HOMEWORK 2 FOR 18.706, FALL 2018
DUE THURSDAY, OCTOBER 4.

(1) Is it true that every indecomposable module over an Artinian ring is a quotient of an indecomposable projective module? Prove or give a counterexample.

(2) (a) Let R be a ring and $I \subset R$ a (2-sided) nilpotent ideal. It was proved in class that every idempotent $e \in R/I$ admits a lifting to an idempotent $\tilde{e} \in R$. Prove that (as was claimed in class) such a lifting is unique up to conjugation by an element in $1 + I$.

(b) Let R be an Artinian ring. Prove that the set of conjugacy classes of idempotents in R is finite and give a formula for the cardinality of that finite set in terms of dimensions of irreducible representations of R as vector spaces over their respective skew fields of endomorphisms.

(c) Show that the following rings have no idempotents except for the unit element.

(i) $k[G]$ where k is a characterisitc p field and G is a finite group of order p^n.

(ii) (optional) $k[G]$ where k is any field and G is a torsion free group (i.e. $g^n \neq 1$ if $g \in G$, $g \neq 1$).

(3) Let R be an Artinian ring. For a module M let $C(M)$ be the set of cyclic elements in M, i.e. $m \in C(M)$ iff $Rm = M$. Let M be an R module such that $C(M) \neq \emptyset$.

Show that the following are equivalent.

i) The complement $M \setminus C(M)$ is a submodule.

ii) M is a quotient of an indecomposable projective R-module.

(4) Let R be a ring and F be the forgetful functor from the category of R-modules to abelian groups. Describe $End(F)$.

More generally, let $A \to R$ be a ring homomorphism. Describe the endomorphism ring of the pull-back functor $R - mod \to A - mod$.

(5) An Artinian ring is called self-injective if the free module is injective. A finite dimensional algebra A over a field k is called Frobenius if there exists a linear functional τ on A such that the bilinear pairing $A \times A \to k$, $(x, y) \mapsto \tau(xy)$ is non-degenerate.[1]

(a) Prove that a Frobenius algebra is self-injective.

(b) Let A be a finite dimensional algebra over a field. Assume that for every simple A-module L the multiplicity of L in the co-socle of A viewed as a left A-module equals the multiplicity of L^* in the socle of A viewed as a right A-module. Show that A is self-injective.

[1]It is easy to see that the group algebra $k[G]$ of a finite group G and exterior algebra of a finite dimensional vector space are examples of Frobenius algebras. Another class of examples is provided by cohomology of compact oriented manifolds, this follows from Poincare duality.

(Here we use that for a left A-module M the dual vector space $M^* = Hom(M, k)$ carries a right A-module structure, the action is given by the adjoint operators).

(6) Let Γ be a finite group acting on a ring R by automorphisms. Then the *smash product*[2] $\Gamma \# R$ or $\Gamma \ltimes R$ is the abelian group $\bigoplus_{\gamma\in\Gamma} R$ with multiplication given by: $(r_{\gamma_1})(r'_{\gamma_2}) = r\gamma_1(r')_{\gamma_1\gamma_2}$ in the self-explanatory notation.

Suppose that R is simple and $|G|$ is invertible in R.

Suppose that one the following two conditions holds.

(a) No nontrivial element $\gamma \in \Gamma$ acts by an inner automorphism of R.

(b) The element $\gamma \in \Gamma$ acts by conjugation by an invertible element $r_\gamma \in R$, where the elements $r_\gamma \in R$ are linearly independent over the center of R.

Prove that the rings $\Gamma \# R$-modules and R^Γ are Morita equivalent.

(7) For each of the following functors determine existence of a left adjoint, of a right adjoint and describe the existing adjoint functors.

(a) Let $Q = (\bullet \longrightarrow \bullet)$ be the quiver with two vertices and one arrow between them. Let $\mathcal{B}$ be the category of representations of Q over a fixed field, $\mathcal{A}$ be the full subcategory consisting of such representations that the map between the two vector spaces is injective, and $F : \mathcal{A} \to \mathcal{B}$ be the embedding.

(b) A graded commutative (or *super-commutative*) ring is a $\mathbb{Z}/2\mathbb{Z}$ graded ring $R = R_0 \oplus R_1$ such that $xy = -yx$ for $x, y \in R_1$ and $xy = yx$ for $x \in R_0$, $y \in R$.

Let $\mathcal{A}$ be the category of vector spaces over a field k not of characteristic two, $\mathcal{B}$ be the category of graded commutative k-algebras and let F send a vector space to its exterior algebra.

[2] I prefer the notation which can be described as "semi-tensor product" but I don't know how to reproduce it in LaTeX!