(1) (a) Show that there is no subring $A$ in the Hamilton quaternion ring $\mathbb{H}$ such that $\mathbb{R} \otimes_{\mathbb{Z}} A = \mathbb{H}$ and $A$ is an Azumaya algebra over $\mathbb{Z}$.

(b) Part a) shows that the ring of integer quaternions $\mathbb{H}_2 = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z} \} \subset \mathbb{H}$ is not an Azumaya algebra over $\mathbb{Z}$. Show however that $\mathbb{H}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ is an Azumaya algebra over $\mathbb{Z}[\frac{1}{2}]$.

(2) Let $F$ be a field. For $a, b \in F^\times$ let $A_{a,b}$ be the four dimensional algebra over $k$ with basis $1, i, j, k$, such that $ij = k = -ji, \ i^2 = a, \ j^2 = b$.

(a) Check that $A_{a,b}$ is a c.s.a. over $F$. Let $\left( \frac{a,b}{F} \right)$ denote its class in the Brauer group.

(b) Show that $\left( \frac{a,b}{F} \right) = 1$ iff $b = \text{Nm}_{E/F}(z)$ for some $z \in E = F(\sqrt{a})$.

(Here the operation in the Brauer group is written multiplicatively).

(c) (Optional) Check$^1$ that $\left( \frac{a,b}{F} \right)^2 = 1 = \left( \frac{a,1-a}{F} \right)$ and $\left( \frac{a,bc}{F} \right) = \left( \frac{a,b}{F} \right) \left( \frac{a,c}{F} \right)$

(3) (Optional) Let $G$ be a group and set $A = k[G]$ for a field $k$. Problem 6 of homework 4 gives a homomorphism $HH^*(A) \to H^*(G,k)$.

(a) Construct a homomorphism $H^*(G,k) \to HH^*(A)$ which is right inverse to the above homomorphism.

(b) For $c \in H^2(G,\mathbb{Z})$ let $\tilde{G}_c$ be the corresponding central extension of $G$ by $\mathbb{Z}$, let $\gamma \in \tilde{G}$ be the image of the generator 1 of $\mathbb{Z}$ in $\tilde{G}_c$. Set $\tilde{A}_c = k[\tilde{G}]/\langle \gamma - 1 \rangle^2$.

Check that $\tilde{A}_c$ is a 1-st order deformation of $A$ whose class is the image of $c_k$ under the homomorphism of part (a). Here $c_k$ is the image of $c$ under the natural map $H^*(G,\mathbb{Z}) \to H^*(G,k)$.

(4) An element $x$ in a ring $R$ is said to be ad locally nilpotent if $ad(x) : a \mapsto xa - ax$ is locally nilpotent, i.e. for any $a \in R$ there exists $n$ such that $ad(x)^n(a) = 0$. Show that a multiplicative set consisting of ad locally nilpotent elements satisfies Ore’s condition.

(5) Let $K$ be a skew field and $\phi : K \to K$ a homomorphism. Set $A = K(\phi)/(xa = \phi(a)x)$.

Show that the set of powers of $x$ is a left Ore set, and it is a right Ore set if $\phi$ is surjective. In the latter case prove that $A$ is a right Ore domain, i.e. the set of all nonzero elements is right localizing.

(6) Let us say that an ideal $I \subset R$ is right localizable if the set of elements regular modulo $I$ is a right Ore set. (Recall that an element is called regular if it’s neither left nor right zero divisor).

\footnote{The Milnor $K_2$ group of $F$ is the abelian group generated by symbols $\{ a, b \}$, $a, b \in F^\times$ subject to the relations $\{ a, bc \} = \{ a, b \} \{ a, c \}$, $\{ a, b \} = \{ b, a \}^{-1}$, $\{ a, 1 - a \} = 1$. The identities of this problem yield a homomorphism from $K_2(F)/K_2(F)^2 \to Br(F)[2]$, where $G[2]$ denotes the 2-torsion in an abelian group $G$. A difficult theorem by Merkuriev and Suslin asserts that this homomorphism is onto.}
Let $k$ be a field and $R \subset \text{Mat}_2(k[x])$ be given by $R = \{(a_{ij}) \mid a_{21} = 0, a_{11} \in k, a_{22} - a_{11} \in xk[x]\}$. Show that $R$ is right Noetherian, the ideal of strictly upper triangular matrices is prime, has square zero and is not localizable.

(7) Recall that a module is called uniform if it is nonzero and any two nonzero submodules have a nonzero intersection.

Let $R$ be the ring of continuous $\mathbb{C}$-valued functions on $[0, 1]$ with pointwise operations. Show that $R$ has no uniform ideals.

(8) (Optional, repeated from pset 4) Let $R$ be the ring of real valued continuous functions on the 2-sphere $S^2$. Let $R^+$ and $R^-$ be the ring of continuous functions on the upper and lower closed hemispheres respectively. Let $A \subset \text{Mat}_2(R^+) \times \text{Mat}_2(R^-)$ be the subring given by: $m = (m_+, m_-) \in A$ if $m_+(\theta) = S(\theta)m_-(\theta)S(\theta)^{-1}$. Here $\theta \in [0, 2\pi)$ is the standard coordinate on the equator circle bounding the upper and the lower hemisphere, and

$$S(\theta) = \begin{pmatrix}
\cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\
-\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2})
\end{pmatrix}.$$ 

Prove that $A$ is a non-split Azumaya algebra over $R$.

[Hint: Reduce to the fact that the map $\pi_1(S^1) \to \pi_1(S^1)$ induced by the double cover map $S^1 \to S^1$ is not surjective.]