(1) Let $Z$ be an irreducible closed subset in an algebraic variety $X$. Show that if $\dim(Z) = \dim(X)$ then $Z$ is a component of $X$.

(2) Let $Y$ be a closed subvariety of dimension $r$ in $\mathbb{P}^n$.

(a) Suppose that $Y$ can be presented as the set of common zeroes of $q$ homogeneous polynomials. Show that $r \geq n - q$.

If $Y$ can be presented as the set of common zeroes of $q$ homogeneous polynomials with $q = n - r$ we say that $Y$ is a set-theoretic complete intersection.

If moreover the ideal $I_Y$ can be generated by $n - r$ homogeneous polynomials, then $Y$ is called a (strict) complete intersection.

(b) Show that every irreducible closed subvariety in $\mathbb{P}^n$ is a component in a set theoretic complete intersection of the same dimension.

[Hint: use induction to construct homogeneous polynomials $P_1, P_2, \ldots, P_{n-r}$, such that the set of common zeroes of $P_1, \ldots, P_i$ has dimension $n - i$ and contains our subvariety].

(c) Show that the twisted cubic curve in $\mathbb{P}^3$ (see problem 2 of problem set 2) is a set theoretic complete intersection.

(d) (Optional bonus problem) Show that the twisted cubic curve in $\mathbb{P}^3$ is not a strict complete intersection.

(3) Let $C$ be a curve in $\mathbb{P}^2$, $x$ be a point in $C$ and $L$ a line passing through $x$. Let $m$ be the multiplicity of $C$ at $x$ and $M$ the multiplicity of intersection of $C$ and $L$ at $x$. Show that $m \leq M$ and that for given $C$, $x$ the equality $m = M$ holds for all but finitely many lines $L$ as above.

(4) Prove Bezout Theorem for two curves of degrees $d_1, d_2$ in $\mathbb{P}^2$ with no common components

(a) Assuming $d_1 = 1$.

(b) Assuming $d_1 = 2$ and the first curve is irreducible; you can also assume that characteristic of the base field is different from two.

[Hint: first show that in a special case the multiplicity of intersection of two curves can be interpreted as follows. Assume that the first curve $X$ is isomorphic to $\mathbb{A}^1$ and let $f : \mathbb{A}^1 \to X$ be the isomorphism. Let $P$ be the equation of the second curve $Y$. Then the multiplicity of intersection of $X$ and $Y$ at $x = f(a)$ is the multiplicity of $a$ as a root of the polynomial in one variable $Q(t) = P(f(t))$. Now use the isomorphism of the first curve with $\mathbb{P}^1$, choose coordinates so that the infinite line does not contain intersection points and recall a familiar fact about polynomials in one variable].

(5) (Optional bonus problem) Recall from the lecture that Grassmannian $Gr(2,4)$ is isomorphic to a quadric in $\mathbb{P}^5$. Use this to show that given four lines in $\mathbb{P}^4$, the number of lines intersecting each of the four lines is either infinite or equal to one or two.
[Hint: Check that the for a line \( L \subset \mathbb{P}^3 \) the set of lines intersecting \( L \) is parametrized by \( \text{Gr}(2, 4) \cap H \) for a hyperplane \( H \subset \mathbb{P}^5 \), thus the answer is the number of points in the intersection \( L \cap \text{Gr}(2, 4) \) where \( L \subset \mathbb{P}^5 \) is a linear subspace of dimension one or higher. Check that the intersection is infinite unless \( L \) is a line and refer to problem 3(a) from problem set 2].