(1) Show that a quasicoherent sheaf on a quasi-projective variety \( X \) is a union of its coherent subsheaves.

[Hint: reduce to the case when \( X \) is projective by replacing your sheaf by its direct image under an appropriate open embedding. If \( F \) is a quasicoherent sheaf on \( X \subset \mathbb{P}^n \), show that every section of \( F|_{\mathbb{A}^n \cap X} \) extends to a map \( \mathcal{O}(-d) \to F \) for some \( d \). Now consider the images of the direct sum of several such maps.]

(2) Recall that the arithmetic genus of a connected complete curve \( X \) is the dimension of the space \( H^1(\mathcal{O}_X) \).

Suppose that each component of \( X \) is isomorphic to \( \mathbb{P}^1 \), two components intersect by at most one point and each such intersection point is a nodal singularity (i.e. its completed local ring is isomorphic to \( k[[x, y]]/(xy) \)).

Let \( \Gamma \) be a graph whose vertices are indexed by components of \( X \) and two vertices are connected by an edge when the corresponding components intersect. Show that \( p_a(X) = 1 - \chi(\Gamma) \), where \( p_a \) denotes the arithmetic genus and \( \chi \) is the Euler characteristic.

(3) Let \( X = \text{Spec}(A) \) be a normal affine irreducible surface with the only singular point \( x \in X \). Show that the following three statements are equivalent:

(a) \( \text{Cl}(X) = 0 \), where \( \text{Cl} \) is the divisor class group, i.e. the quotient of the group of Weil divisors by the subgroup of principal divisors.

(b) \( \text{Pic}(X \setminus x) = 0 \).

(c) \( A \) is UFD.

(4) Let \( X \) be as in problem 3 and let \( \pi : Y \to X \) be a resolution of singularities of \( X \), suppose that \( \pi^{-1}(X \setminus x) \) maps isomorphically to \( X \setminus x \). Suppose also that the canonical line bundle \( K_Y \) is trivial and that \( \pi^{-1}(x) \) is a curve of the type described in problem 2, let \( D_1, \ldots, D_n \) be the components of \( \pi^{-1}(x) \). We get a homomorphism \( \text{Pic}(Y) \to \mathbb{Z}^n, L \mapsto (d_i) \), where the restriction of \( L \) to \( D_i \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(d_i) \). Compute the image of (the class of) \( \mathcal{O}(D_i) \) under that homomorphism.

(5) Let \( G \) be a finite subgroup in \( \text{SL}(2, \mathbb{C}) \) and \( X = \mathbb{A}^2/G \), let \( x \in X \) be the image of 0. It can be shown that \( X \) is normal and there exists a unique resolution \( Y \to X \) satisfying the assumptions of problem 4. Moreover, the map \( \text{Pic}(Y) \to \mathbb{Z}^n \) described in problem 4 is an isomorphism. Deduce that \( \mathbb{C}[x, y]^{G} \) is a UFD iff the Cartan matrix constructed from the graph \( \Gamma \) has determinant \( \pm 1 \) (in fact this determinant is always positive, so the option for it to equal \(-1 \) is not realized). Here Cartan matrix \( C = C_{\Gamma} \) is given by:

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1This is in fact true for not necessarily quasi-projective varieties, and even more generally, see e.g. Exercise II.5.15. in Hartshorne.

2We have only discussed how to associate a Weil divisor to a rational function in the cases when \( X \) is a curve or when \( X \) is smooth. In this problem you only need to use that such a construction exists for normal irreducible varieties and that it is compatible with restriction to an open subset.
$C_{ij} = 2$, $C_{ij} = -1$ if $i \neq j$ are connected by an edge in the graph $\Gamma$ and $C_{ij} = 0$ otherwise.

[In fact, the graph $\Gamma$ is necessarily one of the simply-laced Dynkin graphs appearing in the classification of compact connected simple groups. The only such graph for which $\det(C_{ij}) = 1$ (this condition is equivalent to the corresponding simple Lie group being simply-connected) corresponds to the largest simple connected compact Lie group $E_8$. The group $G$ in this case is the binary icosahedral group, i.e. the preimage in the special unitary group $SU(2)$ of the group of symmetries of a regular icosahedron under the homomorphism $SU(2) \to PSU(2) \cong SO(3)$. The surface $X$ is isomorphic to the surface in $\mathbb{A}^3$ given by the equation $x^2 + y^3 + z^5 = 0$, as described by Felix Klein in his book "Lectures on the icosahedron and solution of the fifth degree equations" (1884); the resolution $Y$ can be obtained from $X$ by 8 blow-ups, cf. Exercise V.5.8 in Hartshorne.]