Estimated transversality in symplectic geometry and projective maps

Denis AUROUX

Ample bundles over almost-complex manifolds

GW invariants: holomorphic maps from complex manifolds to a symplectic manifold

Dual point of view: (approx.) holomorphic maps from a symplectic manifold to complex manifolds (Donaldson)

Tool: estimated transversality for approx. holomorphic sections of very ample bundles

($\Rightarrow$ good linear systems, maps to $\mathbb{CP}^m$)

(X^{2n}, J) almost-complex, compact
(L_k, ∇_k) line bundles are asympt. very ample if

$$\text{curvature} \begin{cases} iF_k(v, Jv) > c_k|v|^2, \quad c_k \longrightarrow +\infty \\ F_k^{(0,2)} = O(1) \end{cases}$$

$\omega_k = iF_k$ is symplectic, J is ω_k-tame.

Example: $c_1(L_k) = k[\omega]$ and J is ω-compatible.

Asympt. holomorphic sections of L_k :

$$\begin{cases} |s_k|_{C^r,g_k} = O(1) \qquad (\text{rescaling}: \;\; g_k = c_k\, g) \\ |\bar{\partial} s_k|_{C^r,g_k} = O(c_k^{-1/2}) \end{cases}$$

curvature $\longrightarrow +\infty \Rightarrow$ look into X at small scale
$\Rightarrow$ non-integrability $\longrightarrow 0$.

Estimated transversality of jets

Asympt. holomorphic sections $s_{k,0}, \dots, s_{k,m} \in \Gamma(L_k)$
$\Rightarrow$ approx. holomorphic maps $f_k : X \to \mathbb{CP}^m$

Need estimated transversality for the jets of these maps.

$E_k = \mathbb{C}^{m+1} \otimes L_k$ asympt. very ample vector bundles, holom. jet bundles $\mathcal{J}^r E_k = \bigoplus_{j=0}^{r} \big(T^*X^{(1,0)}\big)^{\otimes j}_{\mathrm{sym}} \otimes E_k$.

$\mathcal{S}_k$ = asympt. holomorphic stratifications of $\mathcal{J}^r E_k$: finite Whitney stratifications, transverse to the fibers ; all strata are asympt. holomorphic submanifolds, with bounded curvature away from lower-dimensional strata.

The jet $j^r s_k$ is η-transverse to $\mathcal{S}_k$ if $\mathrm{dist}(j^r s_k(x), S_{k,a}) < \eta \Rightarrow$ the graph of $j^r s_k$ is transverse at x to $TS_{k,a}$, with minimum angle $> \eta$.

Theorem 1

$\mathcal{S}_k$ asympt. holomorphic stratifications of $\mathcal{J}^r E_k$; $\delta > 0$; s_k asympt. holomorphic sections of E_k $\Rightarrow$ for large enough k, $\exists$ asympt. holomorphic sections σ_k of E_k s.t.

(1) $|\sigma_k - s_k|_{C^{r+1},g_k} < \delta$;

(2) $j^r \sigma_k$ *is* $\eta_{(\delta)}$*-transverse to* $\mathcal{S}_k$.

Estimated transversality of jets

Ingredients of proof :

- transversality is an open property
  $\Rightarrow$ transv. to all strata by successive perturbations
- start with lowest dim. strata ; $\mathcal{S}_k$ are Whitney
  $\Rightarrow$ only work away from lower-dim. strata
- very localized asympt. holomorphic sections of L_k
  in coords.: $s_{k,x,I}(z) = z_1^{i_1} \dots z_n^{i_n} \exp(-\frac{1}{4}c_k|z|^2)$
  $\Rightarrow$ local trivializations of $\mathcal{J}^r E_k$
- local transversality result for functions $\mathbb{C}^n \to \mathbb{C}^p$ (Donaldson)
  $\Rightarrow$ a localized small perturbation of s_k yields estimated transversality to $S_{k,a}$ over a small ball
- globalization argument
  $\Rightarrow$ using openness, combine local perturbations to obtain transversality everywhere

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Theorem 1 also holds for families indexed by $t \in [0,1]$
  $\Rightarrow$ objects are canonical up to isotopy

(and even independent of the chosen J as long as L_k remain asympt. very ample)

Boardman stratifications of holomorphic jet spaces

Boardman stratification of jets of holomorphic maps $\mathbb{C}^n \longrightarrow \mathbb{C}^m$:

$f : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ holomorphic $\Rightarrow$ singular loci

$\Sigma_i(f) = \{x,\ \dim \operatorname{Ker} df(x) = i\}$

$\Sigma_{i_1,\dots,i_r}(f) = \Sigma_{i_r}(f_{|\Sigma_{i_1,\dots,i_{r-1}}(f)})$

$\Rightarrow$ stratification of $\mathcal{J}^r(\mathbb{C}^n, \mathbb{C}^m)$ by Σ_I.

Sections s_k of $E_k = \mathbb{C}^{m+1} \otimes L_k$

$\Rightarrow f_k = \mathbb{P}s_k : X - s_k^{-1}(0) \longrightarrow \mathbb{CP}^m$

$j^r s_k = (s_k, \partial s_k, \partial\partial s_k, \dots)$; use local approx. holom. coordinates to identify $\mathcal{J}^r(X, \mathbb{CP}^m)$ with $\mathcal{J}^r(\mathbb{C}^n, \mathbb{C}^m)$

$\Rightarrow$ Boardman stratification of $\mathcal{J}^r E_k$:

- $S_0 = \{j^r s(x),\ s(x) = 0\}$
- $S_I = \{j^r s(x),\ s(x) \neq 0,\ j^r \mathbb{P}s(x) \in \Sigma_I\}$

These stratifications are asympt. holomorphic

$\Rightarrow$ by Theorem 1, for large k we get $s_k \in \Gamma(E_k)$ s.t. $j^r s_k$ uniformly transverse to Boardman stratifications.

Generic projective maps

s_k asympt. holomorphic sections of $\mathbb{C}^{m+1} \otimes L_k$,
$j^r s_k$ uniformly transverse to Boardman stratifications :

• the base loci $Z_k = s_k^{-1}(0)$ are smooth symplectic codim. $2m+2$ submanifolds.

Local model: $f_k(z_1, \dots, z_n) = (z_1 : z_2 : \dots : z_{m+1})$

• the holomorphic r-jets of $f_k = \mathbb{P}s_k$ behave similarly to those of generic holomorphic maps between complex manifolds

• singular loci $\Sigma_I(f_k)$ = stratified symplectic submanifolds of $X - Z_k$, of the expected codimension

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Away from singular loci, estimated transversality + asympt. holomorphicity $\Rightarrow \bar\partial f_k \ll \partial f_k \Rightarrow$ holomorphic local models for f_k

Near $\Sigma_I(f_k)$, need to ensure $\bar\partial f_k \ll \partial f_k \Rightarrow$ obtain some control over $\bar\partial f_k$.

Idea: the antiholomorphic part of the jet of f_k should vanish in the normal directions to $\Sigma_I(f_k)$.

Generic projective maps

Suitable perturbation to kill the antiholomorphic jet of f_k along normal directions to singular loci

$\Rightarrow$ obtain approx. holomorphic projective maps, topologically conjugate near every point of X to generic holomorphic maps between complex manifolds
(in local approx. holomorphic coordinates).

- $m = 1$: symplectic Lefschetz pencils (Donaldson)
- $m = 2$: maps to $\mathbb{CP}^2$ (D. A.)
- $m \geq 2n$: projective immersions/embeddings (Muñoz-Presas-Sols)
- general case : in progress

Symplectic Lefschetz pencils

$(s_0, s_1) \in \Gamma(\mathbb{C}^2 \otimes L_k)$ suitably chosen
$\Rightarrow$ symplectic Lefschetz pencil :

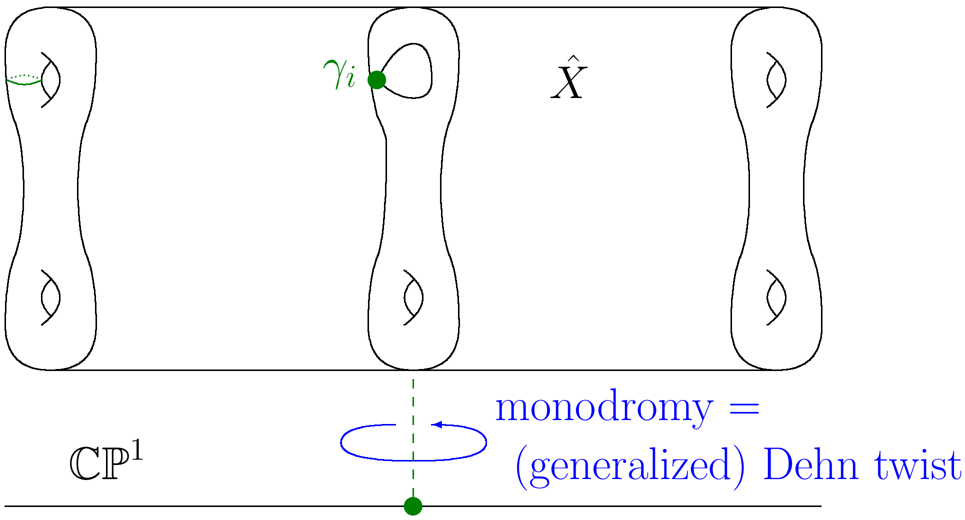
$$\Sigma_\alpha = \{x \in X,\ s_0 + \alpha s_1 = 0\} \quad (\alpha \in \mathbb{CP}^1)$$

symplectic hypersurfaces, smooth except for finitely many singular points.

Base locus $Z = \{s_0 = s_1 = 0\}$ (codim. 4).

Projective map $f = (s_0 : s_1) : X - Z \to \mathbb{CP}^1$:
local model $f(z) = z_1^2 + \cdots + z_n^2$ near critical points.

Blow up $Z \Rightarrow$ Lefschetz fibration $\hat{X} \to \mathbb{CP}^1$



Monodromy = $\theta : \pi_1(\mathbb{C} - \{\text{pts}\}) \to \text{Map}^\omega(\Sigma^{2n-2}, Z)$

$\text{Map}^\omega(\Sigma, Z) := \pi_0(\{\phi \in \text{Symp}(\Sigma, \omega), \phi_{|U(Z)} = \text{Id}\})$

$\Rightarrow$ symplectic invariants.

Symplectic maps to $\mathbb{CP}^2$

$(s_0, s_1, s_2) \in \Gamma(\mathbb{C}^3 \otimes L_k)$ suitably chosen
$\Rightarrow f = (s_0 : s_1 : s_2) : X - Z \to \mathbb{CP}^2$.
Fibers = codimension 4 symplectic submanifolds,
intersecting at the base locus Z (codim. 6),
singular along a smooth symplectic curve $R \subset X$.

Local singular models near R :

1. $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$
   points where R is transverse to the fibers of f
2. $(z_1, \dots, z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$
   cusp points of $D = f(R)$

The critical curve $D = f(R)$ is symplectic, with nodes (both orientations) and cusp singularities.

Fiber above a smooth point of D = obtained by collapsing a vanishing cycle (Lagrangian S^{n-2}) in the generic fiber Σ^{2n-4}.

Monodromy around D :

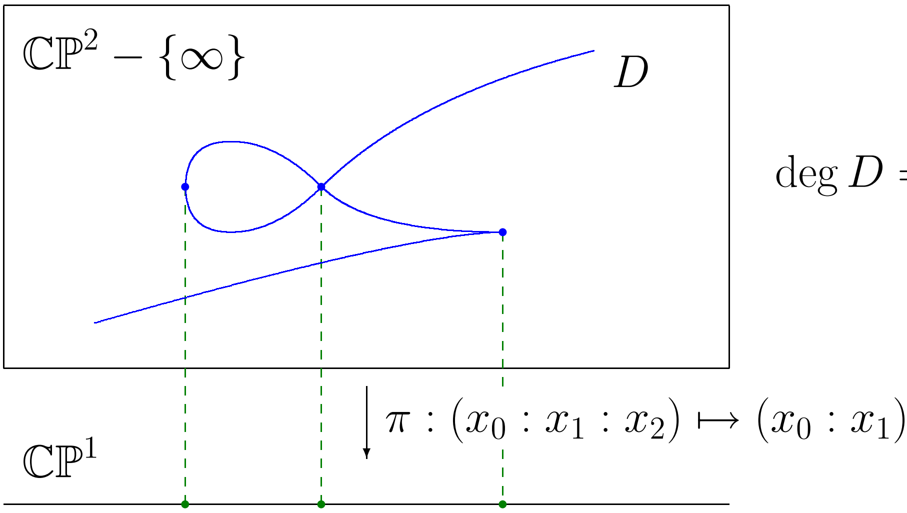
$$\bar{\theta} : \pi_1(\mathbb{C}^2 - D) \to \mathrm{Map}^{\omega}(\Sigma^{2n-4}, Z)$$

$\bar{\theta}$(geometric generator) = generalized Dehn twist.

Up to cancellation of nodes in D, for $k \gg 0$ the topology of f_k is a symplectic invariant.

Monodromy and braid groups

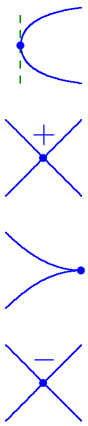
Perturbation $\Rightarrow D =$ singular branched cover of $\mathbb{CP}^1$.



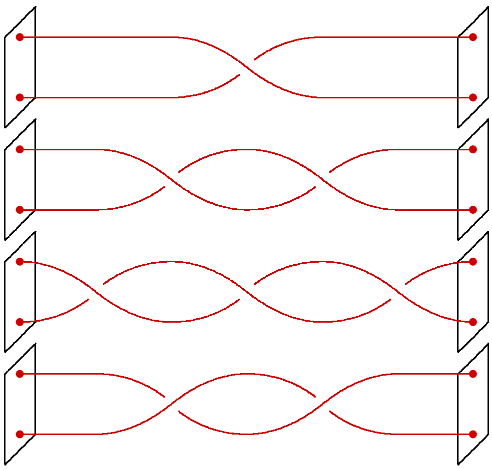
$\deg D = d$

Monodromy $= \rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (braid group)

Monodromy around each crit. point $=$ (half-twist)$^\delta$,
$\delta \in \{-2, 1, 2, 3\}$:



- $\delta = 1$
- $\delta = 2$
- $\delta = 3$
- $\delta = -2$



(Moishezon-Teicher, Auroux-Katzarkov)

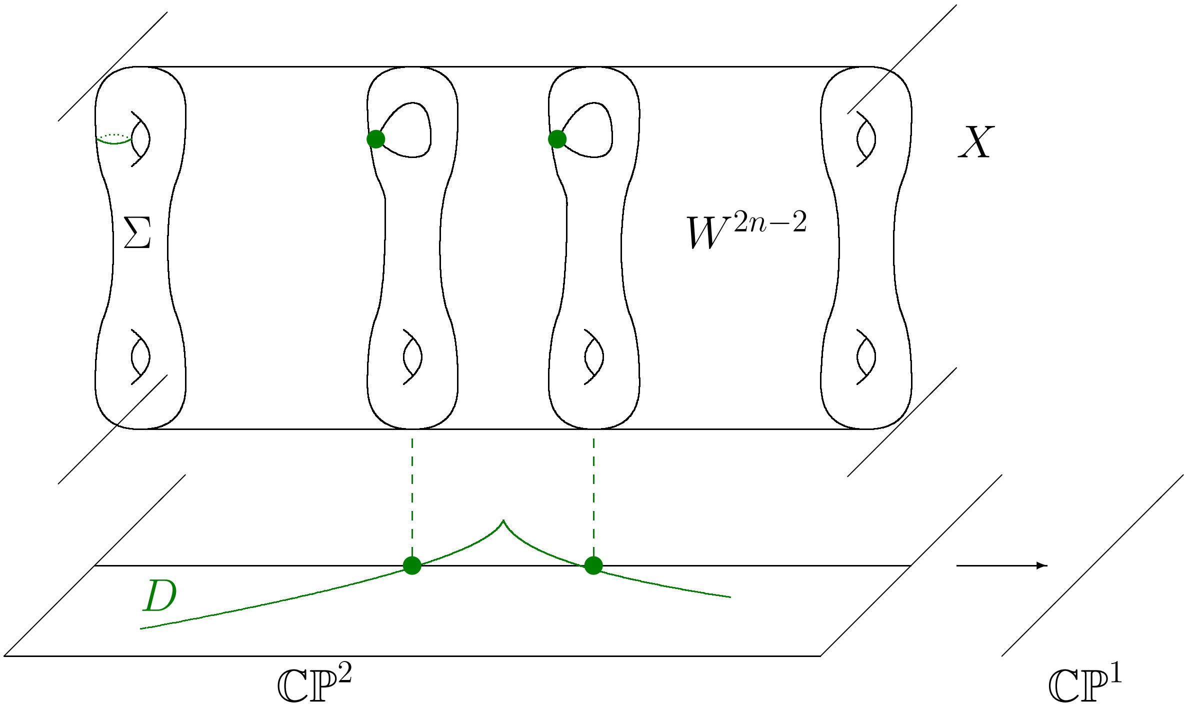
The higher dimensional case

Monodromies of $f : X - Z \to \mathbb{CP}^2$:

- $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (describes D)
- $\bar{\theta} : \pi_1(\mathbb{C}^2 - D) \to \text{Map}^\omega(\Sigma^{2n-4}, Z)$ (describes f)

Restricting to the hypersurface $W^{2n-2} = s_2^{-1}(0)$, $f_{|W} = (s_0 : s_1) : W - Z \to \mathbb{CP}^1$ is a symplectic Lefschetz pencil, monodromy

$$\theta = \bar{\theta} \circ i_* : \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to \text{Map}^\omega(\Sigma^{2n-4}, Z)$$



The monodromy invariants (ρ, θ) determine the manifold X up to symplectomorphism.

Dimensional induction

(X^{2n}, ω) symplectic, $s_0, \dots, s_n \in \Gamma(L_k)$ well-chosen.

• $\Sigma_r = \{s_{r+1} = \dots = s_n = 0\}$ smooth symplectic submanifold, $\dim \Sigma_r = 2r$, $\Sigma_n = X$.

• s_{r-1} and s_r define a SLP on Σ_r, generic fiber Σ_{r-1}, base locus Σ_{r-2}. Monodromy :

$$\theta_r : \pi_1(\mathbb{C} - \{pts\}) \longrightarrow \mathrm{Map}^{\omega}(\Sigma_{r-1}, \Sigma_{r-2})$$

• $(s_{r-2} : s_{r-1} : s_r) : \Sigma_r - \Sigma_{r-3} \longrightarrow \mathbb{CP}^2$, singular locus $D_r \subset \mathbb{CP}^2$, $\deg D_r = d_{r-1}$. Monodromy :

$$\rho_r : \pi_1(\mathbb{C} - \{pts\}) \to B_{d_{r-1}} \qquad \text{and} \quad \theta_{r-1}$$

• ρ_r and θ_{r-1} (description of Σ_r by a map to $\mathbb{CP}^2$) determine θ_r (description of Σ_r by a SLP) explicitly.

Symplectic invariants characterizing X :

$(\theta_r, \rho_{r+1}, \rho_{r+2}, \dots, \rho_n)$, $\forall 1 \le r \le n$.

• $r = n$: SLP

• $r = 2$: $n-2$ braid factorizations + word in $\mathrm{Map}_{g,N}$

• $r = 1$: $n-1$ braid factorizations + word in S_N.