How Robust are Thresholds for Community Detection?

Alex Wein (MIT)

Joint with Ankur Moitra (MIT) and Amelia Perry (MIT)
Two Worldviews

Convex Optimization | Statistical Physics
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Convex Optimization

- Algorithm: semidefinite programming (SDP)
  - Most powerful known algorithm for various worst-case and average-case problems

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- Reasoning about hardness of problems
  - Integrality gaps
  - Extension complexity
  - Sum-of-squares lower bounds

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Do these frameworks always agree?
And if they don’t agree, which one is correct?
Stochastic Block Model (SBM)
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Model for community detection in graphs
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Studied in statistics, information theory, computer science, statistical physics, ...
Sharp Threshold Behavior

\( n \) -- num vertices
\( p \) -- within-community edge prob
\( q \) -- between-community edge prob
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Average degree: \(O(\log n)\)
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**Theorem** [Abbe-Bandeira-Hall ’14, Mossel-Neeman-Sly ‘14]:
Possible to achieve **exact recovery** if
\[
\sqrt{a} - \sqrt{b} \geq \sqrt{2}
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\(^*\) recover communities exactly, with probability \( \rightarrow 1 \) as \( n \rightarrow \infty \)
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Can SDPs reach the threshold in the sparse regime, or are they suboptimal?

Answer: We will give evidence that SDPs cannot reach the threshold! — but only because they are actually solving a harder problem.
Semirandom Models
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Between average-case and worst-case
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2. An adversary can perform any number of \textit{monotone} (‘helpful’) changes:
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Prevents algorithms from over-tuning to specific model statistics (degree distribution, spectrum, etc.)
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Captures some notion of ‘robustness’
A Non-robust Algorithm

Example: \( p = \frac{1}{2}, \, q = \frac{1}{4}, \, n \to \infty, \) exact recovery
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Easy algorithm for the random model: count common neighbors
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Easy algorithm for the **random** model: count common neighbors

- 2 same-side vertices have $\approx \frac{5}{32}n$ common neighbors
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The vast majority of algorithms fail against the semirandom model!
Robust Algorithms

Monotone-robust algorithm: succeeds against the semirandom model
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For *exact recovery*: SDP is robust up to the threshold [Feige–Kilian ’00, Hajek–Wu–Xu ’15]
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For partial recovery: harder…
- In random model, SDP works when $(a - b)^2 > C(a + b)$ [Guédon–Vershynin ‘15]
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- SDP is robust under same condition [Moitra–Perry–W ’15, Makarychev–Makarychev–Vijayaraghavan ’15]
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- SDP is **robust** under same condition [Moitra–Perry–W ’15, Makarychev–Makarychev–Vijayaraghavan ’15]
- **Open**: Can [Montanari–Sen ’15] analysis be made robust?
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*Random:* impossible iff \((a - b)^2 \leq 2(a + b)\)

*Semirandom:* impossible if \((a - b)^2 \leq C_a(a + b)\)

where \(C_a > 2\) for all \(a > 2\)
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- First random-to-semirandom gap
- Gap only exists for partial recovery
Can SDPs reach the threshold?
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- No proof that if SDP succeeds in random model, then it is robust (i.e., succeeds in the semirandom model for the same range of parameters a,b).
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Additional evidence: statistical physics predicts (non-rigorous) that SDP misses the threshold [JMR’15].
Proof Idea: How can ‘helpful’ changes hurt?
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Our adversary: look for degree-2 nodes with 2 opposite-side neighbors; cut both edges
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Sparse graph: this occurs often
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Interpretation: algorithms reaching the threshold (e.g. linearized belief propagation) rely on the distribution of these structures in the noise
Proof Idea

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Use connection to broadcast tree model
Broadcast Tree Model
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2 colors: red, blue (corresponding to 2 communities)
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Recursively, each node gives birth to:
- Pois(a/2) nodes of same color, and
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Answer: when $(a - b)^2 > 2(a + b)$  Look familiar?  
[Kesten-Stigum ’66, Evans-Kenyon-Peres-Schulman ’00]
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Thanks! Questions?