Lecture 20

Goldie's theorem.

<u>Thm.</u> (Goldie) Let A be a semiprime, right Noetherian ring. Then the set S of regular elements of A is a right Ore set, and $Q = AS^{-1}$ is a semisimple ring (direct sum of finitely many matrix algebras over division algebras).

<u>Remark.</u> For commutative rings, semiprime = reduced (no nilpotents), so AS^{-1} is a direct sum of fields corresponding to irred. components of Spec A.

Most of the rest of the lecture will be the proof of this theorem. The main part of the proof is to show that the set of regular elements is an (right) Ore set. It is clear that S is multiplicative so it remains to check the Ore condition

$\forall a \in A,\ s \in S \quad \exists a_1 \in A,\ s_1 \in S$ ~~nonzero~~ such that $sa_1 = as_1$.

For this we will introduce the notion of an essential (bis) submodule.

<u>Def.</u> A submodule E of a right A-module M is essential if for every submodule $X \subset M$ we have $X \cap E \neq 0$

<u>Exer.</u> Let A be prime. Then any nonzero ~~right~~ 2-sided ideal is an essential right submodule. Indeed, if $M \cap I = 0$ then $MI \subset M \cap I = 0$ so $MI = 0 \Rightarrow M = 0$ (as $I \neq 0$). (M right ideal)

<u>Prop. 1</u> (1) Let $M_1 \subset M_2 \subset M_3$ be ~~submodules~~ modules ~~of~~ If M_1 is essential in M_2 and M_2 essential in M_3 then M_1 is essential in M_3

(2) Let $f: M' \to M$ be a morphism of right A-modules. Then if $W \subset M$ is essential then $W' = f^{-1}(W)$ is essential in M'

(3) If $E_i \subset M_i$ are essential submodules then $E_1 \oplus \cdots \oplus E_n$ is an essential submodule of $M_1 \oplus \cdots \oplus M_n$.

<u>Proof.</u> $\boxed{1}$ is clear

$\boxed{2}$. Let X' be a nonzero submodule in M'. We need to show that $X' \cap f^{-1}(W) \neq 0$.

But $f(X' \cap f^{-1}(W)) = f(X') \cap W$, so if $f(X') \neq 0$ then $f(X') \cap W \neq 0$, hence $X' \cap f^{-1}(W) \neq 0$. Otherwise, if $f(X') = 0$ then $X' \subset f^{-1}(W) = W'$ so $X' \cap f^{-1}(W) \neq 0$.

$\boxed{3}$ Let $E \subset M$ be an essential submodule then $E \oplus N$ is essential in $M \oplus N$ by (2). So we can consider the chain

$$E_1 \oplus \cdots \oplus E_n \subset E_1 \oplus \cdots \oplus E_{n-1} \oplus M_n \subset \cdots \subset M_1 \oplus \cdots \oplus M_n$$

and apply (1).

<u>Prop. 2.</u> Let $M_1 \subset M$. Then $\exists M_2 \subset M$ s.t. $M_1 \cap M_2 = 0$ and $M_1 \oplus M_2$ is essential.

<u>Proof.</u> By Zorn's lemma, we may take for M_2 a maximal submodule such that $M_1 \cap M_2 = 0$. Then $M_1 \oplus M_2$ is essential.

<u>Cor.</u> M has no proper essential submodule $\iff$ M is semisimple.

<u>Pf.</u> Suppose M is as stated and $M_1 \subset M$. $\exists M_2$ with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2 = M$.

<u>Def.</u> M is <u>uniform</u> if $M \neq 0$ and every $N \subset M$, $N \neq 0$ is essential.

<u>Ex.</u> If A is a commutative domain, $\mathrm{Frac}(A) = K$ then a torsion-free A-module M is uniform $\iff$ $M \otimes_A K$ is a 1-dim. K-vector space.

Also $k[t]/t^n$ is a uniform $k[t]$-module.

<u>Ex.</u> U is uniform $\iff$ $\forall u_1, u_2 \in U$ nonzero $\exists a_1, a_2 \in A$ s.t. $u_1 a_1 = u_2 a_2$ and they are nonzero.

<u>Pf.</u> Supp. U is uniform. Then $u_1 A$ is essential so $u_1 A \cap u_2 A \neq 0$. Conversely, if

the condition holds and $N, M \subset U$ nonzero submodules ~~and~~ we must show $M \cap N \neq 0$.
Pick $u_1 \in M$, $u_2 \in N \neq 0$, then $\exists\, a_1, a_2$ as needed so $M \cap N \neq 0$.
$(u_1 a_1 = u_2 a_2 \neq 0)$.

<u>Cor.</u> A nonzero non-uniform module M contains essential $M_1 \oplus M_2$, where $M_1, M_2 \neq 0$

<u>Pf.</u> Take $M_1 \neq 0$ non-essential, M_2 = complement.

<u>Def.</u> M has <u>finite Goldie rank</u> if it does not contain an infinite direct sum of nonzero submodules.
E.g. a Noetherian module has finite Goldie rank.

<u>Thm.</u> (Goldie) Let M be a module of finite Goldie rank.
(1) If $M \neq 0$ then it contains a uniform submodule.
(2) M contains an essential submodule which is a finite direct

sum of uniform submodules.

3. Let $E \subset M$ be such essential submodule $E = U_1 \oplus \cdots \oplus U_r$, U_i uniform, and suppose $N = N_1 \oplus \cdots \oplus N_s \subset M$, $N_i \neq 0$. Then $r \geq s$. Thus the number r, called the Goldie rank goldie(M), depends only on M.

4. Also goldie$(M) \leq$ goldie(M'), and equality holds only if M is an essential submodule of M'.

5. Goldie$(M) = 0 \iff M = 0$.

Rem. Goldie rank is also called uniform dimension.

~~Pf~~

Remark. 1. If $A = Mat_n(k)$, k a skew-field, then goldie$(A) = n$.

2. A a comm. domain with fraction field K, M a torsion-free A-module $\Rightarrow$ goldie$(M) = \dim_K(M \otimes_A K)$.

Proof. $\boxed{1,2}$ If M is not uniform then it contains an essential direct

sum of nonzero submodules.
If one of them is not uniform, it contains an essential direct sum, etc. Since M does not contain an infinite direct sum, this process must stop, and all summands are uniform.

$\boxed{3}$ Let $W \subset M$, $W_i = W \cap U_i$. If $W_i \neq 0$ for all i then W_i is essential in U_i hence $W_1 \oplus \cdots \oplus W_r$ is essential in E $\Rightarrow W$ is essential in M.
Now consider $N = N_1 \oplus \cdots \oplus N_s$, $s > 0$. Then $W = N_2 \oplus \cdots \oplus N_s$ is not essential in M, so $W \cap U_i = 0$ for some i, say $i = 1$. So we can replace N_1 by U_1. Continuing, we can replace every N_j by an appropriate U_i, so $s \leq r$

$\boxed{4}$ follows from $\boxed{3}$

$\boxed{5}$ follows from (1).

Now we can proceed with the proof of Goldie's thm.

<u>Lemma 1</u> The left annihilator of an essential right ideal is zero.

<u>Pf.</u> Assume the contrary. Then $I \subset A$ is an essential right ideal s.t. $\mathrm{lann}(I) \neq 0$. So we have a left ideal $J \neq 0$ s.t. $\mathrm{rAnn}(J)$ is essential (e.g. $J = \mathrm{lann}(I)$ then $\mathrm{rann}(J) \supset I$ so it is essential.)

Let us choose J so that its right annihilator R is essential and maximal among right annihilators of nonzero left ideals (exists because A is noetherian), and we will obtain a contradiction by showing that $J=0$.

Since A is semiprime, it suffices to show that $J^2 = 0$.

If $J^2 \neq 0$ then $\exists x, y \in J$ s.t. $xy \neq 0$. Then yA is a nonzero right ideal. So since R is essential, $R \cap yA \neq 0$. So $\exists a \in A$ s.t. $ya \in R$, $ya \neq 0$. But $xya = 0$, so ~~xya~~ So $\operatorname{rann}(xy) \supsetneq \operatorname{rann}(x)$.

But $\operatorname{rann}(y) = \operatorname{rann}(Ay) \supset R$. since $Ay \subset J$. Since R was chosen to be maximal, we get $\operatorname{rann}(xy) = A$, so $xy = 0$, which is a contradiction. ∎

<u>Lemma 2.</u>

Let $s \in A$. If $\operatorname{rann}(s) = 0$ then s is a regular element and sA is an essential right ideal.

<u>Pf.</u> If $\operatorname{rann}(s) = 0$ then $sA \cong A$ as a right A-module, so the goldie rank of A and sA are equal. Hence sA is an essential right ideal. Therefore by the

previous lemma, $\mathrm{lann}(sA) = \mathrm{lann}(s) = 0$, so s is regular.

<u>Lemma 3</u>. Every right ideal N of A contains an element x s.t. $\mathrm{rann}(x) \cap N = 0$.

<u>Proof</u>. Case 1. N is uniform. Since A is semiprime, $N^2 \neq 0$, so can choose $x, y \in N$ s.t. $xy \neq 0$. We claim that then $W = \mathrm{rann}(x) \cap N = 0$. Otherwise if $W \neq 0$ then W is essential in N (as N is uniform). Consider the homom $\lambda_y : A \to N$ of right modules $\lambda_y(\alpha) = y\alpha$. Then $W' = \lambda_y^{-1}(W)$ is an essential right ideal. But $yW' \subset W$ hence $xyW' = 0$. Since $xy \neq 0$, the left annihilator of W' is nonzero, which contradicts Lemma 1.

<u>Case 2</u>: The general case.

Consider the submodules $V \subset N$ which contain $v \in V$ with $\mathrm{rann}(v) \cap V = 0$, and choose V maximal among such submodules. (exists because A is right Noetherian)

We'll show that $V = N$ by showing that $\mathrm{rann}(v) \cap N = 0$ (for that particular v). Otherwise, if $\mathrm{rann}(v) \cap N \neq 0$, choose a uniform submodule $U \subset \mathrm{rann}(v) \cap N$. It has been shown $\exists u \in U$ s.t. $\mathrm{rann}(u) \cap U = 0$. Let $x = u + v$, and claim that $\mathrm{rann}(x) \cap (U+V) = 0$. Since $U + V \supsetneq V$, this will provide a contradiction (as $U \cap V \subseteq \mathrm{rann}(v) \cap V = 0$, so $U + V = U \oplus V$).

Suppose $x' \in \mathrm{rann}(x) \cap (U \oplus V)$. So $xx' = 0$ and $x' = u' + v'$. Thus $ux' = 0$, $vx' = 0$, so $uu' + uv' = 0$, $vu' + vv' = 0$. But $vu' = 0$ as $U \subset \mathrm{rann}(v)$.

Hence $vv' = 0$, and since $\mathrm{rann}(v) \cap V = 0$ we have $v' = 0$. Then $uu' = 0$, hence $u' = 0$ (as $\mathrm{rann}(u) \cap V = 0$), so $x' = 0$, as claimed. ∎

<u>Lemma 4.</u> Every essential right ideal $E \subseteq A$ contains a regular element.

<u>Pf.</u> Let $x \in E$ be as in the previous lemma: $\mathrm{rann}(x) \cap E = 0$. Then, since E is essential, $\mathrm{rann}(x) = 0$, so x is regular by Lemma 2. ∎

Now we are in a position to check Ore condition. If s is regular, sA is an essential right ideal by Lemma 2. Consider the map $\lambda_a : A \to A$, $\lambda_a(\alpha) = a\alpha$. Then $\lambda_a^{-1}(sA)$ is also an essential right ideal, so contains a reg. element, say s_1, by Lemma 4.

So $as_1 \in sA$, i.e. $as_1 = sa_1$, as claimed.

Finally, to prove Goldie's thm, we need to show that the ring AS^{-1} is semisimple.

<u>Lemma 5.</u> Let S be a right Ore set in a ring A, $Q = AS^{-1}$.

1. A is an essential submodule of Q (as a right module).
2. If N is an essential right ideal of Q then $N \cap A$ is an essential right ideal in A.

<u>Pf.</u> 1. Since $A \hookrightarrow Q$ is a ring homom., A is a submodule of Q. Let X be a nonzero submod of Q, and $q = as^{-1} \in X$, $q \neq 0$. Then $a = qs \in X \cap A$ and $a \neq 0$. So $A \subset Q$ is essential.

2. Let $X \subset A$ be a nonzero right ideal. Then XS^{-1} is a nonzero right ideal of Q, so $XS^{-1} \cap N \neq 0$. Clearing denom. $X \cap N \neq 0$, and $X \cap N = X \cap (N \cap A)$. So $N \cap A$ is essential.

Now to show that Q is semisimple, show that Q has no proper essential right ideal. (this suffices by what was shown above) Let $N = N_Q$ be an essential right ideal of Q. Then $N \cap A$ is an essential right ideal in A by Lemma 5, so contains a regular element s by Lemma 4. Then s is a unit in Q, so N contains a unit and $N = Q$.