Research Statement

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My research interests are in mathematical physics and applied mathematics. I focus on the areas of electromagnetics, material science, and dissipative systems. My mathematical specializations are in functional analysis, spectral and scattering theory, perturbation theory (with a focus on non-self-adjoint linear operators and nonlinear eigenvalue problems), and linear response theory (with a focus on passive linear systems). I apply the mathematical methods from these areas to study problems involving wave propagation in complex and periodic media [e.g., metamaterials, composites, photonic crystals, materials with defects, slow and fast light, guided modes (i.e., embedded eigenvalues), resonance phenomena].

Currently, I am actively engaged in three different research projects each with a separate group of collaborators. My collaborators are: (1) Alex Figotin from the University of California, Irvine (UCI); (2) Steven G. Johnson and Yehuda Avniel from Massachusetts Institute of Technology (MIT); (3) Stephen P. Shipman from Louisiana State University (LSU).

The following is a brief overview of these three research projects (a more concise description is given in Sections 1, 2, and 3):

(1) The first project is with Alex Figotin from UCI. We are studying the dissipative properties of composite systems when they are composed of a mix of lossless and highly lossy components. In addition, we are developing the framework and theory to understand what the trade-off is between losses and useful functionality inherited by the composite from its components. The results we have achieved so far in this project can be found in our works [1] (published) and [2] (submitted).

(2) The second project is with Steven G. Johnson and Yehuda Avniel from MIT. We are studying how relativistic causality (i.e., information cannot travel faster than $c$ – the speed of light in vacuum) can actually be proved for the propagation of light through periodic media given the broad range material properties that are possible (e.g., isotropy, anisotropy, chirality, nonlocality, dispersive, dissipative). My collaborators and I have found a way to prove this speed-of-light limitation on the electromagnetic energy velocity under very general assumptions, namely, just passivity and linearity of the medium (along with a transparency window, which ensures well-defined energy propagation). This result and our research findings on this project will be found in our work [3].

(3) The third project is with Stephen P. Shipman from LSU. We are studying electromagnetic resonance phenomena and anomalous scattering behavior in composite structures that incorporate anisotropic media. In [4], my collaborator and I identified two different types of resonances (having several important features in common), which I will refer to as (A) a guided-mode resonance and (B) a frozen-mode resonance, involving scattering in photonic crystals with defects. In [5], we study some of the important features associated with the guided-mode resonance such as peaks in energy transmission and amplitude enhancement of scattered fields. In [6], we will continue our study of the frozen-mode resonance which began in [4].

The project with Alex Figotin has produced two papers, one published [1] and one submitted [2]. The work on this project was performed while at UCI (as an Assistant Project Scientist funded by the AFOSR grant FA9550-11-1-0163, “Metamaterials for Miniaturization of Optical Components and Enhancement of Light-Matter Interactions”), LSU, and MIT. The project with Steven G. Johnson and Yehuda Avniel has produced one paper [3], in progress, and the work was performed at MIT. The project with Stephen P. Shipman has produced three papers, two published [4], [5] and one in progress [6]. The work was performed at LSU (as a VIGRE postdoc funded by the National Science Foundation) and at MIT.

The rest of this research statement is organized into the following five sections. Sections 1–3 give a concise description of my active research projects (which were briefly described above) and highlight some of the main

1 Dissipative properties of composite systems

The purpose of this section is to give a brief description of my research project with Alex Figotin from UCI and highlight some of the results we have achieved so far from our works [1] and [2]. I will begin this section by providing motivation for our main objectives and then give a quick summary of the highlights of our publication [1] and our work [2] submitted for publication. Sections [1.1] and [1.2] elaborate on these highlights from [1] and [2], respectively, in order to provide a more precise description of our main results.

Our project can be described as being a study on the dissipative properties of composite systems when they are composed of a mix of lossless and highly lossy components. The main objectives of this project can be summarized as follows:

(i) To understand when and how is significant loss reduction possible (over a broad frequency range) in composite systems having a mix of lossy and lossless components.

(ii) To understand what the trade-off is between the losses and useful properties (or functionality) inherited by a composite from its components.

Motivation An important motivation and guiding examples for our studies come from two component dielectric media composed of a high-loss and a lossless component. Any dielectric medium always absorbs a certain amount of electromagnetic energy, a phenomenon which is often referred to as loss. When it comes to the design of devices utilizing dielectric properties, very often a component that carries a useful property, for instance, magnetism has prohibitively strong losses in the frequency range of interest. Often this precludes the use of such a lossy component with otherwise excellent physically desirable properties. But one reason that composites are so useful is they have the ability to inherit drastically altered properties compared to their components. Hence, one of the objectives of our project is to understand what the trade-off is between the losses and useful properties inherited by the composite from its components.

An important question then arises: Is it possible to design a composite material/system which would have a desired property comparable with a naturally occurring bulk substance but with significantly reduced losses? It is quite remarkable that the answer to the above question is affirmative and an example of a simple layered structure having magnetic properties comparable with a natural bulk material but with 100 times lesser losses in wide frequency range has been constructed in [9]. This question and the example leads us to another objective of our project, namely, to understand when and how is significant loss reduction possible (over a broad frequency range) in composite systems having a mix of lossy and lossless components.

Highlights from our publication [1] The following highlights (in a broad scope) our main results and achievements from [1] (a more precise description is given in [1.1] below):

- In our paper [1], we introduce a general framework to study dissipative properties of two-component systems composed of a high-loss and a lossless component. This framework covers conceptually any dissipative physical system governed by a linear evolution equation. Such systems include, in particular, damped mechanical systems and electric networks or any linear Lagrangian system with a nonnegative Hamiltonian and losses accounted by the Rayleigh dissipative function [10] (that such Lagrangian systems do indeed fit within our framework is a principle result of our submitted paper [2]).

- Based on this framework, we proceed to study in [1] the energy dissipation features of systems comprised of two components one of which is highly lossy and the other lossless. An exhaustive analytical study of the energy, dissipated power, and quality factor for such composite systems is given.
Two principal results from these studies, which address objective (i) above, are

(a) We found that a two-component system involving a high-loss component can be significantly low loss in a wide frequency range provided, to some surprise, that the lossy component is sufficiently lossy. An explanation for this phenomenon is that if the lossy part of the system has losses exceeding a critical value it goes into essentially an overdamping regime, that is, a regime with no oscillatory motion.

(b) The general mechanism of this phenomenon is the modal dichotomy: all the eigenmodes of any such system split into two distinct classes, high-loss and low-loss, according to their dissipative properties. Interestingly, this splitting is more pronounced the higher the loss of the lossy component. In addition to that, the real frequencies of the high-loss eigenmodes can become very small and even can vanish entirely, which is the case of overdamping.

An example of the two-component model using electric circuits is discussed and analyzed in detail in [1], both theoretically and numerically, using the results of our studies.

Highlights from our submitted paper [2] The following highlights (in a broad scope) some select results from our submitted paper [2] (a more precise description is given in §1.2 below):

- In [2], using a Lagrangian mechanics approach, we construct a “Lagrangian” framework to further study the dissipative properties of two-component composite systems composed of a high-loss and a lossless component that was started in [2]. The Lagrangian framework allows one to take into account more significant physical aspects of dissipative systems and thus represents progress towards the objective (ii) above.

- The framework covers any linear Lagrangian system provided it has a finite number of degrees of freedom, a nonnegative Hamiltonian, and losses accounted by the Rayleigh dissipative function [10].

- An important result of our paper is that such a Lagrangian system fits within our framework and model introduced in [1] in the following sense: there exists a two-component composite system whose states are solutions of the linear evolution equations in the canonical form (1) such that by a linear transformation any state in the Lagrangian system becomes a state in the composite system (and vice versa).

- The advantage of this “equivalence” between systems is that the results from our previous work [1], on the dissipative properties of two-component composite systems with a high-loss and a lossless component, can be used to study the energy dissipation properties of such Lagrangian systems.

- A principle result of our paper [2] is that for any Lagrangian system (within our framework) which is a two-component composite system with a high-loss and a lossless component, there is modal dichotomy as in [1] (i.e., the eigenmodes split into two distinct classes, high-loss and low-loss, according to their dissipative behavior) and a rather universal phenomenon occurs, namely, selective overdamping: The high-loss modes are all overdamped (i.e., non-oscillatory) as are an equal number of low-loss modes, but the rest of the low-loss modes remain oscillatory (i.e., underdamped) each with an extremely high quality factor that actually increases as the loss of the lossy component increases.

- These results on the selective overdamping phenomenon are proved in [2] using a new time dynamical characterization of overdamping in terms of a virial theorem for dissipative systems and the breaking of an equipartition of energy.

- As applications and practical importance of “selective overdamping” lie in effective suppression of more dissipative (overdamped) modes with a consequent enhancement of the role of low-loss oscillatory modes, we will give a detailed analysis of this selective overdamping phenomenon in Lagrangian systems in our paper [2].
1.1 The model and main results

In this section I give a brief introduction to the model that we introduced in [1] to study energy dissipation in composites and highlight (in a more precise manner then was done above) some of our main results from [1].

The model The foundational framework to our studies of energy dissipation is as in [11, 12]. It is based on an abstract model of an oscillator damped by a retarded friction. Our primary subject is a linear system whose state is described by a time-dependent velocity \( v(t) \) taking values in a Hilbert space \( H \) with inner product \((\cdot,\cdot)\). The dynamics of \( v \) are governed by a linear evolution equation with external force \( f \) and system losses incorporated via convolution with an operator-valued friction function \( a \),

\[
\tag{1}
\partial_t v(t) = -iA(\beta) v(t) + f(t), \quad \text{where } A(\beta) = \Omega - i\beta B, \ \beta \geq 0,
\]

with \( \Omega \) a self-adjoint operator. The derived equation that gives the balance of energy is

\[
\frac{dU}{dt} = -W_{\text{dis}}[v(t)] + W[v(t)],
\]

\[
U[v(t)] = \frac{1}{2}(v(t), v(t)), \quad W_{\text{dis}}[v(t)] = \beta(v(t), Bv(t)), \quad W[v(t)] = \text{Re}(v(t), f(t)),
\]

where the latter terms are the system energy, dissipated power, and rate of work done by the force \( f \) per unit time, respectively. Keeping in mind our motivations, we associate the operator \( B \) with the lossy component of the composite, its range \( H_B = \text{Ran} B \), which we call the loss subspace, with the degrees of freedom susceptible to losses, and its kernel \( H_B^⊥ = \text{Ker} B \) (i.e., nullspace) with the lossless degrees of freedom. We assume the rank \( N_B \) of the operator \( B \) (i.e., \( N_B = \dim H_B \)) satisfies the loss fraction condition

\[
0 < \frac{N_B}{N} < 1, \quad N = \dim H, \quad N_B = \text{rank} B,
\]

which signifies that only the fraction \( N_B/N \) of the degrees of freedom are susceptible to lossy behavior. The lossy component of the two-component composite is considered high-loss in the regime \( \beta \gg 1 \).

The above represents the model we use for our initial studies in [1] of energy dissipation in two-component composite systems composed of a high-loss and lossless component.

Main results We study in [1] this two-component model of a composite system comprised of a high-loss and a lossless component in the regime \( \beta \gg 1 \). Asymptotic expansions as the loss parameter \( \beta \to \infty \) are derived for the system energy, dissipated power, and quality factor \( Q \) for the system eigenmodes and for oscillatory motions of the system due to harmonic external forces. The exact statements of our main results in [1] are contained in §IV, §V. I will now highlight some of these results and then state two key theorems on the modal dichotomy from [1, §IV] to give a flavor of our achievements.

The following highlights some of our main results from [1]:

- The analysis of the eigenmodes is based on a spectral perturbation analysis of the system operator \( A(\beta) = \Omega - i\beta B \) for \( \beta \) large. This analysis has certain subtleties since the operator \( A(\beta) \) for \( \beta > 0 \) is non-self-adjoint (cf. [8]) and the operator \( B \) does not have full rank. One of our principle results, on the modal dichotomy (see [1, §IV,A, Theorem 5]), is that for \( \beta \) large there are exactly \( N \) linear independent system eigenmodes and they split into two distinct classes, high-loss and low-loss, according to their dissipative behavior with the number of high-loss modes being \( N_B \), the rank of the operator \( B \). In particular, only the loss fraction \( N_B/N \) of the system eigenmodes are effected by the high losses. The distinguishing features of the high-loss modes is each of their damping factor approaches \( \infty \) while the
quality factor $Q$ for these modes is very small with $Q \to 0$ as $\beta \to \infty$, whereas for the low-loss modes each of their damping factor approaches $0$ while the $Q$ factor for some of these modes can be very large with $Q \to \infty$ as $\beta \to \infty$ (see [1, §IV.A, Props. 7, 13, 14]). The underlying reason for this behavior is that the high-loss modes are essentially confined to the loss subspace $H_B$ whereas the low-loss modes are essentially expelled from it (see [1, §IV.A, Cor. 6]).

- The analysis for an oscillatory motion $v(t) = ve^{-i\omega t}$ of the system due to harmonic external forces $f(t) = fe^{-i\omega t}$ ($f \in H, f \neq 0$, real $\omega \neq 0$) is based on the admittance operator $i[\omega I - A(\beta)]^{-1}$. An explicit formula for this operator is found from a block factorization of the operator $\omega I - A(\beta)$ using the Schur complement (see [1, §V, Prop. 21]). An asymptotic expansion for the admittance operator is derived from this formula and used to give the expansions of the energy $U$, dissipated power $W_{\text{dis}}$, and quality factor $Q$ for the oscillatory motion as $\beta \to \infty$ (see [1, §V, Theorems 23, 24, Cor. 26]). The qualitative behavior can be described in terms of the orthogonal projection $P_B$ onto the loss subspace $H_B$. If $P_B f = f$, signifying the driving force/source is located inside the lossy component of the system, then $U \sim \beta^{-2}$, $W_{\text{dis}} \sim \beta^{-1}$, $Q \sim \beta^{-1}$ as $\beta \to \infty$. However, if $P_B f \neq f$, signifying the driving force/source has a portion located inside the lossless component of the system, then $U \sim 1$, $W_{\text{dis}} \sim \beta^{-1}$, $Q \sim \beta$ as $\beta \to \infty$.

**On the modal dichotomy** We will now state two key theorems on the modal dichotomy from [1] to give a flavor of our achievements. We begin by first setting up these theorems.

An *eigenmode* of the system is a damped harmonic oscillation $v(t) = ve^{-i\omega t}$, with frequency $\omega = \text{Re} \zeta$ and *damping factor* $-\text{Im} \zeta$, which is a nontrivial solution of (1) with $f = 0$. The eigenmode corresponds to the eigenpair $\zeta, w$ of the system operator $A(\beta) = \Omega - i\beta B$, i.e.,

$$A(\beta)w = \zeta w, \quad w \neq 0. \quad (5)$$

By the energy balance equation (2), for such a solution, the system energy $U[v(t)] = U[w] e^{2\text{Im} \zeta t}$, dissipated power $W_{\text{dis}}(t) = W_{\text{dis}}[w] e^{2\text{Im} \zeta t}$, and damping factor $-\text{Im} \zeta$ satisfy

$$W_{\text{dis}}[v(t)] = -\frac{d}{dt}U[v(t)] = -\text{Im} \zeta(w, w)e^{2\text{Im} \zeta t}, \quad -\text{Im} \zeta \geq 0. \quad (6)$$

The presentation just given serves to elicit the importance of the damping factor $-\text{Im} \zeta$ in the study of the dissipative properties of the composite system. Another important quantity which characterizes the quality of the damped oscillation is the *quality factor* $Q = Q[w]$ which is defined as the reciprocal of the relative rate of energy dissipation per temporal cycle, i.e.,

$$Q[w] = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per cycle}} = |\omega| \frac{U[v(t)]}{W_{\text{dis}}[v(t)]}. \quad (7)$$

It can be shown that quality factor $Q[w]$ depends only on the eigenvalue $\zeta$, in particular,

$$Q[w] = \frac{1}{2} \frac{|\text{Re} \zeta|}{\text{Im} \zeta}. \quad (8)$$

This presentation elicits the importance of the spectral theory of the system operator $A(\beta)$ and the quantities $U[w], W_{\text{dis}}[w], Q[w]$ in our studies of the dissipative properties of two-component composite systems. One of our main achievements in [1] is a complete asymptotic description of these quantities and the spectrum of the non-self-adjoint operator $A(\beta)$ as $\beta \to \infty$ using perturbation theory. The following highlights some of these results from [1] including the modal dichotomy.

**Theorem 1** (modal dichotomy). Let $A(\beta) = \Omega - i\beta B$, $\beta \geq 0$ be the system operator of the canonical equations (1) in an $N$-dimensional Hilbert space $H$ with inner product $(\cdot, \cdot)$, where $\Omega$ is self-adjoint and $B \geq 0$. Let $\zeta_j, 1 \leq j \leq N_B$ be an indexing of all the nonzero eigenvalues of $B$ (counting multiplicities),
where \( N_B = \text{rank } B \). Then for the high-loss regime \( \beta \gg 1 \), the system operator \( A(\beta) \) is diagonalizable and there exists a complete set of eigenvalues \( \zeta_j(\beta) \) and eigenvectors \( w_j(\beta) \) satisfying

\[
A(\beta) w_j(\beta) = \zeta_j(\beta) w_j(\beta), \quad 1 \leq j \leq N,
\]

which split into two distinct classes of eigenpairs

\[
\begin{align*}
\text{high-loss:} & \quad \zeta_j(\beta), \ w_j(\beta), \quad 1 \leq j \leq N_B; \\
\text{low-loss:} & \quad \zeta_j(\beta), \ w_j(\beta), \quad N_B + 1 \leq j \leq N,
\end{align*}
\]

having the following properties:

(i) The high-loss eigenpairs have the asymptotic expansions as \( \beta \to \infty \):

\[
- \text{Im } \zeta_j(\beta) = \hat{\zeta}_j \beta + O(\beta^{-1}), \quad \hat{\zeta}_j > 0, \quad \text{Re } \zeta_j(\beta) = \rho_j + O(\beta^{-2}), \quad w_j(\beta) = \hat{w}_j + O(\beta^{-1}),
\]

for \( 1 \leq j \leq N_B \). The vectors \( \{\hat{w}_j\}_{j=1}^{N_B} \) form an orthonormal basis of the loss subspace \( H_B = \text{Ran } B \) with

\[
B \hat{w}_j = \hat{\zeta}_j \hat{w}_j, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \quad 1 \leq j \leq N_B.
\]

(ii) The low-loss eigenpairs have the asymptotic expansions as \( \beta \to \infty \):

\[
- \text{Im } \zeta_j(\beta) = d_j \beta^{-1} + O(\beta^{-3}), \quad d_j \geq 0, \quad \text{Re } \zeta_j(\beta) = \rho_j + O(\beta^{-2}), \quad w_j(\beta) = \hat{w}_j + O(\beta^{-1}),
\]

for \( N_B + 1 \leq j \leq N \). The vectors \( \{\hat{w}_j\}_{j=N_B+1}^{N} \) form an orthonormal basis of the no-loss subspace \( H_B^c = \text{Ker } B \) and

\[
B \hat{w}_j = 0, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \quad N_B + 1 \leq j \leq N.
\]

Theorem 2 (eigenmode dissipative behavior). The following asymptotic formulas hold as \( \beta \to \infty \) for the eigenpairs \( \zeta_j(\beta), w_j(\beta) \) in Theorem 1:

\[
U[w_j(\beta)] = \frac{1}{2} + O(\beta^{-1}), \quad 1 \leq j \leq N;
\]

\[
\text{high-loss:} \quad W_{\text{dis}}[w_j(\beta)] = \hat{\zeta}_j \beta + O(1), \quad Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{\zeta_j} \beta^{-1} + O(\beta^{-3}), \quad 1 \leq j \leq N_B;
\]

\[
\text{low-loss:} \quad W_{\text{dis}}[w_j(\beta)] = d_j \beta^{-1} + O(\beta^{-2}), \quad Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{d_j} \beta + O(\beta^{-1}), \quad N_B + 1 \leq j \leq N.
\]

For the low-loss modes quality factor, the asymptotic formula is for the typical case \( d_j \neq 0 \).

1.2 Lagrangian framework and main results

In this section I will first give a brief introduction to the Lagrangian framework from [2] and then, in §1.2.1 highlight some of our main results from [2].

The Lagrangian framework incorporating dissipation As we are interested in dissipative physical systems evolving linearly, the Lagrangian framework starts with a Lagrangian \( \mathcal{L} = \mathcal{L}(q, \dot{q}) \) and a Rayleigh dissipation function \( \mathcal{R} = \mathcal{R}(\dot{q}) \) which are quadratic functions of the coordinates \( q = [q_r]_{r=1}^{N} \) (an \( N \times 1 \) column vector with \( 1 \leq N < \infty \)) and their time derivatives \( \dot{q} \), that is,

\[
\mathcal{L} = \mathcal{L}(q, \dot{q}) = \frac{1}{2} \dot{q}^T \alpha \dot{q} + \dot{q}^T \gamma q - \frac{1}{2} q^T \eta q, \quad \mathcal{R} = \mathcal{R}(\dot{q}) = \frac{1}{2} \dot{q}^T \beta \dot{q}, \quad \beta \geq 0,
\]

where "T" denotes the matrix transposition operation, and \( \alpha, \eta, \zeta, R \) are \( N \times N \)-matrices with real-valued entries. We have introduced here a dimensionless loss parameter \( \beta \) which will be used to scale the intensity.
of dissipation and so we assume \( R \neq 0 \). Now we also assume that \( \alpha \) and \( \eta, R \) are symmetric matrices which are positive definite and positive semidefinite, respectively, i.e.,

\[
\alpha = \alpha^T > 0, \quad \eta = \eta^T \geq 0, \quad R = R^T \geq 0.
\]

Here \( \dot{q}^T \zeta q \) is the gyroscopic term and

\[
\mathcal{T} = \mathcal{T}(\dot{q}) = \frac{1}{2} \dot{q}^T \alpha \dot{q}, \quad \mathcal{V} = \mathcal{V}(q) = \frac{1}{2} q^T \eta q \quad \mathcal{H} = \mathcal{H}(q, \dot{q}) = \mathcal{T}(\dot{q}) + \mathcal{V}(q)
\]

are interpreted as the kinetic, potential, and system energy (i.e., the Hamiltonian), respectively. Using a standard methods from classical mechanics including Rayleigh’s method [10, §46], the dynamics of the system are governed by the general Euler-Lagrange equations of motion with forces:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = -\frac{\partial \mathcal{R}}{\partial \dot{q}} + F,
\]

where \( \frac{\partial \mathcal{R}}{\partial \dot{q}} \) are dissipative forces and \( F = F(t) \) is an external force. These can be rewritten as the following second-order linear differential equation

\[
\alpha \ddot{q} + (\gamma - \gamma^T + \beta R) \dot{q} + \eta q = F.
\]

Now by a linear Lagrangian system with losses we mean a system whose state is described by a time-dependent \( q(t) \) taking values in the Hilbert space \( \mathbb{C}^N \) with the standard complex inner product \( (\cdot, \cdot) \) whose dynamics are governed by the differential equation \( (22) \). The energy balance equation for a state \( q(t) \) of the Lagrangian system with dynamics governed by \( (22) \) is

\[
\frac{d\mathcal{H}}{dt} = -2\mathcal{R}(\dot{q}) + \text{Re}(\dot{q}(t), F(t)),
\]

where \( \mathcal{H}(q, \dot{q}) = \frac{1}{2} (\dot{q}, \alpha \dot{q}) + \frac{1}{2} (q, \eta q) \) and \( \mathcal{R}(\dot{q}) = \frac{1}{2} (\dot{q}, R \dot{q}) \) (which extends the definitions of \( \mathcal{H}, \mathcal{R} \) above from states with values in \( \mathbb{R}^N \) to state with values in \( \mathbb{C}^N \)).

We consider the Lagrangian system to be a model for a two-component composite with a lossy and a lossless component whenever the rank \( N_R \) of \( R \) satisfies the loss fraction condition

\[
0 < \delta_R = \frac{N_R}{N} < 1, \quad N_R = \text{rank} \, R,
\]

where \( N \) is the total degrees of freedom of the Lagrangian system and \( \delta_R \) is referred to as the loss fraction. We then associate the operator \( R \) with the lossy component of the system and consider the lossy component to be highly lossy when \( \beta \gg 1 \).

**Equivalence between systems** An important result of [2], that we will now discuss, is that a linear Lagrangian system as described above with \( N \) degrees-of-freedom having the Hamiltonian \( \mathcal{H} \) and losses accounted by the Rayleigh dissipation function \( \mathcal{R} \) is equivalent (in a sense to be described below) to a two-component composite system with lossy and a lossless component that we modeled in [1] (see §1.1 above).

Let \( \mathcal{H} = \mathbb{C}^{2N} \) be the Hilbert space of \( 2N \times 1 \) column vectors with complex entries and the standard inner product space \( (\cdot, \cdot) \). Then in [2] we show there exists a linear transformation \( C : \mathcal{H} \rightarrow \mathcal{H} \) such that in the new variable \( v = C [\dot{q} \quad q]^T \), the above second-order ODE \( (22) \) becomes the following first-order ODE in the canonical form \( (1) \):

\[
\partial_t v(t) = -iA(\beta)v(t) + f(t), \quad \text{where } A = \Omega - i(\beta B), \quad \beta \geq 0
\]

with

\[
\Omega = \begin{bmatrix} \Omega_p & -i\Phi^T \\ i\Phi & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \hat{R} & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} K_pF \\ 0 \end{bmatrix},
\]

\[
K_p = \sqrt{\alpha}^{-1}, \quad K_q = \sqrt{\eta}, \quad \Omega_p = -iK_p (\gamma - \gamma^T) K_p^T, \quad \Phi = K_qK_p^T, \quad \hat{R} = K_pRK_p^T.
\]
The $2N \times 2N$ matrices $\Omega, B$ are Hermitian and $B \geq 0$. Furthermore, the loss fraction condition \[4\] is satisfied with
\[
0 < \frac{N_B}{2N} \leq \frac{1}{2}, \text{ since } N_B = \text{rank } B = \text{rank } R = N_R \leq N.
\] (28)

In particular, it fits within our previous framework and model for a two-component composite system as described in our previous paper [2] (see \[1\] above). Interpreting the energy balance equation for such a linear system (see Eq. \[2\] above) as before, then the system energy $U[v(t)]$, dissipated power $W_{\text{dis}}[v(t)]$, and $W[v(t)]$ the rate of work done by the force $f(t)$ are
\[
U[v(t)] = \frac{1}{2}(v(t), v(t)), \quad W_{\text{dis}}[v(t)] = \beta(v(t), \dot{v}(t)), \quad W[v(t)] = \text{Re}(v(t), f(t)).
\] (29)

A result of our paper [2] tells us that, in terms of the Lagrangian system, we have
\[
U[v(t)] = \mathcal{H}(q, \dot{q}), \quad W_{\text{dis}}[v(t)] = 2\mathcal{R}(\dot{q}), \quad W[v(t)] = \text{Re}(\dot{q}(t), F(t)).
\] (30)

This shows that the study of the energy dissipative properties of the Lagrangian system with evolution equation \[22\] is reduced to a study of the two-component composite system with the evolution equation \[25\] in canonical form. The advantage of this is that a study of two-component models was already thoroughly carried out in our previous paper [1] for general systems. But, because of the block structure of the system operator $A(\beta)$ in \[25\], much more can be said about this system and hence about the Lagrangian system such as spectral symmetries and overdamping phenomena. All of this is discussed in detail in our paper [2].

1.2.1 Main results

In this section we more precisely state the result from [2] on the selective overdamping phenomenon. The following highlights one of our main results from [2] on this:

- In [2], we show that if the Lagrangian system with $N$ degrees of freedom satisfies the loss fraction condition \[21\] so that $0 < \delta_R = N_R/N < 1$, where $N_R = \text{rank } R$ (i.e., within our Lagrangian framework it is a model of a two-component composite system with a lossy and a lossless component), then when the lossy component is highly lossy, i.e., $\beta \gg 1$, a rather universal phenomenon occurs which we call selective overdamping: The modal dichotomy occurs, i.e., the eigenmodes of the system \[25\] split into two distinct classes, high-loss and low-loss, based on there dissipative properties as described in \[1\] (see Theorems \[1\] \[2\] in this research statement). Moreover, the number of high-loss modes is $N_R$ and they are all overdamped, i.e., non-oscillatory, as are an equal number of low-loss modes, but the rest of the low-loss modes remain oscillatory each with an extremely high quality factor that actually increases as the loss of the lossy component increases. In other words, there is always a positive fraction, namely, the fraction $1 - \delta_R > 0$, of the eigenmodes which are low loss and have high quality factor no matter how high the losses are in the high-loss component of the composite.

On selective overdamping  Let us now discuss this result form [2] on the selective overdamping phenomenon in the high-loss regime $\beta \gg 1$. To begin, an eigenmode $q(t) = q e^{-i\zeta t}$ of the Lagrangian system is a nontrivial solution of \[22\] with $F = 0$. The eigenmode is called overdamped if $\zeta = \zeta(\beta)$ satisfies $\text{Re } \zeta = 0$ for $\beta \gg 1$, that is, the damped harmonic oscillation $q(t)$ is overdamped if it ceases to oscillate when dissipation is sufficiently large. On the other hand, if the mode satisfies $\text{Re } \zeta \neq 0$ for $\beta \gg 1$ then it is said to be underdamped, that is, the damped harmonic oscillation $q(t)$ is underdamped if it remains oscillatory no matter how large the dissipation. One of the main results of our paper [2] is an asymptotic characterization of the selective overdamping phenomenon in terms of the eigenmodes.

But there is a correspondence (via the linear transformation $v = C [\dot{q} \quad q]^T$ discussed above) between the eigenmodes $q(t) = q e^{-i\zeta t}$ and an eigenmodes $v(t) = w e^{-i\zeta t}$ of the system with the linear evolution \[25\], where the system operator $A(\beta) = \Omega - i\beta B$ is given by \[26\], \[27\]. Thus, we call $v(t)$ overdamped or underdamped.
based on the same criteria on its frequency $\text{Re}\zeta$ as above for $q(t)$. Moreover, as discussed in §1.1 above, an eigenmode $v(t) = e^{-i\zeta t}$ corresponds to a solution of the eigenvalue problem $A(\beta)w = \zeta w, w \neq 0$. Now in the high-loss regime $\beta \gg 1$, the modal dichotomy occurs as described in [11] (see Theorems 1, 2 in this research statement), where the eigenmodes split into two classes based on their dissipative properties: high-loss and low-loss.

The following is an asymptotic description of the selective overdamping phenomenon from [2] including the modal dichotomy and quality factor $Q$ as defined by [3].

**Theorem 3** (selective overdamping). Consider the Lagrangian system with $N$ degrees-of-freedom whose states are governed by the second-order ODE in (22). Suppose the loss fraction condition $0 < \delta_R < 1$ is satisfied, where $\delta_R = N\delta, N_R = \text{rank } R$. Assume there is no gyroscopic term, i.e., $\gamma = 0$, and Ker $\eta \cap$ Ker $R = \{0\}$. Let $A(\beta) = \Omega - i\beta B$ be the system operator defined in (26), (27) so that $N_R = N_B = \text{rank } B$. Then for the high-loss regime $\beta \gg 1$, the operator $A(\beta)$ is diagonalizable and there exists a complete set of eigenvalues $\zeta_j(\beta)$ and eigenvectors $w_j(\beta)$ satisfying

$$A(\beta)w_j(\beta) = \zeta_j(\beta)w_j(\beta), \quad 1 \leq j \leq 2N,$$

which split into three distinct classes of eigenpairs

- high-loss, overdamped: $\zeta_j(\beta), \ w_j(\beta), \ 1 \leq j \leq N\delta_R$;
- low-loss, overdamped: $\zeta_j(\beta), \ w_j(\beta), \ N\delta_R + 1 \leq j \leq 2N\delta_R$;
- low-loss, underdamped: $\zeta_j(\beta), \ w_j(\beta), \ 2N\delta_R + 1 \leq j \leq 2N$,

having the following properties:

(i) The high-loss, overdamped eigenpairs have the asymptotic expansions as $\beta \rightarrow \infty$:

$$\text{Re } \zeta_j(\beta) = 0, \quad -\text{Im } \zeta_j(\beta) = \zeta_j + O(\beta^{-1}), \quad \zeta_j > 0, \quad w_j(\beta) = \tilde{w}_j + O(\beta^{-1}),$$

for $1 \leq j \leq N\delta_R$. The vectors $\{\tilde{w}_j\}_{j=1}^{N\delta_R}$ form an orthonormal basis of the loss subspace $H_B = \text{Ran } B$ with

$$B\tilde{w}_j = \zeta_j\tilde{w}_j, \quad 1 \leq j \leq N\delta_R.$$  

(ii) The low-loss, overdamped eigenpairs have the asymptotic expansions as $\beta \rightarrow \infty$:

$$\text{Re } \zeta_j(\beta) = 0, \quad -\text{Im } \zeta_j(\beta) = d_j\beta^{-1} + O(\beta^{-3}), \quad d_j \geq 0, \quad w_j(\beta) = \tilde{w}_j + O(\beta^{-1}),$$

for $N\delta_R + 1 \leq j \leq 2N\delta_R$.

(iii) The low-loss, underdamped eigenpairs have the asymptotic expansions as $\beta \rightarrow \infty$:

$$\text{Re } \zeta_j(\beta) = \rho_j + O(\beta^{-2}), \quad \rho_j = (\tilde{w}_j, \Omega\tilde{w}_j) \neq 0, \quad -\text{Im } \zeta_j(\beta) = d_j\beta^{-1} + O(\beta^{-3}), \quad d_j \geq 0, \quad w_j(\beta) = \tilde{w}_j + O(\beta^{-1}),$$

for $2N\delta_R + 1 \leq j \leq 2N$.

(iv) The zeroth order terms for the eigenvectors of the low-loss modes, i.e., $\{\tilde{w}_j\}_{j=N\delta_R+1}^{2N}$, form an orthonormal basis of the no-loss subspace $H_B^\perp = \text{Ker } B$.

**Theorem 4** (quality factor enhancement). The following asymptotic formulas hold as $\beta \rightarrow \infty$ for the quality factor of the eigenpairs $\zeta_j(\beta), w_j(\beta)$ in Theorem 3:

- high-loss/low-loss, overdamped: $Q[w_j(\beta)] = 0, \quad 1 \leq j \leq 2N\delta_R$;
- low-loss, underdamped: $Q[w_j(\beta)] = \frac{1}{2}\frac{\rho_j}{d_j} \beta + O(\beta^{-1}), \quad \rho_j \neq 0, \quad 2N\delta_R + 1 \leq j \leq 2N$.

For the low-loss underdamped modes, the asymptotic formula is for the typical case $d_j \neq 0$. Regardless of whether $d_j \neq 0$ or not though, it is always true for the low-loss underdamped modes that

$$\lim_{\beta \rightarrow \infty} Q[w_j(\beta)] = \infty, \quad 2N\delta_R + 1 \leq j \leq 2N.$$
2 Speed-of-light limitations in passive linear media

The purpose of this section is to give a brief description of my research project with Steven G. Johnson and Yehuda Avniel from MIT and highlight some of the results we have achieved so far from our paper [3] in progress.

Our project can be described as being a study on how relativistic causality, i.e., information cannot travel faster than $c$ (the speed of light in vacuum), can actually be proved for the propagation of light through periodic media given the broad range material properties that are possible (e.g., isotropy, anisotropy, chirality, nonlocality, dispersive, dissipative). The main objective of this project can be summarized as follows:

(i) To find the broadest range of materials and weakest mathematical conditions which guarantee the speed-of-light restriction $|v_e| \leq c$ on the electromagnetic energy velocity $v_e$ (also known as the energy transport velocity $v_T$).

Motivation   One can get a sense of the range and obscurity of conditions that have been used previously to prove $|v_e| \leq c$ by various authors [13], [14, §84], [15], [16], [17, p. 41]. In all cases these authors assume local linear media, strong regularity properties of the permittivity $\epsilon(\omega)$ and permeability $\mu(\omega)$ of the medium as a function of frequency $\omega$, and usually negligible loss in a ”transparency window.” For instance, Brillouin [13] assumed a homogeneous, isotropic, linear dispersive medium with a Lorentzian model of material dispersion and proved in the case of negligible loss that $|v_e| \leq c$. Landau and Lifshitz [14, §84] generalized Brillouin’s result to wider range of models of material dispersion by assuming only homogeneous, isotropic, linear dispersive materials satisfying causality, positivity of energy density (due to thermodynamic considerations), and some regularity conditions on the permittivity and permeability as a function of frequency. Their derivation of $|v_e| \leq c$ is then based on the Kramers-Kronig relations and positivity of the energy density. Yaghjian [15], [16] was able to give an alternative proof of the result of Landau and Lifshitz for homogeneous, isotropic, linear dispersive materials by using only the assumption of passivity of the medium and some regularity conditions (which were not explicitly specified) on the permittivity and permeability. All these previous results just mentioned, however, were for light propagating in a homogeneous medium. For periodic media (which includes homogeneous media as a special case), the only proof that we know of that $|v_e| \leq c$ is in Joannopoulus et al. [17, p. 41] under the assumptions of isotropic, dispersionless, linear media with real permittivity and permeability bounded below by one.

Highlights from our paper [3] in progress    The following highlights (in a broad scope) some select results from [3] (a more precise description is given in the next section below):

- We prove that well-known speed-of-light restrictions on electromagnetic energy velocity can be extended to a new level of generality, encompassing even nonlocal chiral media in periodic geometries, while at the same time weakening the underlying assumptions to only passivity and linearity of the medium (along with a transparency window, which ensures well-defined energy propagation). Surprisingly, passivity alone is sufficient to guarantee causality and positivity of the energy density (with no thermodynamic assumptions), in contrast to prior work which typically assumed the latter properties. Moreover, our proof is general enough to include a very broad range of material properties, including anisotropy, bianisotropy (chirality), nonlocality, dispersion, periodicity, and even delta functions or similar generalized functions.

- The results in this paper are proved using deep results from linear response theory, harmonic analysis, and functional analysis. The key elements of our proof are derived using some important “representation” theorems from (a) linear response theory on passive convolution operators [18] and (b) the theory of Herglotz functions (which is well-known to spectral theorists [19], [20 Vol. II, §VI], [21], [22]).

- We strengthen the bound $|v_e| \leq c$ by showing that $|v_e| = c$ can only hold for a very special class of materials and field solutions. For instance, for homogeneous passive linear media, we prove that the vacuum is the only realistic isotropic material where the upper bound can be achieved.

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2.1 Mathematical setup and main results

In this section I will highlight some of the main results from our paper [3]. To do this I provide enough mathematical background to properly state these results such as defining the terms “passive linear media” and “transparency window” in Section 2.1.1 and defining the “energy velocity” in Section 2.1.2. These main results have been formulated in this research statement as Theorems 5, 7, 9 and our principal result in Theorems 10 and 11 on the speed-of-light restriction $|v_e| \leq c$ for the electromagnetic energy velocity $v_e$ in periodic media.

2.1.1 Mathematical setup

In order to state more precisely our main results by formulating them as theorems in this research statement, I will briefly define in this section what we mean by a passive linear medium with a transparency window and describe some important consequences of these assumptions.

**Passive linear media** The macroscopic Maxwell equations (in Gaussian units) without sources are the following PDEs for the electromagnetic fields $E, H$,

\begin{align}
\nabla \cdot B &= 0 & \nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} &= 0, \\
\nabla \cdot D &= 0 & \nabla \times H - \frac{1}{c} \frac{\partial D}{\partial t} &= 0,
\end{align}

(c denotes the speed of light in a vacuum) which also include the constitutive relations. In linear (time-translation invariant) media, the constitutive relations are $D = E + 4\pi P$ and $B = H + 4\pi M$, where the electric and magnetic polarizations $P$ and $M$, respectively, are given in terms of convolution with a susceptibility $\chi$:

\[
\begin{bmatrix}
P \\
M
\end{bmatrix}
= \chi * \begin{bmatrix}
E \\
H
\end{bmatrix} = \int_{-\infty}^{\infty} \chi(t') \begin{bmatrix}
E(t - t') \\
H(t - t')
\end{bmatrix} dt.'
\]

Mathematically, in order to include the most general linear media, the convolution is defined in the distribution sense [18, Chap. 5] allowing the susceptibility to be a generalized function (e.g., a delta function for instantaneous polarization response), that is, $\chi$ is a distribution on the test functions $\mathcal{D}(\mathcal{F})$ – infinitely differentiable functions from $\mathbb{R}$ into a Hilbert space $\mathcal{F}$ with compact support. This Hilbert space $\mathcal{F}$ depends on whether the material is homogeneous or periodic. For homogeneous media, the Hilbert space is the complex 6-dimensional vector space $\mathcal{F} = \mathbb{C}^6$ with standard inner product. For a periodic media with unit cell $V$, the Hilbert space is $\mathcal{F} = (L^2(V))^6$ – vector-valued functions from $V$ into $\mathbb{C}^6$ which are square-integrable in $V$.

The key property of the susceptibility that we use to derive the energy velocity bound $|v_e| \leq c$ and other properties is *passivity*. The exact statement will depend on whether the medium is homogeneous or periodic.

In the case of homogeneous media, *passivity* is the statement that *polarization currents don’t do work*. Mathematically, this is the following condition: the inequality

\[
0 \leq \text{Re} \int_{-\infty}^{t} E(t') \frac{dP(t')}{dt'} + H(t')^\dagger \frac{dM(t')}{dt'} dt',
\]

must hold for all time $t$ and every $[E \ H]^T \in \mathbb{C}^6$ (here $^\dagger$ denotes the conjugate transpose of a vector and Re denotes the real part of a complex number). Physically, the integral in (43) represents the work that the fields do on the bound currents. Thus a passive medium is one satisfying the passivity condition (43) or, in other words, is one in which energy can only be absorbed by the material but never generated by it.

In the case of periodic media, *passivity* is the weaker condition: *polarization currents don’t do work on average within the unit cell $V$*. Mathematically this means our condition is like that of (43) but with an
integral over the unit cell as well as over time. More precisely, this is the following condition: the inequality
\[ 0 \leq \text{Re} \int_{-\infty}^{t} \int_{V} E(t') \frac{dP(t')}{dt'} + H(t') \frac{dM(t')}{dt'} \, dr dt', \]
(suppressing the spatial dependency \( r \) of the fields) holds for all time \( t \) and every \([E \ H]^T \in (L^2(V))^6\).

The following theorem proved in our paper \cite{3} tells us that the assumption of a passive linear medium implies causality of the polarization response, strong regularity of the Fourier transform of the susceptibility \( \hat{\chi}(\omega) \) as a function of frequency \( \omega \) in the complex upper half-plane, and positivity of the imaginary part of the operator \( h(\omega) = \omega \hat{\chi}(\omega) \). In particular, an important consequence of the regularity and positivity is that the function \( h(\omega) \) belongs to a class of functions known as Herglotz functions (defined precisely below). This is important because the theory of Herglotz functions (which is well known to spectral theorists \cite{19, 20, Vol. II, §VI, 21, 22}) plays a key role in proving the speed-of-light limitation on energy velocity.

**Theorem 5** (consequences of passivity). For a passive linear medium with susceptibility \( \chi \) the following statements are true:

1. (causality) Polarizations only depend on fields in the past, in particular, \( \chi(t) = 0 \) for \( t < 0 \).
2. (analyticity) Fourier transform of \( \frac{d\chi}{d\omega} \), i.e., \( -i\omega \hat{\chi}(\omega) \), is analytic for \( \text{Im} \omega > 0 \) in the norm topology on \( \mathcal{L}(\mathcal{H}) \) – bounded linear operators.\[
\text{(positivity)} \text{ The function } h(\omega) = \omega \hat{\chi}(\omega) \text{ satisfies } \text{Im} \, h(\omega) \geq 0 \text{ for } \text{Im} \omega > 0.
\]

**Proof (Sketch).** Linearity and passivity means \( \frac{d\chi}{d\omega} \) is a “passive” convolution operator on \( \mathcal{D}(\mathcal{H}) \). Deep results from linear response theory \cite{13} on such operators implies these results. \( \square \)

**Definition 6** (Herglotz function). Let \( \mathcal{H} \) be a Hilbert space. An analytic function from \( \mathbb{C}_+ \) (open complex upper half-plane) into \( \mathcal{L}(\mathcal{H}) \) (bounded linear operators) whose values have positive semidefinite imaginary part is called a bounded-operator-valued Herglotz function [e.g., \( h(\omega) = \omega \hat{\chi}(\omega) \) or the resolvent \( (A - \omega I)^{-1} \) of a self-adjoint \( A \)].

**Transparency window** A transparency window for a passive linear medium is defined as a frequency interval \( (\omega_1, \omega_2) \subset \mathbb{R} \) where losses are negligible. A precise definition in terms of the Herglotz function \( h(\omega) = \omega \hat{\chi}(\omega) \) will be given in this section. Two important properties of the function \( h(\omega) \) in the transparency window will also be given. In particular, we have a theorem from \cite{3} which tells us that the Herglotz function \( h(\omega) \) is a monotonically increasing and differentiable bounded-operator-valued function of frequency in the window. These properties are critical since they imply the positivity of the energy density, which is used to prove the speed-of-light limitation on the energy velocity as described in the following section.

Now there is a sense in which electromagnetic losses (i.e., transfer of electromagnetic energy into matter by absorption) are quantified in terms of the boundary values of \( \text{Im} \, h(\omega) \) on the real axis \( \text{Im} \omega = 0 \) and, in typical circumstances [e.g., if \( \chi(t) \) was integrable], the condition of a transparency window is simply that \( \text{Im} \, h(\omega) = 0 \) in the interval. But more generally the definition of a transparency window is given in terms of a measure induced by \( \text{Im} \, h(\omega) \) which we now describe briefly from a result proven in \cite{3}.

**Theorem 7** (measuring EM losses). For a passive linear medium with susceptibility \( \chi \), electromagnetic losses in any bounded frequency interval \( (\omega_1, \omega_2) \subset \mathbb{R} \) can be quantified by a nonnegative bounded-operator-valued measure \( \Omega(\cdot) \) on the bounded Borel subsets of \( \mathbb{R} \) via the limits (in the strong operator topology):
\[
\Omega((\omega_1, \omega_2)) = \lim_{\delta \downarrow 0} \lim_{\delta \downarrow 0} \int_{\omega_1 + \delta}^{\omega_2 - \delta} \frac{1}{\pi} \text{Im} \, h(\omega + i\eta) \, d\omega.
\]
Proof (Sketch). The total energy dissipated by a field \( F(t) = [E(t), H(t)] \in \mathcal{D}(\mathcal{S}) \) is the integral \( \mathcal{I} \) (for homogeneous media) or \( \mathcal{I} \) (for periodic media) with \( t = \infty \). We show in [3], using results from linear response theory and from harmonic analysis [18], that this time integral is given in terms of a frequency integral involving only the measure \( \Omega(\cdot) \) and the Fourier transform \( \mathcal{F}(\omega) \) of the field. The Stieltjes inversion formula for \( \Omega(\cdot) \) from the theory of Herglotz functions [19], [20, Vol. II, §VI], [21], [22] implies existence of the limits and equality [45].

**Definition 8** (transparency window). For a passive linear medium with susceptibility \( \chi \), a transparency window is a bounded frequency interval \( (\omega_1, \omega_2) \subseteq \mathbb{R} \) in which
\[
\Omega((\omega_1, \omega_2)) = 0. \quad (46)
\]

**Theorem 9** (consequences of a transparency window). For a passive linear medium with susceptibility \( \chi \) and a transparency window \( (\omega_1, \omega_2) \subseteq \mathbb{R} \), i.e., \( \Omega((\omega_1, \omega_2)) = 0 \), the following statements are true:

1. (analytic continuation) The function \( h(\omega) = \omega \chi(\omega) \) can be analytically continued through \( (\omega_1, \omega_2) \).
2. (self-adjointness with monotonicity) For all \( \omega \in (\omega_1, \omega_2) \), \( \text{Im} \ h(\omega) = 0 \) and \( h'(\omega) \geq 0 \).

Proof (Sketch). The theory of Herglotz functions [19], [20] Vol. II, §VI], [21], [22] implies the integral representation
\[
h(\omega) = h_0 + h_1 \omega + \int d\Omega_\lambda \left( \frac{1}{\lambda - \omega} - \frac{\lambda}{1 + \lambda^2} \right), \quad \text{Im} \ \omega > 0.
\]
where \( \text{Im} \ h_0 = 0 \), \( h_1 \geq 0 \), and \( (1 + \lambda^2)^{-1} \) is integrable with respect to the measure \( \Omega(\cdot) \). The hypothesis \( \Omega((\omega_1, \omega_2)) = 0 \) and the integral representation implies these results. In particular, differentiating under the integral sign implies monotonicity since
\[
0 \leq h_1 \leq h'(\omega) = h_1 + \int_{\mathbb{R} \setminus (\omega_1, \omega_2)} d\Omega_\lambda \frac{1}{(\lambda - \omega)^2}, \quad \text{for all } \omega \in (\omega_1, \omega_2).
\]

### 2.1.2 Main result: speed-of-light limitation

In this section we highlight the principal results of our paper [3], namely, Theorems [10, 11] on the speed-of-light limitation \( ||v_e|| \leq c \) on the energy velocity \( v_e \) in a transparency window and positivity of the energy density. I do this first with Theorem [10] in the simplified case of homogeneous media after we have reviewed planewaves and the definition of energy velocity. Next, I cover the more general case of periodic media with Theorem [11] after we have reviewed Bloch waves and the definition of energy velocity in this case.

**Homogeneous media** In a homogeneous medium, because of the continuous translational symmetry, Maxwell equations can admit nontrivial solutions which are time-harmonic electromagnetic planewaves
\[
E e^{i(k \cdot r - \omega t)}, \quad H e^{i(k \cdot r - \omega t)}, \quad (47)
\]
where \( E, H \in \mathbb{C}^3 \) are constant vectors, with real frequency \( \omega \) and real wavevector \( k \) (for negligible loss) satisfying a dispersion relation \( \omega = \omega(k) \). The group velocity is defined as \( v_g = \nabla_\mathbf{k} \omega(k) \), where \( \nabla_\mathbf{k} \) denotes the gradient, which is the velocity of narrow-bandwidth wavepackets. The energy velocity \( v_e \) of such a field is defined as the ratio of the time-averaged energy flux to the energy density \( U \), i.e.,
\[
v_e = \frac{\text{Re} \ S}{U}, \quad \text{where} \quad S = \frac{c}{8\pi} E \times \mathbf{H}^* \quad (48)
\]
is the complex Poynting vector in Gaussian units (here \( * \) denotes complex conjugation and \( \times \) denotes the cross product). It is known that \( v_g = v_e \) for transparent media under certain reservations. The energy
density $U$, as shown in our paper [3] for frequencies in a transparency window with susceptibility $\chi$, is given in terms of the Herglotz function $h(\omega) = \omega \hat{\chi}(\omega)$ by the formula

$$U = \frac{1}{16\pi} \mathbf{E}^\dagger \frac{d}{d\omega} \left[ 1 + 4\pi \hat{\chi}(\omega) \right] \mathbf{E} = \frac{1}{16\pi} \left( ||\mathbf{E}||^2 + ||\mathbf{H}||^2 \right) + \frac{1}{4\pi} \mathbf{E}^\dagger h'(\omega) \mathbf{E} \quad (49)$$

which generalizes the energy density formula of Brillouin [13, p. 92, (17)], [14, p. 275, (80.11) & p. 332, (96.6)].

The following theorem, proved in [3], is our main result in the case of homogeneous passive linear media with a transparency window.

**Theorem 10** (speed-of-light limitation: homogeneous media). *For any homogeneous passive linear medium and for any time-harmonic EM planewave with frequency in a transparency window, the following statements are true:*

1. *(energy positivity)* The energy density is positive, i.e., $U > 0$.

2. *(speed-of-light limitation)* The energy velocity $v_e$ (as defined in (48)) has the limitation

$$||v_e|| \leq c. \quad (50)$$

**Proof (Sketch).** Lagrange’s identity implies

$$||\mathbf{E} \times \mathbf{H}^*||^2 = ||\mathbf{E}||^2 ||\mathbf{H}||^2 - ||\mathbf{E}^T \mathbf{H}||^2 \Rightarrow ||\text{Re} \mathbf{S}|| \leq \frac{c}{16\pi} \left( ||\mathbf{E}||^2 + ||\mathbf{H}||^2 \right).$$

The transparency window implies monotonicity $h'(\omega) \geq 0$ (see Theorem 9 of this research statement with the Hilbert space $\mathcal{S} = \mathbb{C}^6$) implies, by the definition of $U$ in (49), that

$$cU \geq \frac{c}{16\pi} \left( ||\mathbf{E}||^2 + ||\mathbf{H}||^2 \right) \geq ||\text{Re} \mathbf{S}||. \quad \square$$

**Periodic media** In a periodic medium, because of the discrete translational symmetry, Maxwell equations can admit nontrivial solutions which are time-harmonic electromagnetic Bloch waves

$$\mathbf{E}(r, t) = \mathbf{E}(r) e^{i(k \cdot r - \omega t)}, \quad \mathbf{H}(r, t) = \mathbf{H}(r) e^{i(k \cdot r - \omega t)},$$

where $F(r) = [\mathbf{E}(r), \mathbf{H}(r)]^T$ is periodic function on the lattice with unit cell $V$ with $F \in (L^2(V))^6$, and real frequency $\omega$ and real wavevector $k$ (for negligible loss) satisfying a dispersion relation $\omega = \omega(k)$. The group velocity is defined as $v_g = \nabla_k \omega(k)$. The energy velocity $v_e$ of such a field is similarly to that of a planewave except now we must spatial averaging over the unit cell and hence is the ratio

$$v_e = \frac{\int_V \text{Re} \mathbf{S}(r) dr}{\int_V U(r) dr}, \quad (51)$$

where $\text{Re} \mathbf{S}$ is the time-averaged Poynting vector defined in (48) (but now a function of position $r$). It is known that $v_g = v_e$ for transparent media under certain reservations (cf. [23, Appendix B]), a fact that we review in our paper [3]. The energy density $U$, as shown in our paper [3] for frequencies in a transparency window with susceptibility $\chi$, is given in terms of the Herglotz function $h(\omega) = \omega \hat{\chi}(\omega)$ by the formula:

$$U = \frac{1}{16\pi} \mathbf{E}^\dagger \frac{d}{d\omega} \left[ 1 + 4\pi \hat{\chi}(\omega) \right] \mathbf{T}(k) \left[ \mathbf{E} \right] = \frac{1}{16\pi} \left( ||\mathbf{E}||^2 + ||\mathbf{H}||^2 \right) + \frac{1}{4\pi} \mathbf{E}^\dagger h'(\omega) \mathbf{T}(k) \left[ \mathbf{E} \right] \quad (52)$$

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where $T(k)$ is the unitary operator [with adjoint $T(-k)$] defined by multiplication
\[
(T(k)\psi)(r) = e^{ikr}\psi(r), \quad \psi \in (L^2(V))^6.
\] (53)
This formula (52) for the energy density $U$ generalizes the previous formula (49) to the case of periodic media.

The following theorem, proved in [3], is our main result in the case of periodic passive linear media with a transparency window.

**Theorem 11** (speed-of-light limitation: periodic media). For any periodic passive (in a unit cell $V$) linear medium and for any time-harmonic EM Bloch wave with frequency in a transparency window, the following statements are true:

1. (energy positivity) The spatially averaged energy density is positive, i.e., $\int_V U(r)\,dr > 0$.
2. (speed-of-light limitation) The energy velocity $v_e$ (as defined in 51) has the limitation

\[
||v_e|| \leq c.
\] (54)

**Proof (Sketch).** An elementary inequality and Lagrange’s identity implies
\[
\left\| \int_V \text{Re} S(r)\,dr \right\| \leq \int_V ||\text{Re} S(r)||\,dr \leq \frac{c}{16\pi} \int_V (||E(r)||^2 + ||H(r)||^2)\,dr.
\]
The transparency window implies monotonicity $h'(\omega) \geq 0$ [see Theorem 9 of this research statement with the Hilbert space $H = (L^2(V))^6$] implies, by the definition of $U$ in (52), that
\[
c \int_V U(r)\,dr \geq \frac{c}{16\pi} \int_V (||E(r)||^2 + ||H(r)||^2)\,dr \geq \left\| \int_V \text{Re} S(r)\,dr \right\|.
\]

3 Resonant electromagnetic scattering in anisotropic layered media

![Figure 1: (Left) A periodic layered medium with anisotropic layers. (Right) A defective layer, or slab, embedded in an ambient periodic layered medium. In this example, the structure is symmetric about the centerline of the slab.](image)

The purpose of this section is to highlight the most significant results and ideas of my research with Stephen P. Shipman from LSU based on our works [4], [5], and to indicate briefly in our discussion what the next step of our research is based on our work in progress [6].

Our ongoing research can be described broadly as a study (using linear media) on electromagnetic resonance phenomena and the anomalous scattering behavior in periodic anisotropic layered media (a 1D photonic crystal, e.g., Fig. 1 on left) when a slab defect is introduced as depicted in Fig. 2 (e.g., Fig. 1 on right). We are currently focused on lossless layered media which have continuous translation symmetry in the two directions determined by a given plane (e.g., in Figs. 2 on the xy plane), for our purposes its the xy plane. Such media permits a rigorous analytical study of EM scattering and quick numerical computations, yet such systems are complex enough to support more diverse types of resonance and slow light regimes then would be possible if only isotropic layers were used. Since many of these regimes have yet to be fully explored from a scattering theory perspective, they merit further studies.
Our studies in [4, 5, 6] concern resonance of a particular nature which can be described loosely as follows. We consider the situation in which the layered structure supports a time-harmonic electromagnetic field at a specific frequency in the absence of any source field originating from outside the slab (a source-free field). Such a field is dynamically decoupled from energy-transporting time-harmonic waves (external radiations) in the ambient medium. But when the parameters of the system are perturbed (e.g., structure, frequency, angle(s) of incidence), there is a coupling between these energy-transporting waves to the source-free field, resulting in a resonant interaction and anomalous scattering. This type of resonance requires that the ambient medium admit both evanescent and propagating waves at the same frequency $\omega$ (and wavevector $\kappa$ parallel to the slab) and therefore must include anisotropic layers.

Highlights of some of our achievements include:

- In [4], using periodic anisotropic layered media with a slab defect, we introduce by example two different cases [which were called (A) Resonance with a guided mode and (B) A unidirectional ambient medium] in which the type of resonance with a source-free field described above can actually occur. I will refer to the resonance phenomena in these two cases as a (A) guided-mode resonance and a (B) frozen-mode resonance, respectively. While both resonances involve anomalous scattering behavior, only the frozen-mode resonance is associated with a specific type of slow light regime and, although sharing some similarities to a guided-mode resonance, it is a qualitatively new wave phenomenon—it does not reduce to any known electromagnetic resonance. For this reason it needs to be explored as it may have potential applications. The principal result of this paper was the introduction of this new type of resonance and to identify some of the similarities to the more familiar guided-mode resonance [24, 25]. A thorough study of case (A), the guided-mode resonance, is carried out in our paper [5]. A thorough study of case (B), the frozen-mode resonance, will be carried out in our next paper [6] using the tools, techniques, and ideas developed in [4, 5, 8, 7] on scattering, perturbation theory, and slow light.

- Excitation of a guided mode is typically achieved by periodic variation (i.e., discrete translation symmetry) of the dielectric properties of the slab (embedded in air) in directions parallel to it. The periodicity couples radiating Rayleigh-Bloch waves with evanescent ones composing a guided mode [24]. But in our paper [4], an explicit example of a guided mode is constructed for the purposes of 1) to show that coupling can be achieved without periodicity of the slab by replacing the air with a homogeneous anisotropic ambient medium that supports radiation and evanescent modes at the same frequency and wavevector (parallel to the slab) and 2) to demonstrate numerically some of the important features of (A) the guided-mode resonance.

- In [5], we carry out a thorough perturbation analysis [in wavevector-frequency ($\kappa, \omega$)] of the resonance and anomalous scattering behavior for the (A) guided-mode resonance, including the sharp peak-dip shape of the resonant energy-transmission anomalies (often called a “Fano resonance” [24] and field amplification around the frequency of the guided mode. This work extends significantly the results of Shipman and Venakides [25, 26]. For instance, the asymptotic descriptions of the transmission anomalies and field amplification in our work (described in §IV.D.1 and §IV.D.2, respectively, of [5]) extend their previous formulae, essentially involving one angle of incidence, to two angles of incidence of the source field. Moreover, we show that by varying these two degrees of freedom this permits independent control over the width and central frequency of a resonance; this is important, for example, in the tuning of LED structures [27].

- Our analysis in [5] of the scattering problem (described in [5] §IV.A) and the resonance phenomena is based solely on the Maxwell equations and the full scattering matrix $S = S(\kappa, \omega)$, without invoking...
a heuristic model (for instance, [24] uses a heuristic model). The term "full" is used here because it is the (nonunitary) scattering matrix that arises when solving the scattering problem of finding the outgoing modes from given incoming modes which includes the full set of modes, i.e., the propagating and the evanescent modes (Bloch waves) in the periodic ambient medium.

- The reduction of the problem to the scattering matrix and the analysis of its poles as they move off the real frequency axis under a perturbation is an expression of the universal applicability of the formulae and our techniques to very general linear scattering problems.

- Principal results (described below) of our paper, which are needed in the analysis of resonance, are on two nondegeneracy conditions (i), (ii) of the dispersion relation \( \ell(\kappa, \omega) = 0 \) for guided modes which relates to (iii) the poles of the scattering matrix. The two nondegeneracy conditions were assumed in previous works [23], [20] but are now proved in our work for layered media and appear to be applicable to other scattering problems as well. The facts (ii) and (iii) rely on formulas in [5] Theorems 4.4, 4.5 for \( \partial \ell / \partial \omega \), \( S \) involving the total energy of the guided mode (i.e., the integral over a cross section perpendicular to the layers of the energy density of the guided mode) which is finite and positive by Theorem 4.2 of [5]. These principal results can be described as follows: Assume that a guided mode exists at a real wavevector-frequency pair \((\kappa_0, \omega_0)\) in which the periodic (or homogeneous) ambient medium admits both propagating and evanescent waves. Then, for \((\kappa, \omega)\) near \((\kappa_0, \omega_0)\),

  (i) By Theorems 4.1, 4.3 of [5], there exists a matrix \( B(\kappa, \omega) \) analytic in \((\kappa, \omega)\) with a simple eigenvalue \( \ell(\kappa, \omega) \) such that \( \ell(\kappa_0, \omega_0) = 0 \) defines the dispersion relation for guided modes and the nullspace of the matrix corresponds to the guided modes.

  (ii) By Theorem 4.4 of [5], the eigenvalue \( \ell(\kappa, \omega) \) is analytic in \((\kappa, \omega)\) and nondegenerate in the sense:

  \[
  \ell(\kappa_0, \omega_0) = 0, \quad \frac{\partial \ell}{\partial \omega}(\kappa_0, \omega_0) \neq 0.
  \]

  (iii) By Theorem 4.5 of [5], the scattering matrix \( S \) is meromorphic in \((\kappa, \omega)\). Furthermore, the matrix \( \ell S \) is analytic in \((\kappa, \omega)\) and nonzero. Moreover, the scattering matrix \( S(\kappa_0, \omega) \) has a simple pole in frequency at \( \omega = \omega_0 \), i.e.,

  \[
  \lim_{\omega \to \omega_0} (\omega - \omega_0) S(\kappa_0, \omega) \neq 0.
  \]

- Although our results in [5] are proved for lossless, nondispersive, anisotropic layered media by a judicious use of the energy conservation law and positivity of the energy density (cf. [5] §III.A, Ineq. (3.3)], [5] §III.B, Eq. (3.13)], and [5] §III.B, Theorems 3.1 & 3.2), the techniques we used to prove our results were done in a way so that they could be generalized to include dispersive and bianisotropic layered media as well, provided we work with just passive linear media in a transparency window (i.e., a real frequency interval where losses are negligible).

## 4 Past Research

In this section I briefly describe my past research and accomplishments from my paper [8] and my thesis [7] in Sections 4.1 and 4.2 respectively.

### 4.1 Spectral Perturbation Analysis of a Jordan Block

A fundamental problem in the perturbation theory for non-self-adjoint matrices with a degenerate spectrum is the determination of the perturbed eigenvalues and eigenvectors. Formulas for the higher order terms of these perturbation expansions are often needed in problems which require an accurate asymptotic analysis.

Let me give one such example which has been a major motivation for my research which lead to my first paper [8]. My Ph.D. advisor Alex Figotin and his colleague, I. Vitebskiy, considered scattering problems
involving slow light in one-dimensional semi-infinite photonic crystals \cite{28, 29, 30, 31, 32}. They found that only in the case of the frozen mode regime could incident light enter a photonic crystal with little reflection and be converted into a slow wave. This frozen mode regime was found to correspond to a stationary inflection point of the dispersion relation and a $3 \times 3$ Jordan block in the Jordan normal form of the unit cell transfer matrix (the monodromy matrix of the reduced Maxwell’s equations \cite{31} §5, i.e., the transfer matrix $T(0,d) = e^{iKd}$ of the $d$-periodic Maxwell ODEs as described in my paper \cite{3} §III.C)). In this setting, the eigenpairs of the unit cell transfer matrix corresponded to Bloch waves and their Floquet multipliers. Thus in order for them to rigorously prove the physical results and provide a better understanding of the very essence of the frozen mode regime, they needed an asymptotic analytic description of the frozen mode regime which required a sophisticated mathematical framework based on the spectral perturbation theory of a Jordan block. Unfortunately, at the time when \cite{31} was written such a theory did not exist and hence this was a big motivating factor for my paper \cite{8}. In fact, such a theory will be used in my paper \cite{6} with Stephen Shipman to study the frozen-mode resonance as first introduced in our paper \cite{4} and described in Section 3 of this research statement.

The goal of \cite{8} was to develop the spectral perturbation theory of a Jordan block to address the following:

1. Determine a generic condition that allows a constructive spectral perturbation analysis for non-self-adjoint matrices with degenerate eigenvalues.
2. What connection is there between that condition and the local Jordan normal form of the matrix perturbation corresponding to a perturbed eigenvalue?
3. Does there exist explicit recursive formulas to determine the perturbed eigenvalues and eigenvectors for non-self-adjoint perturbations of matrices with degenerate eigenvalues?

The statement of my main results regarding those three issues are contained in Theorem 2.1 and Theorem 3.1 of \cite{8}. The following highlights some of the main elements of those results:

**Theorem 12** (Aaron Welters 2010). Let $A(\varepsilon)$ be a matrix-valued function having a range in $\mathbb{C}^{n \times n}$ such that its matrix elements are analytic functions of $\varepsilon$ in a neighborhood of the origin. Let $\lambda_0$ be an eigenvalue of the unperturbed matrix $A(0)$ and denote by $m$ its algebraic multiplicity. Suppose that the generic condition

$$\frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon)) \bigg|_{(\varepsilon, \lambda)=(0, \lambda_0)} \neq 0,$$

is true. Then the Jordan normal form of $A(0)$ corresponding to the eigenvalue $\lambda_0$ consists of a single $m \times m$ Jordan block. Furthermore, all the perturbed eigenvalues near $\lambda_0$ (the $\lambda_0$-group) and their corresponding eigenvectors are each given by exactly one convergent Puiseux series whose branches are given by

$$\lambda_h(\varepsilon) = \lambda_0 + \sum_{k=1}^{\infty} \alpha_k \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^k$$

$$x_h(\varepsilon) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^k$$

for $h = 0, \ldots, m - 1$ and any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi i}{m}}$ with

$$\alpha_1^m = -\frac{\frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon)) \bigg|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{m! \frac{\partial^{m}}{\partial \lambda^m} \det(\lambda I - A(\varepsilon)) \bigg|_{(\varepsilon, \lambda)=(0, \lambda_0)}}.$$

Moreover, explicit recursive formulas exist \cite{8} Theorem 3.1, (3.12)–(3.14) to compute $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ using only the derivatives of the perturbation $A(\varepsilon)$ at $\varepsilon = 0$. In particular, the coefficients of those perturbed eigenvalues and eigenvectors, up to second order, are conveniently listed in \cite{8} Corollary 3.3.

**Corollary 13.** For $0 < |\varepsilon| < 1$, $\lambda_h(\varepsilon)$ is a simple eigenvalue of $A(\varepsilon)$. 

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4.2 Slow Waves and Periodic Differential-Algebraic Equations

The core of my thesis [7] was research on slow waves and periodic differential-algebraic equations (DAEs), also called singular or implicit differential equations. It was an effort to initiate a rigorous study that goes beyond one-dimensional photonic crystals to other physical systems where slow waves would be important. The choice to use periodic DAEs to begin such a study was a well thought out one because, using the time-harmonic Maxwell equations, electromagnetic wave propagation in one-dimensional photonic crystals is modeled by periodic DAEs (cf. [7, Chap. 1.2] and [5, Appendix]). That they are periodic DAEs can be seen as a consequence of the curl operator having a nontrivial kernel and the one-dimensional periodicity of the crystal.

As is often stated, DAEs are not ODEs. Their differences make it much more difficult to analyze the behavior of DAEs especially with respect to their spectral properties. My achievements from my thesis [7] in this direction of research can be summarized as follows:

- I developed a Floquet and spectral theory for first-order linear periodic (index-1) DAEs depending nonlinearly on a spectral parameter, using Sobolev space theory and block operator matrix methods [33, 34].

- By using this theory, I generalized some of the spectral perturbation theorems of [35, 36] on linear periodic Hamiltonian equations to linear periodic (index-1) DAEs depending holomorphically on a spectral parameter.

I will describe the highlights of my results from [7] after the next section. In what follows, I briefly describe the type of periodic DAEs I studied as a model. This model is general enough to derive rigorously the results for electromagnetic slow-wave propagation through one-dimensional photonic crystals whose constituent layers include lossless anisotropic, bianisotropic, and/or frequency dependent materials.

4.2.1 The Periodic Differential-Algebraic Equations

The equations I consider, called *implicit canonical equations* by S. G. Krein [37], are the first-order linear periodic differential-algebraic equations

\[ \mathcal{G} y'(x) = H(x, \omega) y(x), \]  
where \( \det(\mathcal{G}) = 0 \), \( G := i\mathcal{G} \) is a constant \( N \times N \) Hermitian matrix, and the \( d \)-periodic Hamiltonian \( H \) belongs to \( \mathcal{O}(\Omega, M_N(L^2(0, d))) \), the set of all holomorphic matrix-valued functions of the spectral parameter \( \omega \) in the frequency domain \( \Omega \) with square integrable functions on \((0, d)\) as matrix entries. We assume the real frequency domain \( \Omega_R = \Omega \cap \mathbb{R} \) is nonempty and the matrix \( H(x, \omega) \) is Hermitian for each fixed \( \omega \in \Omega_R \) and for almost every (a.e.) \( x \) in \( \mathbb{R} \).

The domain and definition of the DAEs in (55) are given by

\[ \mathcal{D} = \{ y \in (L^2_{\text{loc}}(\mathbb{R}))^N : (I - P)y \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^N \}, \]
\[ \mathcal{G} y' := \mathcal{G} \frac{d}{dx}((I - P)y). \]  
(56)

where \( P \) denotes the unique projection onto the kernel of \( G \).

The DAEs given by (55) and (56) are not ODEs. They have to satisfy an algebraic equation as well, i.e., \( PHy = 0 \), and hence do not always have solutions. Thus in order to guarantee that they do have solutions we consider an additional hypothesis, namely

\[ (G + PHP)^{-1} \in \mathcal{O}(\Omega, M_N(L^\infty(0, d))). \]  
(57)

Differential-algebraic equations satisfying the conditions of (55)–(57) are called *index-1 DAEs*.

In this setting, Bloch waves are solutions of these periodic DAEs which satisfy

\[ y(x + d) = e^{ikd} y(x) \]
Theorem 14 (Reduced ODEs). Let $n$ be the dimension of the range of $G$. Then there exists a constant invertible hermitian $n \times n$ matrix $J$, $A \in \mathcal{O}(\Omega, M_n(L^1(0, d)))$, $Q \in \mathcal{O}(\Omega, M_n(W^{1, 1}(0, d)))$, and $\Psi \in \mathcal{O}(\Omega, M_n(W^{1, 1}(0, d)))$ with $A(x + d, \omega) = A(x, \omega)$, $Q(x + d, \omega) = Q(x, \omega)$, $\Psi(x + d, \omega) = \Psi(x, \omega)\Psi(d, \omega)$, such that if $\gamma \in \mathbb{C}^n$ then $y = Q\Psi \gamma$ is a solution of index-1 DAEs given in (55) where $\Psi(0, \omega) = I_n$ and $\Psi(\cdot, \omega)$ is a fundamental matrix of the periodic ODEs (the transfer matrix of the reduced ODEs)

$$i^{-1}J \frac{d\psi}{dx} = A(\cdot, \omega)\psi, \quad \psi \in (W^{1, 1}_{loc}(\mathbb{R}))^n.$$

Conversely, if $y$ is a solution of index-1 DAEs given in (55) then there exists a unique $\gamma \in \mathbb{C}^n$ such that $y = Q\Psi \gamma$. Moreover, $A(x, \omega)$ is a Hermitian matrix for each fixed $\omega \in \Omega$ and for a.e. $x$ in $\mathbb{R}$.

Theorem 15 (Floquet theory for DAEs). If $\gamma$ is an eigenvector of the monodromy matrix $\Psi(d, \omega)$ (the unit cell transfer matrix) with corresponding eigenvalue $\lambda$ then $y = Q\Psi \gamma$ is a nontrivial Bloch wave of the index-1 DAEs given in (55) with frequency $\omega$ and Floquet multiplier $\lambda$. Conversely, if $y$ is a nontrivial Bloch wave of the index-1 DAEs given in (55) with frequency $\omega$ and Floquet multiplier $\lambda$ then $y = Q\Psi \gamma$ with $\gamma$ an eigenvector of the monodromy matrix $\Psi(d, \omega)$ with corresponding eigenvalue $\lambda$. Moreover, a (non-Bloch) Floquet wave exists with Floquet multiplier $\lambda$ and frequency $\omega$ if and only if the Jordan normal form corresponding to the eigenvalue $\lambda$ of the monodromy matrix $\Psi(d, \omega)$ has a Jordan block of dimension $\geq 2$.

Theorem 16 (Bloch variety). The Bloch variety $\mathcal{B}$ is the set of zeros of the analytic function

$$F(k, \omega) = \det(e^{ikd}I_n - \Psi(d, \omega)), \quad (k, \omega) \in \mathbb{C} \times \Omega.$$
4. What is the connection between stationary points on the dispersion relation and the Jordan normal form of the monodromy matrix?

5. How does the frequency perturbed spectra of the monodromy matrix correspond to its unperturbed spectra?

6. In terms of the energy density and the Jordan structure of the monodromy matrix, what is the “physical” meaning of the generic condition \( \frac{\partial}{\partial \omega} \det(\lambda I_n - \Psi(d, \omega))|_{(\omega, \lambda) = (\omega_0, \lambda_0)} \neq 0 \) (from my paper [8] and described in Section 4.1 of this research statement) when \( \lambda_0 = e^{ik_0d} \) is a Floquet multiplier (i.e., \( \frac{\partial F}{\partial \omega} \neq 0 \) at \((k_0, \omega_0)\))?

In order to answer these questions we make the following definition, which I prove is well-defined in [7], and then give as theorems our main results from my thesis, namely, [7, Theorem 50, §4.3.1, p. 108], [7, Theorem 47, §4.2.3, p. 105], [7, Theorem 48, §4.2.4, p. 105], and [7, Corollary 49, §4.2.4, p. 106].

**Definition 17.** We say that the index-1 DAEs in (55) are of definite type at \((k_0, \omega_0) \in \mathcal{B} \cap \mathbb{R}^2\) provided the averaged energy density of any nontrivial Bloch wave \( y \) with wavenumber \( k_0 \) and frequency \( \omega_0 \) is nonzero, i.e.,

\[
\frac{1}{d} \int_0^d \langle H(x, \omega_0)y(x), y(x) \rangle dx \neq 0.
\]

**Theorem 18** (local band structure and Jordan structure). Suppose the index-1 DAEs in (55) are of definite type at \((k_0, \omega_0) \in \mathcal{B} \cap \mathbb{R}^2\). Let \( g \) be the number of Jordan blocks (geometric multiplicity) in the Jordan form of the monodromy matrix \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( \lambda_0 := e^{ik_0d} \) and \( m_1 \geq \cdots \geq m_g \geq 1 \) the dimensions of each of those Jordan blocks (partial multiplicities). Define \( m = m_1 + \cdots + m_g \) (algebraic multiplicity). Then

i. The order of the zero of \( F(k_0, \omega) \) at \( \omega_0 \) is \( g \) and the order of the zero of \( F(k, \omega_0) \) at \( k_0 \) is \( m \).

ii. The Bloch variety \( \mathcal{B} \) is locally the graph of a (multivalued) analytic function \( \omega = \omega(k) \) (the dispersion relation) whose branches are given by \( g \) (counting multiplicities) single-valued nonconstant real analytic functions \( \omega_1(k), \ldots, \omega_g(k) \), with \( \omega_j(k_0) = \omega_0 \) (the band functions).

iii. The number \( m_j \) is the order of the zero of \( \omega_j(k) - \omega_0 \) at \( k_0 \), for \( j = 1, \ldots, g \).

iv. All points \((k, \omega) \in \mathcal{B} \cap \mathbb{R}^2\) in a sufficiently small neighborhood of \((k_0, \omega_0)\) are of definite type. In particular, this theorem is true for these points as well.

**Theorem 19** (Interpretation of the generic condition). Suppose that \((k_0, \omega_0) \in \mathcal{B} \cap \mathbb{R}^2\). Then the generic condition is true, i.e.,

\[
\frac{\partial}{\partial \omega} \det(\lambda I_n - \Psi(d, \omega))|_{(\omega, \lambda) = (\omega_0, \lambda_0)} \neq 0.
\]

if and only if the index-1 DAEs in (55) are of definite type at \((k_0, \omega_0)\) and the Jordan normal form for the eigenvalue \( \lambda_0 = e^{ik_0d} \) (Floquet multiplier) of the monodromy matrix \( \Psi(d, \omega_0) \) is a single Jordan block (i.e., \( g = 1 \) in Theorem 18).

In which case, the results of my paper [8] apply where I give explicit recursive formulas to compute all the Puiseux series coefficients for the perturbed eigenvalues which split from \( \lambda_0 \) and their eigenvectors of the monodromy matrix \( \Psi(d, \omega) \).

**References**


