Semiprime rings

Let $R$ be a ring and $I \subseteq R$ an ideal.
Recall that $I$ is prime if for any ideals $A, B \subseteq I$ we have $A \subseteq I$ or $B \subseteq I$.

**Def.** $R$ is prime if the zero ideal is prime, i.e., if $A, B, C \subseteq R$ are ideals and $AB = 0$ then $A = 0$ or $B = 0$.

**Remark.**
1. $I$ is prime $\iff$ $R/I$ is prime.
2. If $MCR$ is a maximal ideal then $R/M$ is a simple ring, hence prime.
Thus $M$ is prime, so prime ideals always exist.

3. If $R$ is a domain then $\text{Mat}_n(R)$ is prime for all $n$ but not a domain (it is false if $R$ is a field).

4. If $R$ is called completely prime

It's easy to see that $I$ is completely prime if $R/I$ is a domain. So every completely prime ideal is prime but not vice versa.
Prop 1. TFAE:

1. \( R \) is prime.
2. \( \forall \text{ left ideals } A, B \subseteq R \text{ s.t. } AB = 0 \text{ we have } A = 0 \text{ or } B = 0.\)
3. Same as (2) for right ideals.
4. If \( a, b \in R \) and \( ab = 0 \) then \( a = 0 \) or \( b = 0.\)

Proof: \( 1 \Rightarrow 2, 3. \) Let \( A, B \subseteq R \) be left ideals.
Then \( (AR)(BR) = A(ARB)R = ABR = 0.\)
Since \( AR, BR \) are 2-sided ideals, we have \( AR = 0 \text{ or } BR = 0, \) so \( A = 0 \text{ or } B = 0.\) Same for right ideals.

\( 2, 3 \Rightarrow 4. \) Let \( a, b \in R, \) \( ab = 0, \) then \( RaRb = 0.\)
Since \( Ra, Rb \) are left ideals, this implies \( Ra = 0 \text{ or } Rb = 0, \) so \( a = 0 \text{ or } b = 0.\)

\( 4 \Rightarrow 1. \) Let \( A, B \subseteq R \) be 2-sided ideals.
\( A, B \neq 0. \) So there are \( a \in A, b \in B, ab \neq 0.\)
Then \( \exists c \in R, \) \( axb \neq 0, \) so \( AB \neq 0 \) or \( ax \in A, bx \in B. \)

Cor. If \( R \) is commutative then \( R \) is prime \( \iff \) \( R \) is a domain.
Proof: Apply (4).
An ideal $I$ is semiprime if for any ideal $A$ of $R$ such that $A^2 \subseteq I$ we have $A \cap I = 0$.

Corollary 1. $I$ is semiprime if and only if $R/I$ is semiprime.

2. A prime ideal or zig are semiprime.

3. $R$ is semiprime if and only if it has no nonzero nilpotent ideals. Indeed, suppose $R$ is semiprime and $I \subseteq R$ is a nilpotent ideal. Then there is a smallest integer $N \geq 0$ such that $I^N \neq 0$. Let $J = I^{2^N}$. Then $J^2 = 0$ but $J \neq 0$, a contradiction.

Proposition 2. TFAE:

1. $R$ is semiprime.

2. For any left ideal $A$ of $R$ such that $A^2 = 0$ we have $A = 0$.

3. Same as (2) for right ideal.

4. If $a \in R$ and $Ra = 0$ then $a = 0$.

Proof: Same as prop. 1 with $a = b$, $A = B$. 
Corollary. If $R$ is commutative then $R$ is semiprime iff $R$ is reduced (i.e. $0$ is a radical ideal). So a semiprime ideal in a commutative ring is the same as a radical ideal.

Prop 3. The intersection of any collection of semiprime ideals is semiprime.

Proof. Let $I_\alpha$ be semiprime ideals in $R$, $I = \cap I_\alpha$, let $A \in R$ with $A^2 \in I$. Then $ACI_\alpha$ for all $\alpha$, so $A \in I_\alpha$ for all $\alpha$, so $A \in I$.

Cor. If $R_\alpha$ are semiprime rings then $\prod R_\alpha$ is semiprime.

Note however that if $R_1, R_2$ are prime then $R_1 \times R_2$ is not prime (only semiprime).

Ex. If $R_\alpha$ are domains and $n_\alpha$ are positive integers then $\prod \text{Mat}_{n_\alpha}(R_\alpha)$ is semiprime.
The prime radical:
Def. The prime radical of $R$, $N(R)$ (also called the Baer radical) is the intersection of all prime ideals in $R$.

Rem. By Prop 3, $N(R)$ is semiprime.

The following theorem characterizes elements of $N(R)$.

Def. An element $a \in R$ is strongly nilpotent if there sequence $a_0 = a, a_1, a_2, \ldots \in R$ such that $a_{n+1} \in a_n R a_n$ for $n \geq 0$, we have $a_n = 0$ for some $N$ (and hence for all $n \geq N$).

It is clear that any strongly nilpotent element is nilpotent, but not vice versa ($\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R})$ is not strongly nilpotent).

Thm. $N(R)$ is the set of all strongly nilpotent elements of $R$.

Pf. Suppose $a \notin N(R)$. Then $I$ a prime ideal $P \subset R$ such that $a \notin P$. Now define
a function $\sigma : R \times R \to R \times R$ such that $\sigma(b) \leq bRb$ for each $b$. Such a function exists since $bRb \equiv 0 \mod p$ if $b \neq 0 \mod p$ (as $p$ is prime, hence semiprime). Now define the sequence $a_0 = a, a_1, a_2, \ldots$ recursively by $a_{n+1} = \sigma(a_n)$. Then $a_{n+1} \in a_n Ran$ and $a_n \to 0$ for all $n$, so $a$ is not strongly nilpotent.

Conversely, suppose $a$ is not strongly nilpotent and let $a_0 = a, a_1, a_2, \ldots \in R$ be a sequence such that $a_{n+1} \in a_n Ran$ and $a_n \to 0$ for all $n$. Let $P$ be an ideal maximal among those not containing $a_n \forall n$ (they exist, e.g. 0). We claim that $P$ is prime. Indeed, let $A, B \subseteq R$, $A, B \neq P$. Then $A + P \neq P$, $B + P \neq P$, so $\exists m a_m \in A + P$ and $\exists n a_n \in B + P$. Then $\forall N \geq \max(m, n), a_N \in (A + P) \cap (B + P) \Rightarrow \forall N \geq \max(m, n) + 1$ we have
\[ a_{n} \in (A + B) \cdot (B + B) = AB + B. \]

Thus \( AB \neq 0 \) (as \( a_{n} \neq 0 \)).

Now, since \( a \neq 0 \), we have \( a \neq N(R) \).

The theorem is proved.

Cor: The prime radical is a nil ideal.

i.e. consists of nilpotent elements.

The nil radical \( N(R) \) is the smallest semiprime ideal of \( R \).

Prop. TFAE: (intersection of all semiprime ideals)

1. \( R \) is semiprime
2. \( N(R) = 0 \)
3. If \( A, B \) are left, right, or two-sided ideals of \( R \) such that \( AB = 0 \) then \( A \cap B = 0 \).

Proof: \[ 2 \Rightarrow 1: \] Since \( 0 = N(R) \), it is semiprime, so \( R \) is semiprime.

\[ 1 \Rightarrow 2: \] Since \( R \) is semiprime, for each \( a \neq 0 \) we have \( aRa \neq 0 \). So \( 0: R \cdot 0 \to R \cdot 0 \) with \( a(b) \in bRa \land a \in R_{0} \).

Now given \( a \in R \cdot 0 \), form the sequence \( a_{0} = a, \ a_{1} = \delta(a), \ a_{2} = \delta^{2}(a) \) etc.

Then \( a_{n} \to 0 \) and \( a_{n} \in R_{0} \), so
a, is not strongly nilpotent. Thus $a \notin N(R)$ by the theorem. Hence, $N(R) = 0$.

2 $\Rightarrow$ 1 Let $I = R$, $I^2 = 0$, then by (4) $I \cap I = I = 0$, so $R$ is semiprime.

2 $\Rightarrow$ 3 Let $R$ be semiprime and $AB = 0$, for right, left or 2-sided ideals $A, B$. Say, $A, B$ are left ideals. Then $\overline{A} \cap \overline{B} = \emptyset$, hence $A_0 \cdot B_0 \subset \emptyset$ for each prime $P$. But then $A_+ \text{ or } B_+ \subset \emptyset$ for each prime $P$, so $A_0 \cap B_0 \subset \emptyset$ for each $P$, hence $A \cap B = 0$.

Prop. Suppose $R$ is left noetherian. Then $N(R)$ is the largest nilpotent ideal in $R$, which is the sum of all nilpotent ideals in $R$.

Proof. Let $\overline{N}(R)$ be the union of all nilpotent ideals in $R$. Then $\overline{N}(R)$ is an ideal. Also, since $R$ is left noetherian, $\overline{N}(R)$ is
finitely generated, so
\[ N(R) = I_1 + \cdots + I_m \]
where we may assume WLOG that
\[ I_k = R a_k R. \]
The element \( a_k \) lies in some nilpotent ideal, so \( I_k \) is nilpotent: \( I_k^{2n} = 0 \). Now if
\[ I^n = 0 \text{ and } J^m = 0, \]
it's easy to see that \( (I+J)^{n+m} = 0 \). Thus \( N(R) \) is nilpotent, hence it's the largest nilpotent ideal in \( R \). Also, \( N(R) < N(R) \), since \( R/N(R) \) is semiprime and can't contain nilpotent ideals \( \neq 0 \).

It remains to show that \( N(R) = N(R) \).

For this we show that \( N(R) \) is semiprime (it's enough since \( N(R) \) is the smallest semiprime ideal) let \( I \subset R \) such that \( I^2 \subset N(R) \). Then \( I^2 \) is nilpotent, so \( I \) is nilpotent
\[ \Rightarrow I < N(R) \]
(as \( N(R) \) is the sum of all nilpotent ideals). Thus \( N(R) \) is semiprime by desired.
Def. A 2-sided ideal $I \triangleleft R$ is primitive if it is the annihilator of a simple $R$-module. $R$ is primitive if $0 \triangleleft R$ is a primitive ideal, i.e. $R$ has a faithful simple module.

Remark. 1. $I$ is primitive ($\iff$) $R/I$ is primitive.

2. Primitive ideals exist. Indeed, any maximal ideal $m$ is primitive, namely $R/m$ is a simple algebra. So if $J \subseteq R/m$ is a maximal left ideal then $(R/m)/J$ is a simple $R$-module with annihilator $m$.

3. A primitive ideal $I$ is prime.

Indeed, let $A, B \triangleleft R$ be two-sided ideals s.t. $AB \subseteq I$. Let $V$ be a simple $R$-module with annihilator $I$.

Then $ABV = 0$. So $BV = 0$ or $BV = V$.

In the first case $B \subseteq I$, and in
the second case $ABV = AV = 0$

so $A \subseteq I$. Hence $I$ is prime.

However, a prime ideal need not be primitive,

E.g. $0 \subset \mathbb{Z}$ is prime but not primitive.

**Def.** The Jacobson radical $J(R)$ is

the intersection of all primitive ideals, i.e. the common annihilator of all simple $R$-modules.

Since a primitive ideal is prime,

we have $N(R) \subseteq J(R)$.

However, $N(R) \neq J(R)$ in general,

E.g. if $R = \mathbb{C}[\lbrack \lbrack t \rceil \rceil]$ then

$N(R) = 0$ but $J(R) = (t)$.

**Remark.** If $R$ is commutative

then $N(R)$ and $J(R)$ are as usual.

In particular, $N(R)$ is the set of

all nilpotent elements of $R$

(as the notions of nilpotent and

strongly nilpotent coincide in this

case).