HOMEWORK 5 FOR 18.706, FALL 2018
DUE MONDAY, NOVEMBER 19.

(1)  (a) Show that there is no subring \( A \) in the Hamilton quaternion ring \( H \)
such that \( \mathbb{R} \otimes_{\mathbb{Z}} A = \mathbb{H} \) and \( A \) is an Azumaya algebra over \( \mathbb{Z} \).
(b) Part a) shows that the ring of integer quaternions
\( \mathbb{H}_{\mathbb{Z}} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z} \} \subset \mathbb{H} \) is not an Azumaya algebra
over \( \mathbb{Z} \). Show however that \( \mathbb{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \) is an Azumaya algebra over
\( \mathbb{Z}[\frac{1}{2}] \).

(2)  Let \( F \) be a field of characteristic different from 2. For \( a, b \in F^\times \) let \( A_{a,b} \)
be the four dimensional algebra over \( k \) with basis 1, \( i, j, k \), such that
\( ij = k = -ji, i^2 = a, j^2 = b \).
(a) Check that \( A_{a,b} \) is a c.s.a. over \( F \). Let \( \left( \frac{a,b}{F} \right) \) denote its class in the
Brauer group.
(b) Show that \( \left( \frac{a,b}{F} \right) = 1 \) iff \( b = Nm_{E/F}(z) \) for some \( z \in E = F(\sqrt{a}) \).
(Here the operation in the Brauer group is written multiplicatively).
(c) (Optional) Check\(^1\) that \( \left( \frac{a,b}{F} \right)^2 = 1 = \left( \frac{a,1-a}{F} \right) \) and
\( \left( \frac{a,b,c}{F} \right) = \left( \frac{a,b}{F} \right) \left( \frac{a,c}{F} \right) \).

(3)  (Optional) Let \( G \) be a group and set \( A = k[G] \) for a field \( k \). Problem 6 of
homework 4 gives a homomorphism \( HH^*(A) \to H^*(G,k) \).
(a) Construct a homomorphism \( H^*(G,k) \to HH^*(A) \) which is right inverse to
the above homomorphism.
(b) For \( c \in H^2(G, \mathbb{Z}) \) let \( \tilde{G}_c \) be the corresponding central extension of \( G \)
by \( \mathbb{Z} \), let \( \gamma \in \tilde{G} \) be the image of the generator 1 of \( \mathbb{Z} \) in \( \tilde{G}_c \). Set
\( \tilde{A}_c = k[\tilde{G}]/(\gamma - 1)^2 \).
Check that \( \tilde{A}_c \) is a 1-st order deformation of \( A \) whose class is the image
of \( c_k \) under the homomorphism of part (a). Here \( c_k \) is the image of \( c \)
under the natural map \( H^*(G, \mathbb{Z}) \to H^*(G,k) \).

(4)  An element \( x \) in a ring \( R \) is said to be ad locally nilpotent if \( ad(x) : a \mapsto xa - ax \) is locally nilpotent, i.e. for any \( a \in R \) there exists \( n \) such that
\( ad(x)^n(a) = 0 \). Show that a multiplicative set consisting of ad locally
nilpotent elements satisfies Ore’s condition.

(5)  Let \( K \) be a skew field and \( \phi : K \to K \) a homomorphism. Set \( A = K(x)/(xa = \phi(a)x) \).
Show that the set of powers of \( x \) is a left Ore set, and it is a right Ore
set iff \( \phi \) is surjective. In the latter case prove that \( A \) is a right Ore domain,
i.e. the set of all nonzero elements is right localizing.

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\(^1\)The Milnor \( K_2 \) group of \( F \) is the abelian group generated by symbols \( \{ a, b \} \), \( a, b \in F^\times \)
subject to the relations \( \{ a, bc \} = \{ a, b \} \{ a, c \}, \{ a, b \} = \{ b, a \}^{-1}, \{ a, 1 - a \} = 1 \). The identities
of this problem yield a homomorphism from \( K_2(F)/K_2(F)^2 \to Br(F)[2] \), where \( G[2] \) denotes the
2-torsion in an abelian group \( G \). A difficult theorem by Merkuriev and Suslin asserts that this
homomorphism is onto.
(6) Let us say that an ideal \( I \subset R \) is right localizable if the set of elements regular modulo \( I \) is a right Ore set. (Recall that an element is called regular if it’s neither left nor right zero divisor).

Let \( k \) be a field and \( R \subset Mat_2(k[x]) \) be given by \( R = \{(a_{ij}) \mid a_{21} = 0, a_{11} \in k, a_{22} - a_{11} \in xk[x]\} \). Show that \( R \) is right Noetherian, the ideal of strictly upper triangular matrices is prime, has square zero and is not localizable.

(7) Recall that a module is called uniform if it is nonzero and any two nonzero submodules have a nonzero intersection.

Let \( R \) be the ring of continuous \( \mathbb{C} \)-valued functions on \([0,1]\) with pointwise operations. Show that \( R \) has no uniform ideals.

(8) (Optional, repeated from pset 4) Let \( R \) be the ring of real valued continuous functions on the 2-sphere \( S^2 \). Let \( R^+ \) and \( R^- \) be the ring of continuous functions on the upper and lower closed hemispheres respectively. Let \( A \subset Mat_2(R^+) \times Mat_2(R^-) \) be the subring given by: \( m = (m_+, m_-) \in A \) if \( m_+(\theta) = S(\theta)m_-(\theta)S(\theta)^{-1} \). Here \( \theta \in [0,2\pi] \) is the standard coordinate on the equator circle bounding the upper and the lower hemisphere, and

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S(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}.
\]

Prove that \( A \) is a non-split Azumaya algebra over \( R \).

[Hint: Reduce to the fact that the map \( \pi_1(S^1) \rightarrow \pi_1(S^3) \) induced by the double cover map \( S^3 \rightarrow S^1 \) is not surjective.]