HOMEWORK 4 FOR 18.706, FALL 2018
DUE THURSDAY, NOVEMBER 1.

(1) Let $R \subset Mat_2(\mathbb{Q})$ consist of matrices $A = (a_{ij})$ such that $a_{11} \in \mathbb{Z}$ and $a_{21} = 0$. Show that the homological dimension of R equals two but the homological dimension of R^{op} (i.e. the homological dimension of the category of right R-modules) equals one.

(2) Let $D \supset K$ be skew fields. Let d_l be the dimension of D as a left K module and d_r be the dimension of D as a right K-module. Show[1] that if D is finite over its center then $d_l = d_r$.

(3) Let D be a skew field which is not algebraic over its center k. Show that $D \otimes_k k(t)$ is a simple Noetherian ring which is not isomorphic to a matrix ring over a skew field.

(4) Let $k = \mathbb{F}_p(t_1, t_2)$ be the field of rational functions in two variables over $\mathbb{F}_p$. Set $D = k\langle x, y\rangle/(xy - yx = 1,\ x^p = t_1,\ y^p = t_2)$.
   (a) Show that D is a skew field of dimension p^2 over its center k. Give an example of a splitting field of D.

   (Hint: Check that $\mathbb{F}_p\langle x, y\rangle/(xy - yx = 1)$ and hence D has no zero divisors.)
   (b) (Optional) Let us generalize the definition of D as follows. For $P \in \mathbb{F}_p[t]$ let $D(P)$ be given by $x^p = P(t_1)$, $y^p = t_2$, $xy - yx = 1$.
   Check that $P \to [D(P)]$ is a homomorphism from the additive group of polynomials to $Br(k)$.
   (c) (Optional) Show that the kernel of this homomorphism is $\mathbb{F}_p[t^p]$. Conclude that p-torsion in the Brauer group of k is infinite.

(5) Let A_0 be an algebra over a field k. An n-th order deformation of A_0 is an associative algebra A over $k[t]/t^{n+1}$, free as a module over $k[t]/t^{n+1}$, together with an isomorphism of k-algebras $f : A/tA \to A_0$. Two such deformations (A, f) and (A', f') are said to be equivalent if there exists an algebra isomorphism $g : A \to A'$ such that $f'g = f$. As has been explained in class, first order deformations are parametrized by $HH^2(A_0)$.

   For the next two questions one can assume that k has characteristic zero.
   (a) Compute Hochschild cohomology of the polynomial algebra $A_0 = Sym(V)$ where V is a finite dimensional vector space over k. More precisely, show that it is isomorphic to the space of polynomial polyvector fields on the V^*.
   [Hint: do NOT use the bar complex]
   (b) According to the above, a first order deformation of $A_0 = Sym(V)$ is determined by a bivector field $\alpha \in Sym(V) \otimes \wedge^2 V^*$. This bivector field defines a skew-symmetric bilinear operation on A_0, given by $\{f, g\} = \langle \alpha, df \otimes dg\rangle$. Show that the first order deformation defined by α lifts

[1] According to T.Y. Lam, the question whether $d_l = d_r$ was raised by E. Artin. The answer is negative in general: there exist skew fields $D \supset K$ for which d_l, d_r is an arbitrary prescribed pair of integers greater than 1 [Schofield, 1985].

to a second order deformation if and only if this operation is a Lie bracket (satisfies the Jacobi identity). In this case α is said to be a *Poisson* bivector field.

(6) Let A be a k-algebra for a field k, let $H = HH^*(A)$ be its Hochschild cohomology and $H^{ev} = \oplus_n HH^{2n}(A)$ be the even part of H. Let $D(A)$ be the category whose objects are A-modules and $Hom_{D(A)}(M,N) = Ext^*(M,N)$ with the usual composition maps.
   (a) Define a natural homomorphism $H^{ev} \to End(Id_{D(A)})$.
   (b) (Optional) Extend the homomorphism in (a) to a homomorphism from H to a modification of $End(Id_{D(A)})$ involving the appropriate sign rule.
   (c) (Optional) Show that for $A = k[t]/(t^2)$ where k is a characteristic two field the map in (a) is not injective on $HH^2(A)$.
   (d) (Optional) Assume now that $A = k\langle V\rangle/(I)$, $I \subset V \otimes V$ is a Koszul quadratic algebra.
   Prove that the homomorphism $H^{ev} \to A^! = Ext^*_A(k,k)$ defined in part (a) induces an isomorphism $\bigoplus\limits_n HH^{2n}(A)_{(-2n)} \to Z(A^!)_{ev}$, while $HH^i(A)_{(j)} = 0$ for $j < -i$. Here $Z(A^!)$ denotes the center of $A^!$.
   Optionally, extend this to an isomorphism $\bigoplus\limits_n HH^n(A)_{(-n)} \to S(A^!)$ where $S(A^!)$ stands for the *supercenter*:
$$S(A^!)^i = \{a \in (A^!)^i \mid ab = (-1)^{ij} ba \forall b \in (A^!)^j\}.$$
   [Hint: Show that the i-th term in the minimal resolution for the regular bimodule over A has the form $((A^!)^i)^* \otimes (A \otimes A^{op})$ and identify a component of the differential in the corresponding complex for $HH^*(A)$ with the map $(A^!)^i \to V \otimes (A^!)^{i+1}$ obtained from the (super)commutator map $V^* \otimes (A^!)^i \to (A^!)^{i+1}$ by "lowering the index".]

(7) (Optional) Let R be the ring of real valued continuous functions on the 2-sphere S^2. Let R^+ and R^- be the ring of continuous functions on the upper and lower closed hemispheres respectively. Let $A \subset Mat_2(R^+) \times Mat_2(R^-)$ be the subring given by: $m = (m_+, m_-) \in A$ if $m_+(\theta) = S(\theta) m_-(\theta) S(\theta)^{-1}$. Here $\theta \in [0, 2\pi)$ is the standard coordinate on the equator circle bounding the upper and the lower hemisphere, and
$$S(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}.$$
Prove that A is a non-split Azumaya algebra over R.
   [Hint: Basic topology can be used in this problem. Reduce the statement to the fact that the map $\pi_1(S^1) \to \pi_1(S^1)$ induced by the double cover map $S^1 \to S^1$ is not surjective.]