(1) Let $R \subset \text{Mat}_2(\mathbb{Q})$ consist of matrices $A = (a_{ij})$ such that $a_{11} \in \mathbb{Z}$ and $a_{21} = 0$. Show that the homological dimension of $R$ equals two but the homological dimension of $R^{op}$ (i.e. the homological dimension of the category of right $R$-modules) equals one.

(2) Let $D \supset K$ be skew fields. Let $d_l$ be the dimension of $D$ as a left $K$ module and $d_r$ be the dimension of $D$ as a right $K$-module. Show\(^1\) that if $D$ is finite over its center then $d_l = d_r$.

(3) Let $D$ be a skew field which is not algebraic over its center $k$. Show that $D \otimes_k k[t]$ is a simple Noetherian ring which is not isomorphic to a matrix ring over a skew field.

(4) Let $k = \mathbb{F}_p(t_1, t_2)$ be the field of rational functions in two variables over $\mathbb{F}_p$. Set $D = \mathbb{F}_p \langle x, y \rangle / (xy - yx = 1, x^p = t_1, y^p = t_2)$.

(a) Show that $D$ is a skew field of dimension $p^2$ over its center $k$. Give an example of a splitting field of $D$.

(Hint: Check that $\mathbb{F}_p(x, y)/(xy - yx = 1)$ and hence $D$ has no zero divisors.)

(b) (Optional) Let us generalize the definition of $D$ as follows. For $P \in \mathbb{F}_p[t]$ let $D(P)$ be given by $x^p = P(t_1)$, $y^p = t_2$, $xy - yx = 1$.

Check that $P \to [D(P)]$ is a homomorphism from the additive group of polynomials to $\text{Br}(k)$.

(c) (Optional) Show that the kernel of this homomorphism is $\mathbb{F}_p[t^p]$. Conclude that $p$-torsion in the Brauer group of $k$ is infinite.

(5) Let $A_0$ be an algebra over a field $k$. An $n$-th order deformation of $A_0$ is an associative algebra $A$ over $k[t]/t^n+1$, free as a module over $k[t]/t^n+1$, together with an isomorphism of $k$-algebras $f : A/IA \to A_0$. Two such deformations $(A, f)$ and $(A', f')$ are said to be equivalent if there exists an algebra isomorphism $g : A \to A'$ such that $f'g = f$. As has been explained in class, first order deformations are parametrized by $HH^2(A_0)$.

For the next two questions one can assume that $k$ has characteristic zero.

(a) Compute Hochschild cohomology of the polynomial algebra $A_0 = \text{Sym}(V)$ where $V$ is a finite dimensional vector space over $k$. More precisely, show that it is isomorphic to the space of polynomial poly-vector fields on the $V^*$.

[Hint: do NOT use the bar complex]

(b) According to the above, a first order deformation of $A_0 = \text{Sym}(V)$ is determined by a bivector field $\alpha \in \text{Sym}(V) \otimes \wedge^2 V^*$. This bivector field defines a skew-symmetric bilinear operation on $A_0$, given by $\{f, g\} = \langle \alpha, df \otimes dg \rangle$. Show that the first order deformation defined by $\alpha$ lifts

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\(^1\)According to T.Y. Lam, the question whether $d_l = d_r$ was raised by E. Artin. The answer is negative in general: there exist skew fields $D \supset K$ for which $d_l$, $d_r$ is an arbitrary prescribed pair of integers greater than 1 [Schofield, 1985].
(6) Let $A$ be a $k$-algebra for a field $k$, let $H = HH^*(A)$ be its Hochschild cohomology and $H^{ev} = \oplus_n HH^{2n}(A)$ be the even part of $H$. Let $D(A)$ be the category whose objects are $A$-modules and $Hom_{D(A)}(M, N) = Ext^*(M, N)$ with the usual composition maps.

(a) Define a natural homomorphism $H^{ev} \to \text{End}(I_d_D(A))$.

(b) (Optional) Extend the homomorphism in (a) to a homomorphism from $H$ to a modification of $\text{End}(I_d_D(A))$ involving the appropriate sign rule.

(c) (Optional) Show that for $A = k[t]/(t^2)$, where $k$ is a characteristic two field the map in (a) is not injective on $HH^2(A)$.

(d) (Optional) Assume now that $A = k(V)/(I)$, $I \subset V \otimes V$ is a Koszul quadratic algebra.

Prove that the homomorphism $H^{ev} \to A' = Ext^*_A(k, k)$ defined in part (a) induces an isomorphism $\bigoplus_n HH^{2n}(A)_{(-2n)} \to Z(A')^{ev}$, while $HH^i(A)(j) = 0$ for $j < -i$. Here $Z(A')$ denotes the center of $A'$.

Optional, extend this to an isomorphism $\bigoplus_n HH^n(A)(-n) \to S(A')$ where $S(A')$ stands for the supercenter:

$$S(A')^i = \{a \in (A')^i \mid ab = (-1)^{|ab|} ba + h \in (A')^i \}.$$ 

[Hint: Show that the $i$-th term in the minimal resolution for the regular bimodule over $A$ has the form $(A')^i \otimes (A \otimes A^{op})$ and identify a component of the differential in the corresponding complex for $HH^*(A)$ with the map $(A')^i \to V \otimes (A')^{i+1}$ obtained from the (super)commutator map $V^* \otimes (A')^i \to (A')^{i+1}$ by “lowering the index”.]

(7) (Optional) Let $R$ be the ring of real valued continuous functions on the 2-sphere $S^2$. Let $R^+$ and $R^-$ be the ring of continuous functions on the upper and lower closed hemispheres respectively. Let $A \subset Mat_2(R^+) \times Mat_2(R^-)$ be the subring given by: $m = (m_+, m_-) \in A$ if $m_+(\theta) = S(\theta)m_-(\theta)S(\theta)^{-1}$. Here $\theta \in [0, 2\pi)$ is the standard coordinate on the equator circle bounding the upper and the lower hemisphere, and

$$S(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}.$$ 

Prove that $A$ is a non-split Azumaya algebra over $R$.

[Hint: Basic topology can be used in this problem. Reduce the statement to the fact that the map $\pi_1(S^1) \to \pi_1(S^1)$ induced by the double cover map $S^1 \to S^1$ is not surjective.]