Lecture 20

Goldie's Theorem.

Thm. (Goldie) Let $A$ be a semiprime, right Noetherian ring. Then the set $S$ of regular elements of $A$ is an ore set, and $Q = AS^{-1}$ is a semisimple ring (direct sum of finitely many matrix algebras).

Remark. For commutative rings, semiprime $\iff$ reduced (no nil/potents), so $AS^{-1}$ is a direct sum of fields corresponding to irreducible components of Spec $A$.

Most of the rest of the lecture will be the proof of this theorem. The main part of the proof is to show that the set of regular elements is an ore set. It is clear that $S$ is multiplicative so it remains to check the Ore condition.
\[ \forall a \in A, s \in S \exists a_1, a_2, s_1, s_2 \text{ such that } sa_1 = as_2. \]

For this we will introduce the notion of an essential submodule.

**Def.** A submodule \( E \) of a right \( A \)-module \( M \) is essential if for every submodule \( X \leq M \) we have \( X \cap E \neq 0 \).

**Exerc.** Let \( A \) be prime. Then any 2-sided nonzero right ideal is an essential right submodule. Indeed, if \( MNI = 0 \) then \( MI \leq MNI = 0 \) so \( MI = 0 \) \( \implies M = 0 \) (as \( I \neq 0 \)).

**Prop. 1** (1) Let \( M_1 \leq M_2 \leq M_3 \) be \( A \)-modules.

If \( M_1 \) is essential in \( M_2 \) and \( M_2 \) essential in \( M_3 \) then \( M_1 \) is essential in \( M_3 \).

(2) Let \( f : M' \to M \) be a morphism of right \( A \)-modules. Then if \( W \leq M \) is essential then \( W' = f^{-1}(W) \) is essential in \( M' \).
(3) If $E_i \subseteq M_i$ are essential submodules then $E_1 \oplus \ldots \oplus E_n$ is an essential submodule of $M_1 \oplus \ldots \oplus M_n$.

Proof. [1] is clear

2. Let $X'$ be a nonzero submodule in $M$. We need to show that $X' \cap f'(w) \neq 0$.

But $f(X' \cap f'(w)) = f(X') \cap w$.

So if $f(x') \neq 0$ then $f(x') \cap w \neq 0$, hence $X' \cap f'(w) \neq 0$. Otherwise, if $f(x') = 0$ then $X' \subseteq f'(w) = w$ so $X' \cap f'(w) = 0$.

3. Let $E \subseteq M$ be an essential submodule then $E \oplus N$ is essential in $M \oplus N$ by (2).

So we can consider the chain

$E \oplus \ldots \oplus E_n \subseteq E_1 \oplus \ldots \oplus E_n \subseteq \ldots \subseteq M_n \subseteq \ldots \subseteq M_1 \oplus \ldots \oplus M_n$.

and apply (4).

Prop. 2. Let $M, CM$. Then $FM_2 \subseteq M$

s.t. $M_1 \cap M_2 = 0$ and $M_1 \oplus M_2$ is essential.
Proof. By Zorn's lemma, we may take for $M_2$ a maximal submodule such that $M_1 \cap M_2 = 0$. Then $M_1 \oplus M_2$ is essential.

Cor. $M$ has no proper essential submodule $\iff M$ is semisimple.

PF. Suppose $M$ is as stated and $M_1 \subseteq M$. Find $M_2$ with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2 = M$.

Def. $M$ is uniform if $M \neq 0$ and every $N \subseteq M$, $N \neq 0$ is essential.

Ex. If $A$ is a commutative domain, $\text{Frac}(A) = K$ then a torsion-free $A$-module $M$ is uniform $\iff M \otimes K$ is a 1-dim $A$-vector space.

Also $k[t]/f(t)$ is a uniform $k[t]$-module.

Ex. $U$ is uniform $\iff \forall u_1, u_2 \in U$ nonzero $\exists a, a_2 \in A$ s.t. $u_1a_1 = u_2a_2$ and they are nonzero.

Def. Supp. $U$ is uniform. Then $u_1A$ is essential so $u_1A \cap u_2A \neq 0$. Conversely, if
The condition holds and $NM \leq U$ are nonzero submodules. Thus we must show $MN \neq 0$.
Pick $u_1 \in M$, $u_2 \in N \neq 0$, then $3a_1, a_2$, as needed so $MN \neq 0$.

Cor. A nonzero non-uniform module $M$ contains $M \oplus M_2$, where $M, M_2 \neq 0$.
Proof. Take $M_1 \neq 0$ non-essential, $M_2$ complement.
Def. $M$ has finite Goldie rank if it does not contain an infinite direct sum of nonzero submodules.
E.g. a Noetherian module has finite Goldie rank.

Thm (Goldie) let $M$ be a module of finite Goldie rank.
(1) If $M \neq 0$ then it contains a uniform submodule.
(2) $M$ contains an essential submodule which is a finite direct
sum of uniform submodules.

3. Let $E \leq M$ be such essential submodule $E = U_1 \oplus \ldots \oplus U_r$, $U_i$ uniform, and suppose $A = N_1 \oplus \ldots \oplus N_s \leq M$, $N_i \neq 0$. Then $r \geq s$. Thus the number $r$, called the Goldie rank $\text{goldie}(M)$, depends only on $M$.

4. Also $\text{goldie}(M) \leq \text{goldie}(M')$, and equality holds only if $M$ is an essential submodule of $M'$.

5. $\text{goldie}(M) = 0 \iff M = 0$.

**Remark.** 1. If $A = \text{Mat}_n(k)$, $k$ a skew-field, then $\text{goldie}(A) = n$.

2. If a comm. domain with fraction field $K$, $M$ a torsion-free $A$-module $\Rightarrow \text{goldie}(M) = \dim_k (A \otimes_A K)$.

**Proof.** 1. If $M$ is not uniform then it contains an essential direct
sum of non-zero submodules. If one of them is not uniform, it contains an essential direct sum, etc. Since $M$ does not contain an infinite direct sum, this process must stop, and all remainders are uniform.

3. Let $W \subseteq M$, $W = W \cap U_i$. If $W \neq 0$ for all $i$, then $W_i$ is essential in $V_i$, hence $W \otimes \cdots \otimes W_r$ is essential in $E$.

$\Rightarrow W$ is essential in $M$.

Now consider $N = N_1 \otimes \cdots \otimes N_s$, $s > 0$.

Then $W = N_2 \otimes \cdots \otimes N_s$ is not essential in $M$, so $W \cap U_i = 0$ for some $i$, say $i = 1$. So we can replace $N_1$ by $U_1$. Continuing, we can replace every $N_j$ by an appropriate $U_j$, so $s \leq r$.

\[ \text{[4]} \text{ follows from [3]} \]

\[ \text{[5]} \text{ follows from (1).} \]
Now we can proceed with the proof of Goldie's theorem.

**Lemma 1.** The left annihilator of an essential right ideal is zero.

**Proof.** Assume the contrary. Then $I\cap A$ is an essential right ideal s.t. $\text{lann}(I) \neq 0$. So we have a left ideal $J \neq 0$ s.t. $\text{rAnn}(J)$ is essential (e.g. $J = \text{lann}(I)$) and $\text{rann}(J) = I$ so it is essential.

Let us choose $J$ so that its right annihilator $R$ is essential and maximal among right annihilators of non-zero left ideals, and we will obtain a contradiction by showing that $J = 0$.

Since $A$ is semiprime, it suffices to show that $J^2 = 0$. 
If \( J \neq 0 \) then \( \exists x,y \in J \) s.t. \( xy \neq 0 \). Then \( yA \) is a nonzero right ideal. So since \( R \) is essentia
\( R \cap yA \neq 0 \). So \( \exists e \in A \) s.t. \( yae \in R \), \( ya \neq 0 \). But \( xya = 0 \), so
\( y \). So \( \text{rann}(xy) \neq \text{rann}(x) \).
But \( \text{rann}(y) = \text{rann}(Ay) \supset R \).
since \( Ay \subset J \). Since \( R \) was chosen
to be maximal, we get \( \text{rann}(xy) = A \),
so \( xy = 0 \), which is a contradiction.

Lemma 2.
Let \( s \in A \). If \( \text{rann}(s) = 0 \) then \( s \) is
a regular element and \( sA \) is an
essential right ideal.
If \( \text{rann}(s) = 0 \) then \( sA \cong A \\
as a right \( A \)-module, so the

goldie rank of \( A \) and \( sA \) are
equal. Hence \( sA \) is an essential
right ideal. Therefore by the
previous lemma, \( \text{lam}(sA) = \text{lam}(s) = 0 \), so \( s \) is regular.

**Lemma 3.** Every right ideal \( N \) of \( A \) contains an element \( x \) s.t. \( \text{rann}(x) \cap N = 0 \).

**Proof.** Case 1. \( N \) is uniform.

Since \( A \) is semi-prime, \( N \neq 0 \), so can choose \( x, y \in N \) s.t. \( xy \neq 0 \). We claim that then \( W = \text{rann}(x) \cap N = 0 \). Otherwise if \( W \neq 0 \) then \( W \) is essential in \( N \) (as \( N \) is uniform). Consider the homomorphism \( \lambda_y : A \to N \) of right modules \( \lambda_y(x) = yx \). Then \( W' = \lambda_y(W) \) is an essential right ideal.

But \( yW' \subseteq W \) hence \( xyW' = 0 \).

Since \( xy \neq 0 \), the left annihilator of \( W' \) is nonzero, which contradicts Lemma 1.
Case 2: The general case.

Consider the submodule $\mathcal{V} \subseteq \mathcal{N}$ which contain $v \in \mathcal{V}$ with $\text{rann}(v) \cap \mathcal{N} = 0$ and choose $\mathcal{V}$ maximal (existence by Zorn's Lemma). We'll show that $\mathcal{V} = \mathcal{N}$ by showing that $\text{rann}(v) \cap \mathcal{N} = 0$. Otherwise, if $\text{rann}(v) \cap \mathcal{N} \neq 0$, choose a uniform submodule $U \subseteq \text{rann}(v) \cap \mathcal{N}$.

It has been shown $\exists u \in U$ s.t. $\text{rann}(u) \cap U = 0$. Let $x = u + v$ and claim that $\text{rann}(x) \cap (U + V) = 0$.

Since $U + V \supseteq V$, this will provide a contradiction (as $U \cap V = 0$ and $\text{rann}(v) \cap \mathcal{N} = 0$, so $U + V = U \oplus V$).

Suppose $x' \in \text{rann}(v) \cap (U + V)$. So $xx' = 0$ and $x' = u' + v'$.

Thus $ux' = 0$, $vx' = 0$.

So $uu' + uv' = 0$, $vu' + vv' = 0$.

But $uv' = 0$ as $U \subseteq \text{rann}(v)$.
Hence $uv' = 0$, and since $\text{ran}(v) \cap v' = 0$, we have $u' = 0$. Then $uu' = 0$, hence $u' = 0$, so $x' = 0$, as claimed (as $\text{ran}(v) \cap u = 0$).

Lemma 4. Every essential right ideal contains a regular element.

Proof. Let $x \in E$ be as in the previous lemma: $\text{ran}(x) \cap E = 0$. Then, since $E$ is essential, $\text{ran}(x) = 0$, so $x$ is regular by Lemma 2. \[\square\]

Now we are in a position to check the condition. If $s$ is regular, $sA$ is an essential right ideal by Lemma 2. Consider the map $\lambda_a : A \rightarrow A$, $\lambda_a(x) = ax$. Then $\lambda_a^{-1}(sA)$ is also an essential right ideal, so contains a regular element, say $s_1$, by Lemma 4.

So $as_1 \in sA$, i.e. $as_1 = sa$, as claimed.
Finally, to prove Goldie's theorem, we need to show that the ring \( AS^1 \) is semisimple.

Lemma 5. Let \( S \) be a right Ore set in a ring \( A \), \( Q = AS^{-1} \).

1. If \( A \) is an essential submodule of \( Q \) (as a right module),

2. If \( N \) is an essential right ideal of \( Q \) then \( NDA \) is an essential right ideal in \( A \).

**Proof.** Since \( ACQ \) is a ring homomorphism, \( A \) is a submodule of \( Q \). Let \( X \) be a nonzero submodule of \( Q \), and \( q = aS^{-1} \in X \), \( q \neq 0 \). Then \( a = qs \in X \cap A \) and \( a \neq 0 \). So \( ACQ \) is essential.

2. Let \( X \subseteq A \) be a nonzero right ideal. Then \( XS^{-1} \) is a nonzero right ideal of \( Q \), so \( XS^{-1} \cap N \neq 0 \).

Clearly denote \( X \cap N \neq 0 \) and \( X \cap N = X \cap N \).

So \( NDA \) is essential.
Now to show that $Q$ is semisimple, show that $Q$ has no proper essential right ideal. (This suffices by what was shown above.) Let $N=Q$ be an essential right ideal of $Q$. Then $NA$ is an essential right ideal in $A$ by Lemma 5, so contains a regular element $s$ by Lemma 4. Then $s$ is a unit in $Q$, so $N$ contains a unit and $N=Q$. 