Lecture 19
November 19

Semiprime rings

Let R be a ring and $I \subset R$ an ideal. Recall that I is <u>prime</u> if for any ideals $A, B \subset I$ we have $A \subset I$ or $B \subset I$.

<u>Def.</u> R is prime if the zero ideal is prime, i.e. if $A, B \subset R$ are ideals and $AB = 0$ then $A = 0$ or $B = 0$.

<u>Remarks.</u> 1. I is prime $\Leftrightarrow$ R/I is prime.

2. If $m \subset R$ is a maximal ideal then R/m is a simple ring, hence prime. Thus m is prime, so prime ideals always exist.

3. If R is a domain then $Mat_n(R)$ is prime for all n (exercise) but not a domain if $n \geq 2$ (it is obvious if R is a field). So while any domain is prime, a prime ring need not be a domain.

4. $I \subset R$ is called <u>completely prime</u> if $\forall a, b \in R$ s.t. $ab \in I$ one has $a \in I$ or $b \in I$. It's easy to see that I is completely prime iff R/I is a domain. So every completely prime ideal is prime but not vice versa.

<u>Prop. 1</u> TFAE:

1. R is prime
2. $\forall$ left ideals $A, B \subset R$ s.t. $AB=0$ we have $A=0$ or $B=0$.
3. Same as (2) for right ideals
4. If $a, b \in R$ and $aRb=0$ then $a=0$ or $b=0$.

<u>Proof.</u> $\boxed{1 \Rightarrow 2,3}$. Let $A, B \subset R$ be left ideals. Then $(AR)(BR) = A(RB)R = ABR = 0$. Since AR, BR are 2-sided ideals, we have $AR=0$ or $BR=0$, so $A=0$ or $B=0$. Same for right ideals

$\boxed{2,3 \Rightarrow 4}$ Let $a, b \in R$, $aRb=0$, then $RaRb=0$. Since Ra, Rb are left ideals, this implies $Ra=0$ or $Rb=0$, so $a=0$ or $b=0$.

$\boxed{4 \Rightarrow 1}$ Let $A, B \subset R$ be 2-sided ideals, $A, B \neq 0$. So there are $a \in A$, $b \in B$, $a, b \neq 0$. Then $\exists x \in R$ $axb \neq 0$, so $AB \neq 0$ as $ax \in A$, $b \in B$. ∎

<u>Cor.</u> If R is commutative then R is prime $\Leftrightarrow$ R is a domain

<u>Pf.</u> Apply (4).

An ideal

<u>Def.</u> $I \subset R$ is semiprime if for any ideal $A \subset R$ such that $A^2 \subset I$ we have $A \subset I$

R is semiprime if for any ideal A such that $A^2 = 0$ we have $A = 0$.

<u>Rem.</u> 1. I is semiprime if and only if R/I is semiprime.

2. A prime ideal or $\mathbb{Z}$ are semiprime.

3. R is semiprime if and only if it has no nonzero nilpotent ideals. Indeed, suppose R is semiprime and $I \subset R$ is a nilpotent ideal. Then there is a smallest integer $n \geq 0$ such that $I^{2^n} \neq 0$. Let $J = I^{2^n}$. Then $J^2 = 0$ but $J \neq 0$, a contradiction.

<u>Proposition 2.</u> TFAE:

1. R is semiprime.
2. For any left ideal $A \subset R$ such that $A^2 = 0$ we have $A = 0$
3. Same as (2) for right ideal
4. If $a \in R$ and $aRa = 0$ then $a = 0$.

<u>Proof.</u> Same as Prop. 1 with $a = b$, $A = B$.

<u>Corollary</u>. If R is commutative then R is semiprime iff R is reduced (i.e. 0 is a radical ideal). So a semiprime ideal in a commutative ring is the same as a radical ideal.

<u>Prop 3.</u> The intersection of any collection of semiprime ideals is semiprime.

<u>Proof</u>. Let I_α be semiprime ideals in R, $I = \bigcap_\alpha I_\alpha$. Let $A \subset R$ with $A^2 \subseteq I$. Then $A^2 \subset I_\alpha$ for all α, so $A \subset I_\alpha$ for all α, so $A \subset I$.

<u>Cor.</u> If R_α are semiprime rings then $\prod_\alpha R_\alpha$ is semiprime.

Note however that if R_1, R_2 are ~~semi~~ prime then $R_1 \times R_2$ is not prime (only semiprime).

<u>Ex.</u> If R_α are domains and n_α are positive integers then $\prod_\alpha \mathrm{Mat}_{n_\alpha}(R_\alpha)$ is semiprime.

The prime radical:

Def. The prime radical of R, $N(R)$ (also called the Baer radical) is the intersection of all prime ideals in R.

Rem. By Prop 3, $N(R)$ is semiprime.

The following theorem characterizes elements of $N(R)$.

Def. An element $a \in R$ is strongly nilpotent if $\forall$ sequence $a_0 = a, a_1, a_2, \ldots \in R$ such that $a_{n+1} \in a_n R a_n$ for $n \geq 0$, we have $a_N = 0$ for some N (and hence for all $n \geq N$).

It is clear that any strongly nilpotent element is nilpotent, ~~but~~ not vice versa ($\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Mat_2(k)$ is not strongly nilpotent).

Thm. $N(R)$ is the set of all strongly nilpotent elements of R.

Pf. Suppose $a \notin N(R)$. Then $\exists$ a prime ideal $\mathfrak{P} \subset R$ such that $a \notin \mathfrak{P}$. Now define

a function $\sigma : R \setminus \mathcal{P} \to R \setminus \mathcal{P}$ such that $\sigma(b) \in bRb$ for each b. Such a function exists since $bRb \neq 0 \bmod \mathcal{P}$ if $b \neq 0 \bmod \mathcal{P}$ (as $\mathcal{P}$ is prime, hence semiprime). Now define the sequence $a_0 = a, a_1, a_2, \ldots$ recursively by $a_{n+1} = \sigma(a_n)$. Then $a_{n+1} \in a_n R a_n$ and $a_n \neq 0$ for all n, so a is not strongly nilpotent.

Conversely, suppose a is not strongly nilpotent and let $a_0 = a, a_1, a_2, \ldots \in R$ be a sequence such that $a_{n+1} \in a_n R a_n$ and $a_n \neq 0$ for all n. Let $\mathcal{P}$ be an ideal maximal among those <u>not</u> containing a_n $\forall n$ (they exist, e.g. 0). We claim that $\mathcal{P}$ is prime. Indeed, let $A, B \subset R$, $A, B \not\subset \mathcal{P}$. Then $A + \mathcal{P} \supsetneq \mathcal{P}$, $B + \mathcal{P} \supsetneq \mathcal{P}$, so $\exists\, m$ $a_m \in A + \mathcal{P}$ and $\exists\, n$ $a_n \in B + \mathcal{P}$. Then $\forall N \geq \max(m,n)$, $a_N \in (A+\mathcal{P}) \cap (B + \mathcal{P})$ $\Rightarrow$ $\forall N \geq \max(m,n)+1$ we have

$a_N \in (A+\mathcal{P})\cdot(B+\mathcal{P}) \subseteq AB+\mathcal{P}$.

Thus $AB \not\subseteq \mathcal{P}$ (as $a_N \notin \mathcal{P}$).

Now, since $a \notin \mathcal{P}$, we have $a \notin N(R)$

The theorem is proved. ∎

<u>Cor.</u> The prime radical is a nil ideal, i.e. consists of nilpotent elements

The next prop shows that $N(R)$ is the smallest semiprime ideal of R (intersection of all semiprime ideals).

<u>Prop.</u> TFAE:

1. R is semiprime
2. $N(R) = 0$
3. If A, B are left, right, or two-sided ideals of R such that $AB = 0$ then $A \cap B = 0$.

<u>Proof.</u> $\boxed{2 \Rightarrow 1}$: Since $0 = N(R)$, it is semiprime, so R is semiprime.

$\boxed{1 \Rightarrow 2}$ Since R is semiprime, for each $a \neq 0$ we have $aRa \neq 0$. So $\exists \sigma : R\setminus 0 \to R\setminus 0$ with $\sigma(b) \in bRb$ $\forall b \in R\setminus 0$.

Now given $a \in R\setminus 0$, form the sequence $a_0 = a$, $a_1 = \sigma(a)$, $a_2 = \sigma^2(a)$ etc.

Then $a_n \neq 0$ and $a_{n+1} \in a_n R a_n$, so

a is not strongly nilpotent.
Thus $a \notin N(R)$ by the theorem.
Hence, $N(R) = 0$.

$\boxed{3 \Rightarrow 1}$ Let $I \subset R$, $I^2 = 0$, then by (4) $I \cap I = I = 0$, so R is semiprime

$\boxed{2 \Rightarrow 3}$ Let R be semiprime and $AB = 0$, for right, left or 2-sided ideals A, B. Say, A, B are left ideals. Then ~~AB=BA=0~~. Hence $A \cdot B \subset \mathcal{P}$ for each prime $\mathcal{P}$. But then A or $B \subset \mathcal{P}$ for each prime $\mathcal{P}$, so $A \cap B \subset \mathcal{P}$ for each $\mathcal{P}$, hence $A \cap B = 0$.

<u>Prop.</u> Suppose R is left noetherian. Then $N(R)$ is the largest nilpotent ideal in R, which is the sum of all nilpotent ideals in R.

<u>Proof.</u> Let $\tilde{N}(R)$ be the sum of all nilpotent ideals in R. Then $\tilde{N}(R)$ is an ideal. Also, since R is left noetherian, $\tilde{N}(R)$ is

finitely generated, so $\tilde{N}(R) = I_1 + \cdots + I_m$ where we may assume WLOG that $I_k = Ra_kR$. The element a_k lies in some nilpotent ideal, so I_k is nilpotent: $I_k^{m_k} = 0$. Now if $I^n = 0$ and $J^m = 0$, it's easy to see that $(I+J)^{n+m} = 0$. Thus $\tilde{N}(R)$ is nilpotent, hence it's the largest nilpotent ideal in R. Also, $\tilde{N}(R) \subset N(R)$, since $R/N(R)$ is semiprime and can't contain nilpotent ideals $\neq 0$.

It remains to show that $N(R) \subset \tilde{N}(R)$. For this we show that $\tilde{N}(R)$ is semiprime (it's enough since $\tilde{N}(R)$ is the smallest semiprime ideal). Let $I \subset R$ such that $I^2 \subset \tilde{N}(R)$. Then I^2 is nilpotent, so I is nilpotent $\Rightarrow I \subset \tilde{N}(R)$ (as $\tilde{N}(R)$ ~~has the maximal nilp. ideals~~ is the sum of all nilpotent ideals). Thus $\tilde{N}(R)$ is semiprime as desired $\blacksquare$

<u>Def.</u> A 2-sided ideal $I < R$ is <u>primitive</u> if it is the annihilator of a simple R-module. R is primitive if $0 < R$ is a primitive ideal, i.e. R has a faithful simple module.

<u>Remark.</u> 1. I is primitive $\Longleftrightarrow$ R/I is primitive

2. primitive ideals exist. Indeed, any maximal ideal m is primitive: namely R/m is a simple algebra. So if $J < R/m$ is a maximal left ideal then $(R/m)/J$ is a simple R-module with annihilator m.

3. A primitive ideal I is prime. Indeed, let $A, B < R$ be two-sided ideals s.t. $AB < I$. Let V be a simple R-module with annihilator I. Then $ABV = 0$. So $BV = 0$ or $BV = V$. In the first case $B < I$, and in

the second case $ABV = AV = 0$ so $A \subset I$. Hence I is prime. However, a prime ideal need not be primitive, e.g. $0 \subset \mathbb{Z}$ is prime but not primitive.

<u>Def.</u> The Jacobson radical $J(R)$ is the intersection of all primitive ideals in R, i.e. the common annihilator of all simple R-modules.

Since a primitive ideal is prime, we have $N(R) \subset J(R)$.

However, $N(R) \neq J(R)$ in general, e.g. if $R = \mathbb{C}[[t]]$ then $N(R) = 0$ but $J(R) = (t)$.

<u>Remark.</u> If R is commutative then $N(R)$ and $J(R)$ are as usual. In particular, $N(R)$ is the set of all nilpotent elements of R (as the notions of nilpotent and strongly nilpotent coincide in this case).