Singular plane curves and symplectic 4-manifolds

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Symplectic manifolds

A symplectic structure on a smooth manifold is a 2-form ω such that $d\omega = 0$ and $\omega \wedge \cdots \wedge \omega$ is a volume form.

Example: $\mathbb{R}^{2n}$, $\omega_0 = \sum dx_i \wedge dy_i$.

(Darboux: every symplectic manifold is locally $\simeq (\mathbb{R}^{2n}, \omega_0)$, i.e. there are no local invariants).

Example: Riemann surfaces (Σ, vol_Σ) are symplectic.

Example: Every Kähler manifold is symplectic.
(includes all complex projective manifolds)

but the symplectic category is much larger.
(Gompf 1994: $\forall G$ finitely presented group, $\exists (X^4, \omega)$ compact symplectic such that $\pi_1(X) = G$).

Symplectic manifolds are not always complex, but they are almost-complex, i.e. there exists $J \in \mathrm{End}(TX)$ such that

$$J^2 = -\mathrm{Id}, \quad g(u,v) := \omega(u, Jv) \text{ Riemannian metric.}$$

At any given point (X, ω, J) looks like $(\mathbb{C}^n, \omega_0, i)$, but J is not integrable ($\nabla J \neq 0$; $\bar\partial^2 \neq 0$). So there are no holomorphic functions (in particular no holomorphic local coordinates).

Symplectic topology

Typical problems:

– Which smooth manifolds admit symplectic structures ?

– Classify symplectic structures on a given smooth manifold.

(Moser: if $[\omega] \in H^2(X, \mathbb{R})$ is fixed then all small deformations are trivial).

Why we care:

– Physics (classical mechanics; string theory; ...)

– Next step after understanding complex manifolds.

Some facts from complex geometry extend to symplectic manifolds; most don't.

A lot is known if $\dim X = 4$. Core ingredient: structure of Seiberg-Witten / Gromov-Witten invariants of symplectic 4-manifolds (Taubes).

For $\dim X \geq 6$, almost nothing is known. E.g., no known non-trivial obstruction to the symplecticity of compact 6-manifolds (except $\exists[\omega] \in H^2(X, \mathbb{R})$ s.t. $[\omega]^{\wedge 3} \neq 0$).

Approximately holomorphic geometry

Idea:

Since we have almost-complex structures, even though there are no holomorphic sections and linear systems, we can work similarly with approximately holomorphic objects.

(Donaldson, ∼1995)

Setup: (X^{2n}, ω) symplectic, compact

- $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$ (not restrictive)
- J compatible with ω ; $g(.,.) = \omega(., J.)$
- L line bundle such that $c_1(L) = \frac{1}{2\pi}[\omega]$
- ∇^L, with curvature $-i\omega$; $\nabla^L = \partial^L + \bar{\partial}^L$.
$$\bar{\partial}^L s(v) = \tfrac{1}{2}(\nabla^L s(v) + i\nabla^L s(Jv)).$$

If X Kähler, then L is a holomorphic ample line bundle, i.e. $L^{\otimes k}$ has many holomorphic sections for k large enough.

⇒ projective embeddings $X \hookrightarrow \mathbb{CP}^N$ (Kodaira).

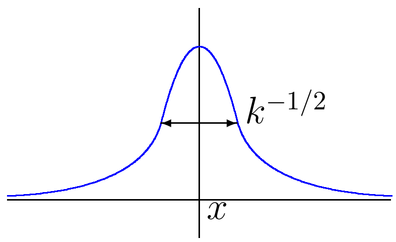
⇒ smooth hypersurfaces (Bertini).

⇒ linear systems, projective maps.

Approximately holomorphic sections

X symplectic: J is not integrable $\Rightarrow$ no holomorphic sections.
However, local approximately holomorphic model:

$$(X,x),\ \omega,\ J \longleftrightarrow (\mathbb{C}^n,0),\ \omega_0,\ (i+\dots)$$
$$L^{\otimes k},\ \nabla \longleftrightarrow \underline{\mathbb{C}},\ d+\tfrac{k}{4}\sum(z_j d\bar z_j - \bar z_j dz_j).$$



$\Rightarrow$ $s_{k,x}(z) = \exp(-\frac{1}{4}k|z|^2)$ is approx. holomorphic !

A sequence of sections $s_k \in \Gamma(L^{\otimes k})$ is approx. holomorphic if $\sup|\bar\partial s_k| < C\,k^{-1/2}\sup|\partial s_k|$ (& higher order derivatives).

Goal: find approx. holom. sections with “generic” behavior.

Theorem 1. (Donaldson, 1996) *If $k \gg 0$, then $L^{\otimes k}$ admits approx. holomorphic sections s_k whose zero sets W_k are smooth symplectic hypersurfaces.*

Make up for loss of holomorphicity by achieving estimated transversality: require $|\partial s_k(x)| \gg \sup|\bar\partial s_k|$ along $s_k^{-1}(0)$.
(uniform lower bound instead of just $\partial s_k(x)\neq 0$)

Also consider linear systems of ≥ 2 sections:

E.g., (s_0,s_1) well-chosen approx. hol. sections of $L^{\otimes k}$ $(k\gg 0)$
$\Rightarrow$ symplectic Lefschetz pencils (Donaldson, 1999)

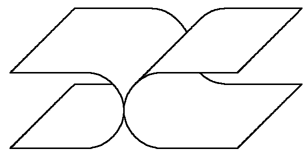
Branched covers of $\mathbb{CP}^2$

Theorem 2. (A., 2000) *For $k \gg 0$, three suitable approx. hol. sections of $L^{\otimes k}$ define a map $X \to \mathbb{CP}^2$ with generic local models, canonical up to isotopy.*

(X^4, ω) symplectic, $s_0, s_1, s_2 \in \Gamma(L^{\otimes k})$ well-chosen
$\Rightarrow f = (s_0 : s_1 : s_2) : X \to \mathbb{CP}^2$.

Local models near branch curve $R \subset X$:

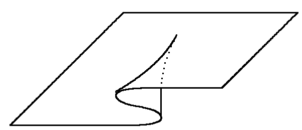
– branched cover : $(x, y) \mapsto (x^2, y)$.



$R:\ x = 0$ $\qquad f(R):\ X = 0$

$X^{2n} \to \mathbb{CP}^2$: $(z_1, \dots, z_n) \mapsto (z_1^2 + \cdots + z_{n-1}^2, z_n)$

– cusp : $(x, y) \mapsto (x^3 - xy, y)$.



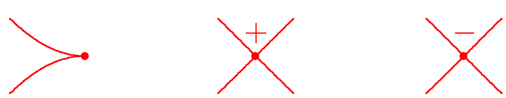
$R:\ y = 3x^2$ $\qquad f(R):\ 27X^2 = 4Y^3$

$X^{2n} \to \mathbb{CP}^2$: $(z_1, \dots, z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \cdots + z_{n-1}^2, z_n)$

R smooth connected symplectic curve in X.
$D = f(R)$ symplectic, immersed except at the cusps.

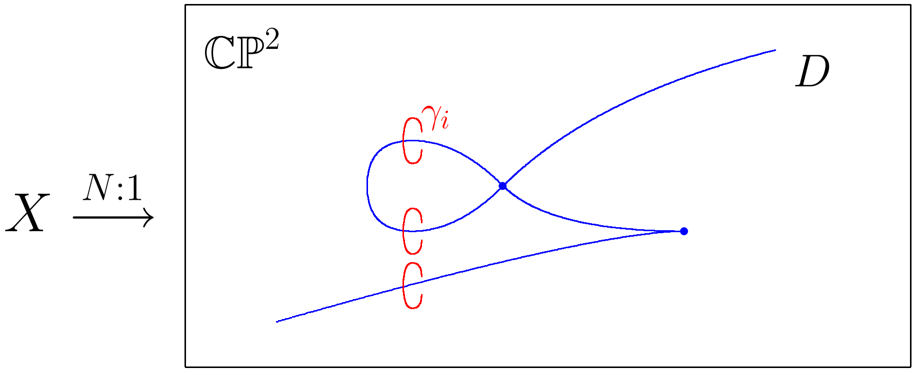
Generic singularities :

complex cusps; nodes (both orientations)



Theorem 2 $\Rightarrow$ up to cancellation of nodes, the topology of D is a symplectic invariant (if k large).

Topological invariants



Topological data for a branched cover of $\mathbb{CP}^2$:

1) Branch curve: $D \subset \mathbb{CP}^2$
(up to isotopy and node cancellations).

2) Monodromy: $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_N \quad (N = \deg f)$
(surjective, maps γ_i to transpositions).

D and θ determine (X, ω) up to symplectomorphism.

When $\dim X > 4$, main difference: θ takes values in the mapping class group of the generic fiber.

This group is complicated; however there is a dimensional induction procedure $\Rightarrow$ given (X^{2n}, ω) and $k \gg 0$ we get

1) $(n-1)$ plane curves $D_n, D_{n-1}, \ldots, D_2 \subset \mathbb{CP}^2$.
2) $\theta_2 : \pi_1(\mathbb{CP}^2 - D_2) \to S_N$.

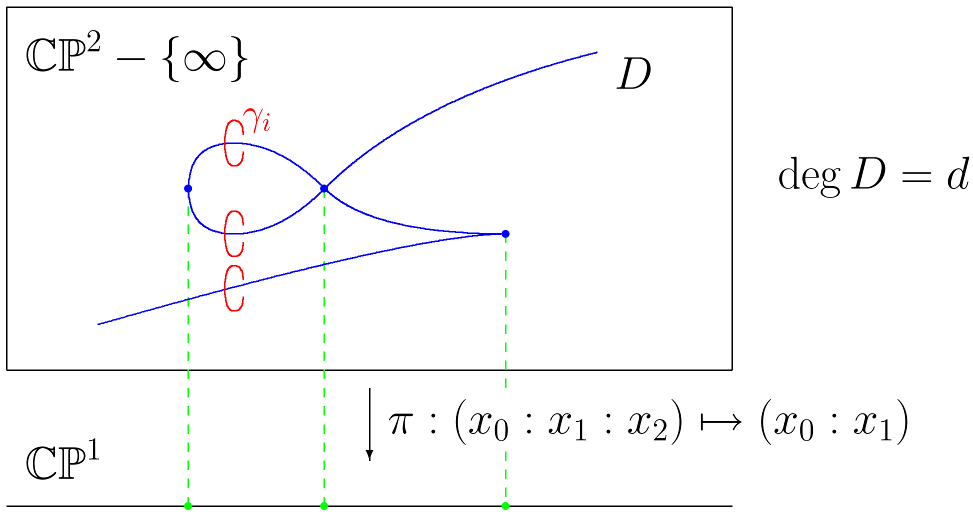
and these data determine (X, ω) up to symplectomorphism.

$\Rightarrow$ In principle it is enough to understand plane curves !

The topology of plane curves

(Moishezon-Teicher; Auroux-Katzarkov)

Perturbation $\Rightarrow$ D = singular branched cover of $\mathbb{CP}^1$.



Monodromy = $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (braid group)

$\Rightarrow$ D is described by a "braid group factorization" (involving cusps, nodes, tangencies).

The braid factorization characterizes D completely.

Problem: once computed, cannot be compared.

$\Rightarrow$ more manageable (incomplete) invariant ?

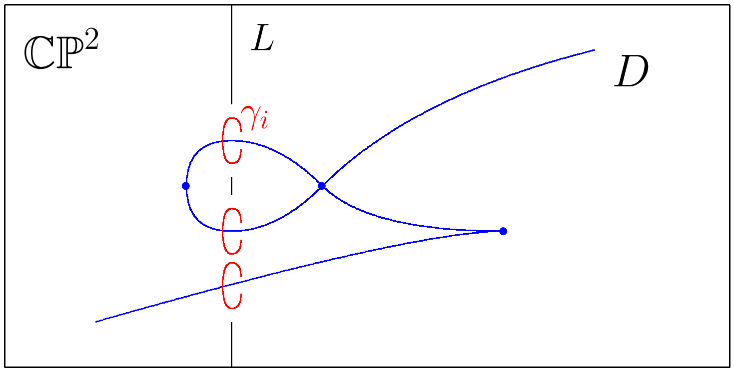
Moishezon-Teicher: $\pi_1(\mathbb{CP}^2 - D)$ to study complex surfaces.

$\pi_1(\mathbb{CP}^2 - D)$ is generated by "geometric generators" $(\gamma_i)_{1 \leq i \leq d}$; relations given by the braid factorization.

But: in the symplectic case, affected by node cancellations.

Stabilized fundamental groups

(Auroux-Donaldson-Katzarkov-Yotov: math.GT/0203183)

$X^4 \xrightarrow{f}$  $\deg D = d$

$L \simeq \mathbb{C} \subset \mathbb{CP}^2$ generic line, $i : L - \{p_1, \dots, p_d\} \hookrightarrow \mathbb{CP}^2 - D$
$\Rightarrow i_* : F_d = \langle \gamma_1, \dots, \gamma_d \rangle \twoheadrightarrow \pi_1(\mathbb{CP}^2 - D)$ surjective.

Geometric generators: $\Gamma = \{\text{conjugates of } i_*\gamma_1, \dots, i_*\gamma_d\}$.

$\theta : \pi_1(\mathbb{CP}^2 - D) \longrightarrow S_N$ maps elements of Γ to transpositions.

$\delta : \pi_1(\mathbb{CP}^2 - D) \longrightarrow \mathbb{Z}_d$ linking number $(\delta(\gamma_i) = 1)$.

Relations: for each special point, two elements of Γ s.t.

- tangency: $\gamma = \gamma'$; $\theta(\gamma)$ and $\theta(\gamma')$ identical.
- node: $\gamma\gamma' = \gamma'\gamma$; $\theta(\gamma)$ and $\theta(\gamma')$ disjoint.
- cusp: $\gamma\gamma'\gamma = \gamma'\gamma\gamma'$; $\theta(\gamma)$ and $\theta(\gamma')$ adjacent.

$K =$ normal subgroup $\langle [\gamma, \gamma'], \gamma, \gamma' \in \Gamma, \theta(\gamma), \theta(\gamma') \text{ disjoint} \rangle$.

Add a pair of nodes $\Leftrightarrow$ quotient by an element of K.

Theorem 3. *For $k \gg 0$, $G_k(X, \omega) = \pi_1(\mathbb{CP}^2 - D_k)/K_k$ and $G_k^0(X, \omega) = \mathrm{Ker}\,(\theta_k, \delta_k)/K_k$ are symplectic invariants.*

Stabilized fundamental groups

Fact: $1 \longrightarrow G_k^0 \longrightarrow G_k \xrightarrow{(\theta_k,\delta_k)} S_N \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$

$(N = \deg f_k,\ d = \deg D_k)$

Theorem 4. *If* $\pi_1(X) = 1$ *then we have a natural surjection* $\phi_k : \mathrm{Ab}\, G_k^0 \to (\mathbb{Z}^2/\Lambda_k)^{N-1}$.

$$\Lambda_k = \{(L^{\otimes k} \cdot C, K_X \cdot C),\ C \in H_2(X, \mathbb{Z})\}.$$

Known examples:

- $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ (Moishezon)
- some rational and K3 complete intersections (Robb)
- Hirzebruch surfaces, double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ (ADKY)

$\Rightarrow$ **Conjectures:** for $k \gg 0$,

1) X alg. surface $\Rightarrow K_k = \{1\}$ and $G_k = \pi_1(\mathbb{CP}^2 - D_k)$.
2) $\pi_1(X) = 1 \Rightarrow \phi_k$ is an isomorphism.
3) $\pi_1(X) = 1 \Rightarrow [G_k^0, G_k^0] =$ quotient of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Still looking for how to extract useful invariants from braid factorization...

Non-isotopic singular plane curves

(Auroux-Donaldson-Katzarkov: math.GT/0206005)

Isotopy phenomena: (following Gromov, ...)

• Siebert-Tian (2002): every smooth symplectic curve of degree $\leq$ 17 in $\mathbb{CP}^2$ is isotopic to a complex curve. Also in $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ for connected curves s.t. $[C] \cdot [F] \leq 7$.

• Barraud (2000), Shevchishin (2002): isotopy for certain simple singular configurations in $\mathbb{CP}^2$.

Non-isotopy phenomena:

• Fintushel-Stern (1999), Smith (2001): infinitely many non-isotopic smooth connected symplectic curves in certain 4-manifolds (multiples of classes of square zero).

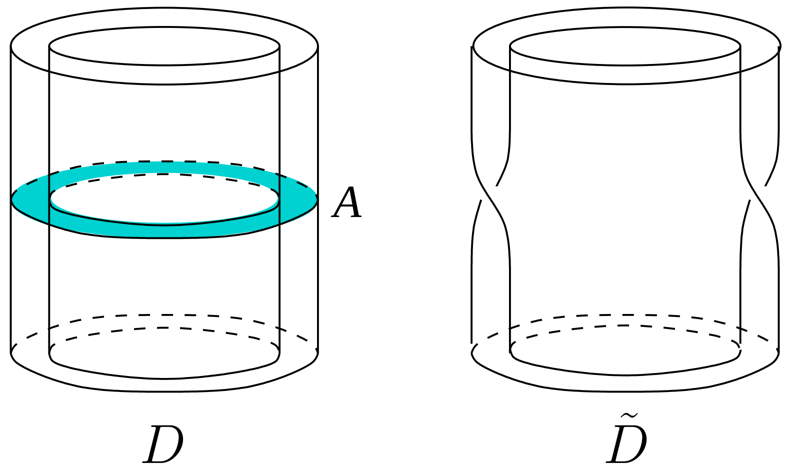
Use braiding constructions on parallel copies; distinguish using topology of branched covers (SW invariants, ...)

• Moishezon (1992): infinitely many non-isotopic singular symplectic curves in $\mathbb{CP}^2$ (fixed number of cusp and node singularities).

Use braid monodromy and π_1 of complement (hard!)

$\Rightarrow$ elementary interpretation of Moishezon ?
It is also a braiding construction !

Non-isotopic singular plane curves



Given $f: X \to Y$ symplectic covering with branch curve D,

Braiding D / Lagrangian annulus $A \Longleftrightarrow$
Luttinger surgery of X / Lagrangian torus $T \subset f^{-1}(A)$.

(i.e. take out a neighborhood of T and glue it back via a symplectomorphism wrapping the meridian around the torus).

Moishezon examples: $D_0 = 3p(p-1)$ smooth cubics in a pencil ($p \geq 2$), remove balls around 9 intersection points, insert branch curve of deg. p polynomial map $\mathbb{CP}^2 \to \mathbb{CP}^2$ in each location. D_j = twist j times in a well-chosen manner.

- Moishezon:
  Before twisting: $\pi_1(\mathbb{CP}^2 - D_0)$ is infinite.
  After twisting: $\pi_1(\mathbb{CP}^2 - D_j)$ finite, of different orders.

- Topological interpretation:
  Before twisting: $c_1(K_{X_0}) = \lambda[\omega_{X_0}]$.
  After twisting: $c_1(K_{X_j}) = \lambda[\omega_{X_j}] + \mu\, j\, [T]^{PD} \quad (\mu \neq 0)$.