(1) Is it true that every indecomposable module over an Artinian ring is a quotient of an indecomposable projective module? Prove or give a counterexample.

(2) (a) Let $R$ be a ring and $I \subset R$ a (2-sided) nilpotent ideal. It was proved in class that every idempotent $e \in R/I$ admits a lifting to an idempotent $\tilde{e} \in R$. Prove that (as was claimed in class) such a lifting is unique up to conjugation by an element in $1 + I$.

(b) Let $R$ be an Artinian ring. Prove that the set of conjugacy classes of idempotents in $R$ is finite and give a formula for the cardinality of that finite set in terms of dimensions of irreducible representations of $R$ as vector spaces over their respective skew fields of endomorphisms.

(c) Show that the following rings have no idempotents except for the unit element.

(i) $k[G]$ where $k$ is a characteristic $p$ field and $G$ is a finite group of order $p^n$.

(ii) (optional) $k[G]$ where $k$ is any field and $G$ is a torsion free group (i.e. $g^n \neq 1$ if $g \in G, g \neq 1$).

(3) Let $R$ be an Artinian ring. For a module $M$ let $C(M)$ be the set of cyclic elements in $M$, i.e. $m \in C(M)$ iff $Rm = M$. Let $M$ be an $R$ module such that $C(M) \neq \emptyset$.

Show that the following are equivalent.

i) The complement $M \setminus C(M)$ is a submodule.

ii) $M$ is a quotient of an indecomposable projective $R$-module.

(4) Let $R$ be a ring and $F$ be the forgetful functor from the category of $R$-modules to abelian groups. Describe $\text{End}(F)$.

More generally, let $A \to R$ be a ring homomorphism. Describe the endomorphism ring of the pull-back functor $R\text{-mod} \to A\text{-mod}$.

(5) An Artinian ring is called self-injective if the free module is injective. A finite dimensional algebra $A$ over a field $k$ is called Frobenius if there exists a linear functional $\tau$ on $A$ such that the bilinear pairing $A \times A \to k$, $(x, y) \mapsto \tau(xy)$ is non-degenerate.\footnote{It is easy to see that the group algebra $k[G]$ of a finite group $G$ and exterior algebra of a finite dimensional vector space are examples of Frobenius algebras. Another class of examples is provided by cohomology of compact oriented manifolds, this follows from Poincare duality.

(a) Prove that a Frobenius algebra is self-injective.

(b) Let $A$ be a finite dimensional algebra over a field. Assume that for every simple $A$-module $L$ the multiplicity of $L$ in the co-socle of $A$ viewed as a left $A$-module equals the multiplicity of $L^*$ in the socle of $A$ viewed as a right $A$-module. Show that $A$ is self-injective.
(Here we use that for a left $A$-module $M$ the dual vector space $M^* = Hom(M, k)$ carries a right $A$-module structure, the action is given by the adjoint operators).

(6) Let $\Gamma$ be a finite group acting on a ring $R$ by automorphisms. Then the smash product $^{\#} \Gamma \# R$ or $\Gamma \ltimes R$ is the abelian group $\bigoplus_{\gamma \in \Gamma} R$ with multiplication given by: $(r_{\gamma_1})(r_{\gamma_2}) = r_{\gamma_1 \gamma_2}$ in the self-explanatory notation.

Suppose that $R$ is simple and $|G|$ is invertible in $R$.

Suppose that one of the following two conditions holds.

(a) No nontrivial element $\gamma \in \Gamma$ acts by an inner automorphism of $R$.
(b) The element $\gamma \in \Gamma$ acts by conjugation by an invertible element $r_\gamma \in R$, where the elements $r_\gamma \in R$ are linearly independent over the center of $R$.

Prove that the rings $^{\#} \Gamma \# R$-modules and $R^\Gamma$ are Morita equivalent.

(7) For each of the following functors determine existence of a left adjoint, of a right adjoint and describe the existing adjoint functors.

(a) Let $Q = (\bullet \rightarrow \bullet)$ be the quiver with two vertices and one arrow between them. Let $\mathcal{B}$ be the category of representations of $Q$ over a fixed field, $\mathcal{A}$ be the full subcategory consisting of such representations that the map between the two vector spaces is injective, and $F: \mathcal{A} \to \mathcal{B}$ be the embedding.

(b) A graded commutative (or super-commutative) ring is a $\mathbb{Z}/2\mathbb{Z}$ graded ring $R = R_0 \oplus R_1$ such that $xy = -yx$ for $x, y \in R_1$ and $xy = yx$ for $x \in R_0, y \in R$.

Let $\mathcal{A}$ be the category of vector spaces over a field $k$ not of characteristic two, $\mathcal{B}$ be the category of graded commutative $k$-algebras and let $F$ send a vector space to its exterior algebra.

\footnote{I prefer the notation which can be described as "semi-tensor product" but I don’t know how to reproduce it in LaTeX!}