HOMEWORK 3 FOR 18.706, FALL 2018
DUE THURSDAY, OCTOBER 18.

(1) Let k be a field and I an uncountable set. Let $R = k^I = \prod_I k$ and let $M \subset R$ consist of such $r = (r_i) \in R$ that $\{i \in I \mid r_i \neq 0\}$ is at most countable. It is clear that M is an ideal in R.
   (a) Check that M is not finitely generated.
   (b) Show that for any sequence $m_1, m_2, \dots, m_n \in M$ there exists $m \in M$ such that m_n belongs to the ideal generated by m.
   (c) Deduce that for a collection of nonzero R-module homomorphisms $f_j : M \to M_j$ either we have $f_j = 0$ for all but finitely many j, or there exists $m \in M$ for which $f_j(m) \neq 0$ for infinitely many j. In other words, the functor $X \mapsto Hom_R(M, X)$ commutes with arbitrary direct sums.

(2) In this problem we consider a finite dimensional algebra A over a field k and the matrix C, $C_{ij} = dim_k(Hom(P_i, P_j))$ where P_i, P_j run over the set of isomorphism classes of indecomposable projective modules.
   (a) Let $k = \mathbb{R}$ and assume that A has finite homological dimension. Show that $det(C) = \pm 2^n$ for some n.
   (b) (Optional) Let k be an algebraically closed field of characteristic $p > 0$ and $A = k[G]$ for a finite group G which has a normal Sylow p-subgroup. Check [1] that $det(C) = p^n$ for some n.
   [Hint. Let $H = G/S$ where S is the Sylow p-subgroup, so that $G = H \ltimes S$ while $k[H]$ is semisimple and irreducible representations of G are in bijection with irreducible representations of H over k and also with irreducible representations of H over a characteristic zero field F. Relate C to the matrix of multiplication by $F[S]$ acting on $K(Rep_F(G))$ and compute this matrix in the basis given by conjugacy classes.]

(3) Show that a self-injective Artinian ring has finite homological dimension iff it is semi-simple.

(4) Let $\mathcal{A}$ be the category of finitely generated modules over $\mathbb{C}[x]$.
   (a) Prove that $\mathcal{A}^{op}$ is not equivalent to the category of finitely generated modules over any algebra.
   (b) Give an example of a Serre subcategory $\mathcal{B} \subset \mathcal{A}$, so that both $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are equivalent to its opposite category.

(5) Let $\mathcal{A}$ be the category of vector spaces over a field k which are at most countably dimensional, and let $\mathcal{B}$ be the Serre subcategory of finite dimensional vector spaces. Prove that the set of isomorphism classes of objects in $\mathcal{A}/\mathcal{B}$ has two elements. Show that the algebra of endomorphisms of a nonzero object in $\mathcal{A}/\mathcal{B}$ is simple.

[1] In fact, this is true for any finite group G but this general result is harder to prove.

(6) Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded algebra over a field k with finite dimensional graded component. Define the Cartan matrix $C \in Mat_d(\mathbb{Z}[[t]])$ by $C_{i,j} = \sum t^n \dim Hom(P_i, P_j(n))$. Here d is the number of isomorphism classes of irreducible modules concentrated in graded degree zero, P_i are projective covers of those irreducibles; also, for a graded module M we let $M(n)$ denote the module M with shifted grading: $M_m(n) = M_{m+n}$.

Assume that A is left Noetherian and has finite homological dimension. Prove that $\det(C)$ is the Taylor series of a nonzero rational function in t.

(7) Let Q be the quiver with two vertices and two edges of opposite orientation connecting them. Set $A = A(Q)/(e_1 e_2)$ where e_1, e_2 are the elements corresponding to the edges.

(a) Show that A is finite dimensional over $\mathbb{C}$ and has finite homological dimension.
[Hint: Recall that an Artinian ring has finite homological dimension provided that each irreducible module has a finite projective resolution.]

(b) (Optional) Show that A is Koszul and the Koszul dual ring satisfies $A^! \cong A$.

(8) (Optional) Let $W = \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$ be the Weyl algebra acting on $M = \mathbb{C}[x_1, \dots, x_n]$. We equip W, M with the usual grading: $deg(x_i) = 1$, $deg(\partial_i) = -1$.

(a) Let $\mathcal{B}$ be the category of graded W-modules such that $e = \sum x_i \partial_i$ acts on the graded component of degree d by $d \cdot Id$. Let $\mathcal{A} = \mathcal{B}/\mathcal{C}$ where $\mathcal{C}$ is the full subcategory of modules where x_i acts locally nilpotently for $i = 1, \dots, n$.
Compute $Ext_{\mathcal{A}}(M, M)$.

(b) Let $\mathcal{B}'$ be the category of all W-modules and $\mathcal{C}'$ the full subcategory of modules where x_i acts locally nilpotently for $i = 1, \dots, n$. Let $\mathcal{A}' = \mathcal{B}'/\mathcal{C}'$.
Compute $Ext_{\mathcal{A}'}(M, M)$.

[Hint: The answer is $H^*(\mathbb{CP}^{n-1}, \mathbb{C})$ and $H^*(\mathbb{C}^n \setminus \{0\}, \mathbb{C})$ respectively. To run the calculation one can use the following ingredients:

– the projective resolution of M in $\mathcal{B}'$ obtained by inducing the Koszul resolution of $\mathbb{C}$ considered as a module over $\mathbb{C}[\partial_1, \dots, \partial_n]$ to W (it is closely related to the De Rham complex of $\mathbb{C}^n$);

– a right resolution of M by modules where one of x_i acts invertibly: notice that for such a module N we have: $Hom_{\mathcal{B}'}(X, N) = Hom_{\mathcal{A}'}(X, N)$ and similarly for $\mathcal{A}$;

– the long exact sequence

$$\cdot Ext^i_{\mathcal{B}'}(X, Y) \to Ext^i_{\mathcal{B}}(X, Y) \to Ext^{i-1}_{\mathcal{B}'}(X, Y) \to Ext^{i+1}_{\mathcal{B}'}(X, Y) \to \cdots$$

for $X, Y \in \mathcal{B}$.]