(1) Let \( k \) be a field and \( I \) an uncountable set. Let \( R = k^I = \prod_I k \) and let \( M \subset R \) consist of such \( r = (r_i) \in R \) that \( \{ i \in I \mid r_i \neq 0 \} \) is at most countable. It is clear that \( M \) is an ideal in \( R \).

(a) Check that \( M \) is not finitely generated.

(b) Show that for any sequence \( m_1, m_2, \ldots, m_n \in M \) there exists \( m \in M \) such that \( m_n \) belongs to the ideal generated by \( m \).

(c) Deduce that for a collection of nonzero \( R \)-module homomorphisms \( f_j : M \to M_j \) either we have \( f_j = 0 \) for all but finitely many \( j \), or there exists \( m \in M \) for which \( f_j(m) \neq 0 \) for infinitely many \( j \). In other words, the functor \( X \mapsto \text{Hom}_R(M, X) \) commutes with arbitrary direct sums.

(2) In this problem we consider a finite dimensional algebra \( A \) over a field \( k \) and the matrix \( C, C_{ij} = \dim_k(\text{Hom}(P_i, P_j)) \) where \( P_i, P_j \) run over the set of isomorphism classes of indecomposable projective modules.

(a) Let \( k = \mathbb{R} \). Show that \( \det(C) = \pm 2^n \) for some \( n \).

(b) Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and \( A = k[G] \) for a finite group \( G \) which has a normal Sylow \( p \)-subgroup. Compute \( C \) and check that \( \det(C) = p^n \) for some \( n \).

(3) Show that a self-injective Artinian ring has finite homological dimension iff it is semi-simple.

(4) Let \( \mathcal{A} \) be the category of finitely generated modules over \( \mathbb{C}[x] \).

(a) Prove that \( \mathcal{A}^{\text{op}} \) is not equivalent to the category of finitely generated modules over any algebra.

(b) Give an example of a Serre subcategory \( \mathcal{B} \subset \mathcal{A} \), so that both \( \mathcal{B} \) and \( \mathcal{A}/\mathcal{B} \) are equivalent to its opposite category.

(5) Let \( \mathcal{A} \) be the category of vector spaces over a field \( k \) which are at most countably dimensional, and let \( \mathcal{B} \) be the Serre subcategory of finite dimensional vector spaces. Prove that the set of isomorphism classes of objects in \( \mathcal{A}/\mathcal{B} \) has two elements. Show that the algebra of endomorphisms of a nonzero object in \( \mathcal{A}/\mathcal{B} \) is simple.

(6) Let \( A = \bigoplus_{n=0}^\infty A_n \) be a graded algebra over a field \( k \) with finite dimensional graded component. Define the Cartan matrix \( C \in \text{Mat}_d(\mathbb{Z}[\![t]\!]) \) by \( C_{ij} = \sum t^n \dim \text{Hom}(P_i, P_j(n)) \). Here \( d \) is the number of isomorphism classes of irreducible modules concentrated in graded degree zero, \( P_i \) are the projective covers of those irreducibles and \( M(n) \) for a graded module \( M \) denotes the module \( M \) with shifted grading: \( M_m(n) = M_{m+n} \).

Assume that \( A \) is of finite homological dimension. Prove that \( \det(C) \) is the Taylor series of a nonzero rational function in \( t \).

\footnote{In fact, this is true for any finite group \( G \) but this general result is harder to prove.}
(7) Let $Q$ be the quiver with two vertices and two edges of opposite orientation connecting them. Set $A = A(Q)/(e_1e_2)$ where $e_1$, $e_2$ are the elements corresponding to the edges.

(a) Show that $A$ is finite dimensional over $\mathbb{C}$ and has finite homological dimension.

[Hint: Recall that an Artinian ring has finite homological dimension provided that each irreducible module has a finite projective resolution.]

(b) (Optional) Show that $A$ is Koszul and the Koszul dual ring satisfies $A^! \cong A$.

(8) (Optional) Let $W = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ be the Weyl algebra acting on $M = \mathbb{C}[x_1, \ldots, x_n]$. We equip $W$, $M$ with the usual grading: $\deg(x_i) = 1$, $\deg(\partial_i) = -1$.

Let $\mathcal{B}$ be the category of graded $W$-modules such that $e = \sum x_i \partial_i$ acts on the graded component of degree $d$ by $d \cdot Id$. Let $\mathcal{A} = \mathcal{B}/\mathcal{C}$ where $\mathcal{C}$ is the full subcategory of modules where $x_i$ acts locally nilpotently for $i = 1, \ldots, n$.

Compute $\text{Ext}_\mathcal{A}(M, M)$. 