanned by $k + 1$ points of $X$.

For $n = 2k$ let $C \subset \mathbb{P}^n$ be the image of the $n$-th Veronese embedding of $\mathbb{P}^1$. Show that $S_{k−1}(C)$ is a hypersurface of degree $k + 1$ in $\mathbb{P}^n$.

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1 A theorem by Noether and Castelnuovo asserts that the Cremona group of birational automorphisms of $\mathbb{P}^2$ is generated by $\phi$ and the subgroup $\text{PGL}_3(k)$ of linear automorphisms of $\mathbb{P}^2$.

2 Formula borrowed from Noam Elkies’ homepage.
[Hint: For a two-dimensional vector space $V$ one needs to find an equation singling out elements in $Sym^n(V)$ which are sums of at most $k$ monomials $\sum_{i=1}^{k} v_i^n$. An element $\sigma \in Sym^n(V)$ determines a map $Sym^k(V^*) \to Sym^k(V)$. Check that if $\sigma$ is of the form $\sum_{i=1}^{k} v_i^n$ then the map has zero determinant, while for some element in $Sym^n(V)$ the determinant is nonzero. This shows $S_{k-1}(C)$ is contained in the zero set of a degree $k+1$ polynomial. One can also check directly that $S_{k-1}(C)$ is an irreducible hypersurface, the above shows its degree is at most $k+1$. To show it can’t be smaller than $k+1$ one can write down a line intersecting it in $k+1$ points.]