Surfaces from a polygon

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What is it about?

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Gives a surface + a drawing on it.
Maps

Definition: A map is a surface + drawing obtained by gluing together polygons.

Maps are considered up to homeomorphism (bicontinuous mappings), that is, the drawing can be deformed (as long as it does not change the gluings between polygons).
Why should we study maps?

- Practical need for algorithms (coding, sampling) as maps are structure underlying **meshed surfaces** (used in video games, scientific imaging, . . . ).
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- Gives a way of defining **random surfaces** (for instance, look at every possible ways of obtaining the sphere by gluing 1000 squares). Mathematical notion of random surfaces are needed for 2D quantum gravity (string theory).
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- Practical need for algorithms (coding, sampling) as maps are structure underlying **meshed surfaces** (used in video games, scientific imaging, . . .).

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- For the beauty of it: mathematical objects with very nice properties appearing at the intersection of many mathematical fields: **graph theory**, **statistical mechanics**, **probability**, **representation theory** . . .
Outline

I. Maps and Euler relation.
   • Classification of surfaces.
   • Euler relation.

II. Surfaces from one polygon.
   • All possible gluings.
   • Gluings giving the sphere and Catalan numbers.
   • Gluings giving other surfaces.
I. Surfaces and Euler relation
1. Classification of surfaces.

We consider the (orientable) surfaces that can be obtained by gluing together polygons. Again, the surface is considered up to homeomorphism (up to deformation).
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We consider the (orientable) surfaces that can be obtained by gluing together polygons. Again, the surface is considered up to homeomorphism (up to deformation).

**Theorem:** The set of all possible surfaces is indexed by the integers. This set is made of the sphere (or 0-torus), the 1-torus, the 2-torus, the 3-torus, etc. The $k$-torus is said to have **genus** $k$. 
2. Euler relation

For a map $M$, we consider the number $v(M)$ of vertices, $e(M)$ of edges, and $f(M)$ of faces.

For the cube:

- $v(M) = 8$
- $e(M) = 12$
- $f(M) = 6$
2. Euler relation

For a map $M$, we consider the number $v(M)$ of vertices, $e(M)$ of edges, and $f(M)$ of faces.

**Definition:** The **Euler Characteristic** of a map is

$$
\chi(M) = v(M) - e(M) + f(M).
$$

For the cube:

$$
\chi(M) = 8 - 12 + 6 = 2
$$
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**Examples.**

\[\begin{array}{ccc}
\chi(M) = 2 & \chi(M) = ? & \chi(M) = ? \\
\end{array}\]
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$$\chi(M) = v(M) - e(M) + f(M).$$

**Examples.**

$$\chi(M) = 11 - 18 + 7 = 0$$
2. Euler relation

**Theorem:** If a map has genus $g(M)$, then its Euler characteristic is

$$\chi(M) = 2 - 2g(M).$$

**Examples.**

- $\chi(M) = 2$
- $\chi(M) = 0$
2. Euler relation

Proof for the sphere (genus 0):
We want to prove that any map $M$ on the sphere satisfies

$$v(M) - e(M) + f(M) = 2.$$
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- True for maps which are **plane trees** (maps having a unique face).

Induction (see blackboard):

add vertices one by one.
2. Euler relation

Proof for the sphere (genus 0):
We want to prove that any map $M$ on the sphere satisfies

$$v(M) - e(M) + f(M) = 2.$$ 

- True for maps which are plane trees.
- True in general.

Induction (see blackboard):
add faces one by one.
2. Euler relation

Proof for other surfaces:
Strategy: make an induction on the genus by cutting the surface.
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Strategy: make an induction on the genus by cutting the surface.

Proof for the torus: We consider a map $M$ on the torus.
By definition, $\chi(M) = v(M) - e(M) - f(M)$.

We want to prove $\chi(M) = 0$. 
2. Euler relation

**Proof for other surfaces:**
Strategy: make an induction on the genus by cutting the surface.

**Proof for the torus:** We consider a map $M$ on the torus. By definition, $\chi(M) = v(M) - e(M) - f(M)$.

We will cut the torus to get the sphere.
2. Euler relation

Proof for other surfaces:
Strategy: make an induction on the genus by cutting the surface.

Proof for the torus: We consider a map $M$ on the torus. By definition, $\chi(M) = v(M) - e(M) - f(M)$.

Before cutting we transform the cutting line by edges. The Euler characteristic is unchanged: $\chi(M') = \chi(M)$. 
2. Euler relation

Proof for other surfaces:
Strategy: make an induction on the genus by cutting the surface.

Proof for the torus: We consider a map $M$ on the torus. By definition, $\chi(M) = v(M) - e(M) - f(M)$.

Cutting increases the Euler characteristic by 2: $\chi(N) = \chi(M) + 2$.

Since $N$ is a map on the sphere we have $\chi(N) = 2$, hence $\chi(M) = 0$. 
II. Surfaces from one polygon
1. All possible gluings.

We consider all the gluings possible from a $2n$-gon (a polygon with $2n$ vertices).

**Example:** gluings from the square.
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**Example:** gluings from the square.
1. All possible gluings.

**Question:** How many ways are there to pair the edges of the $2n$-gon?

**Answer:** $(2n - 1)(2n - 3)(2n - 5) \cdots 3 \cdot 1$ ways.
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2. Gluings giving the sphere

We want to count the gluings of the $2n$-gon giving the sphere.

In other words, we want to count the plane trees (maps with a single face on the sphere) with $n$ edges.
2. Gluings giving the sphere

**Question:** What is the number $\text{Cat}(n)$ of plane trees with \( n \) edges?

\[
\begin{align*}
n = 1 & \quad \text{Cat}(1) = 1 \\
n = 2 & \quad \text{Cat}(2) = 2 \\
n = 3 & \quad \text{Cat}(3) = 5
\end{align*}
\]
2. Gluings giving the sphere

**Theorem:** The number of plane trees with \( n \) edges is

\[
Cat(n) = \frac{(2n)!}{(n+1)! \cdot n!}.
\]

**Notation:** The notation \( n! \) stands for \( n(n-1)(n-2)\cdots2\cdot1 \).

\[
Cat(3) = 5
\]
Some vocabulary about trees

It is customary to see plane trees as genealogical trees:

The vertices $v_1, v_2, v_3$ are called the **children** of $u$.  
The vertex $v_1$ is called the **leftmost child** of $u$. 
A vertex with no child is called a **leaf**.

![Diagram of a tree with vertices $v_1, v_2, v_3$ and an arrow pointing to the ancestor of $v_1$.]
Some vocabulary about trees

A **corner** is defined by two consecutive edges around a vertex.
Some vocabulary about trees

A **corner** is defined by two consecutive edges around a vertex.

By convention, one considers that there is a corner at the left and a corner at the right of the arrow pointing the ancestor.

With this convention, a plane tree with *n* edges has $2n + 1$ corners.
2. Gluings giving the sphere

**Theorem:** The number of plane trees with \( n \) edges is

\[
Cat(n) = \frac{(2n)!}{(n + 1)! n!}.
\]

**Proof:**
It suffices to show that \( (n + 1)Cat(n) = 2(2n - 1)Cat(n - 1) \).

Indeed, in this case,

\[
Cat(n) = \frac{(2n)(2n - 1)}{(n + 1)n} Cat(n - 1)
\]
\[
= \frac{(2n)(2n - 1)(2n - 2)(2n - 3)}{(n + 1)n n(n - 1)} Cat(n - 2)
\]
\[
= \ldots = \frac{(2n)!}{(n + 1)! n!}
\]
2. Gluings giving the sphere

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**Remark:**
\begin{itemize}
  \item \((n + 1)Cat(n)\) is the number of trees with \( n \) edges + a marked vertex.
  \item \(2(2n - 1)Cat(n - 1)\) is the number of trees with \( n - 1 \) edges + a corner marked either green or red.
\end{itemize}

\(\Rightarrow\) We need to prove that these 2 sets have the same number of elements.
2. Gluings giving the sphere

**Theorem:** The number of plane trees with \( n \) edges is

\[
Cat(n) = \frac{(2n)!}{(n + 1)!n!}.
\]

**Def:** A *bijection* is a function from a set \( A \) to a set \( B \) such that every element of \( B \) is the image of exactly one element of \( A \).

**Remark.** In this case, the sets \( A \) and \( B \) have the same number of elements.
2. Gluings giving the sphere

**Theorem:** The number of plane trees with \( n \) edges is

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\text{Cat}(n) = \frac{(2n)!}{(n+1)!n!}.
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**Def:** A *bijection* is a function from a set \( A \) to a set \( B \) such that every element of \( B \) is the image of exactly one element of \( A \).

**Remark.** In this case, the sets \( A \) and \( B \) have the same number of elements.

\[\Rightarrow \] It suffices to find a *bijection* between

- the set of trees with \( n \) edges + a marked vertex.

and

- the set of trees with \( n - 1 \) edges + a corner marked either green or red.
The bijection

**Rule 1:** If the marked vertex is a leaf (no children), then the image of the tree is obtained by deleting the leaf. + the corresponding corner is marked **green**.
The bijection

**Rule 1:** If the marked vertex is a leaf (no children), then the image of the tree is obtained by deleting the leaf. + the corresponding corner is marked **green**.

**Rule 2:** If the marked vertex has some children, then the image of the tree is obtained by contracting the edge toward the leftmost child. + the corresponding corner is marked **red**.
The bijection

**Rule 1:** If the marked vertex is a leaf (no children), then the image of the tree is obtained by deleting the leaf. + the corresponding corner is marked **green**.

**Rule 2:** If the marked vertex has some children, then the image of the tree is obtained by contracting the edge toward the leftmost child. + the corresponding corner is marked **red**.

This defines a bijection $\phi$ between:

- the set of trees with $n$ edges + a marked vertex.
- the set of trees with $n - 1$ edges + a corner marked either **green** or **red**.

This proves $(n + 1)\text{Cat}(n) = 2(2n - 1)\text{Cat}(n - 1)$, hence the theorem. □
2. Gluings giving the sphere

We have proved the following:

Theorem: There are \( \text{Cat}(n) = \frac{(2n)!}{(n+1)!n!} \) ways of gluing the \( 2n \)-gon into the sphere.

This is \( \text{Cat}(6) = 132 \) ways of gluing the dodecagon into the sphere.
3. Gluings giving the torus

**Theorem:** The number of gluings of the $2n$-gon giving the torus is

$$\epsilon_1(n) = \frac{(2n)!}{12n!(n-2)!}.$$
Concluding remarks

1. The Catalan numbers count many other natural classes of objects.
   See the list compiled by Richard Stanley: http://math.mit.edu/~rstan/ec/catadd.pdf

2. Bijections are important for better understanding maps, and for coding algorithms.

3. Bijections can give access to statistical properties.
   For instance, what is the average number of leaves in a plane tree with $n$ edges?