Symplectic 4-manifolds, singular plane curves, and isotopy problems

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Symplectic manifolds

A symplectic structure on a smooth manifold is a 2-form ω such that $d\omega = 0$ and $\omega \wedge \cdots \wedge \omega$ is a volume form.

Example: $\mathbb{R}^{2n}$, $\omega_0 = \sum dx_i \wedge dy_i$.

(Darboux: every symplectic manifold is locally $\simeq (\mathbb{R}^{2n}, \omega_0)$, i.e. there are no local invariants).

Example: Riemann surfaces (Σ, vol_Σ); $\mathbb{CP}^n$; complex projective manifolds.

The symplectic category is strictly larger (Thurston 1976).

Gompf 1994: G finitely presented group $\Rightarrow \exists (X^4, \omega)$ compact symplectic such that $\pi_1(X) = G$.

Symplectic manifolds are not always complex, but they are almost-complex, i.e. there exists $J \in \mathrm{End}(TX)$ such that

$$J^2 = -\mathrm{Id}, \quad g(u, v) := \omega(u, Jv) \text{ Riemannian metric.}$$

At any given point (X, ω, J) looks like $(\mathbb{C}^n, \omega_0, i)$, but J is not integrable ($\nabla J \neq 0$; $\bar\partial^2 \neq 0$; no holomorphic coordinates).

Symplectic topology

Hierarchy of compact oriented 4-manifolds:

$$\text{COMPLEX PROJ.} \subsetneq \text{SYMPLECTIC} \subsetneq \text{SMOOTH}$$

$\Rightarrow$ Classification questions!

Symplectic manifolds retain some (not all!) features of complex proj. manifolds; yet (almost) every smooth 4-manifold admits a "near-symplectic" structure (sympl. outside circles).

Many new developments in the 1990s:

– J-holomorphic curves (Gromov-Witten invariants, Floer homology, ...)

– obstructions to existence of ω in dim. 4 (Taubes: Seiberg-Witten invariants)

– constructions of new examples (symplectic surgeries: Fintushel-Stern, Gompf)

– structure results (e.g., Donaldson: Lefschetz pencils)

Focus of the talk: symplectic branched covers in dimension 4.

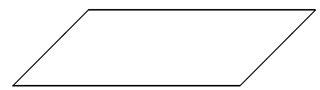
Symplectic branched covers

X^4 compact oriented, (Y^4, ω_Y) compact symplectic.

$f : X \to Y$ is a symplectic branched covering if $\forall p \in X$, $\exists$ local coordinates

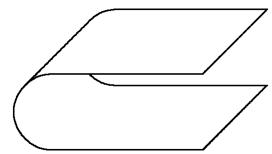
$$\left.\begin{array}{ll} \phi : X \supset U_p \to \mathbb{C}^2 & \text{(oriented)} \\ \psi : Y \supset V_{f(p)} \to \mathbb{C}^2 & \text{(compatible: } \omega_Y(v, iv) > 0) \end{array}\right\} \text{ in which } f \text{ is one of:}$$

- local diffeomorphism: $(x, y) \mapsto (x, y)$.



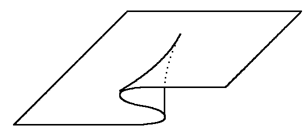
- simple branching: $(x, y) \mapsto (x^2, y)$.

  $R: \ x = 0 \qquad f(R): \ z_1 = 0$



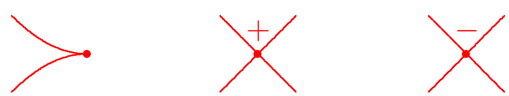
- cusp: $(x, y) \mapsto (x^3 - xy, y)$.

  $R: \ y = 3x^2 \qquad f(R): \ 27z_1^2 = 4z_2^3$



$R = \{\det(df) = 0\} \subset X$ is the ramification curve (smooth).
$D = f(R)$ is the branch curve (symplectic: $\omega_{|TD} > 0$), with singularities:

complex cusps; nodes (both orientations)



Proposition. *X carries a natural symplectic structure.*

Branched covers of $\mathbb{CP}^2$

Proposition. *$f : X^4 \to (Y^4, \omega_Y)$ symplectic branched cover $\Rightarrow$ X carries a natural symplectic structure.*

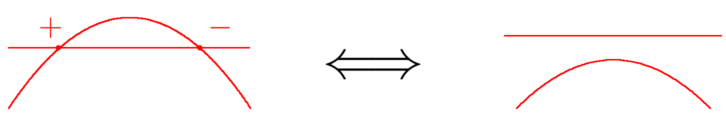
$[\omega_X] = [f^*\omega_Y]$, ω_X is canonical up to symplectomorphism.

Theorem. *(X^4, ω) compact symplectic, $[\omega] \in H^2(X, \mathbb{Z})$ $\Rightarrow$ X can be realized as symplectic branched cover of $\mathbb{CP}^2$.*

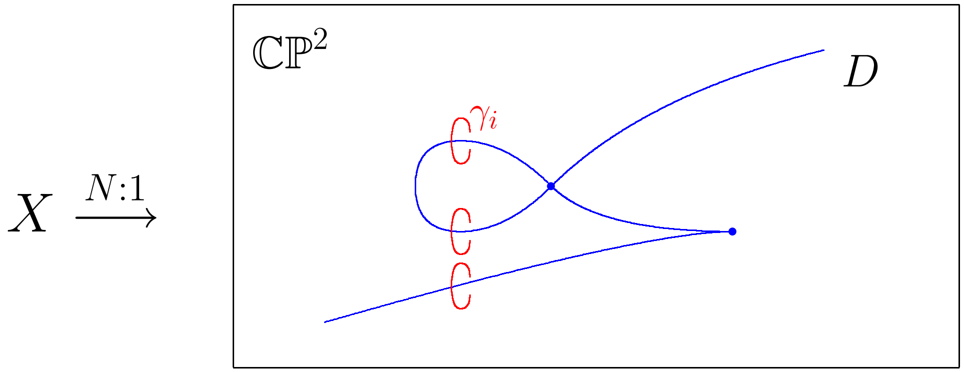
$\exists\ f_k : X \to \mathbb{CP}^2$, inducing $\omega_k \sim k\omega$, canonical up to isotopy for $k \gg 0$. The topology of f_k, e.g. the branch curve $D_k \subset \mathbb{CP}^2$, yields invariants of (X, ω).

Tool: "approx. hol. geometry": sections of $L^{\otimes k}$, $c_1(L^{\otimes k}) = k[\omega]$, with $|\bar\partial s|_{C^0} \ll |\partial s|_{C^0}$.

$D_k \subset \mathbb{CP}^2$ symplectic, with generic singularities = complex cusps, and nodes (both orientations)
Theorem $\Rightarrow$ up to cancellation of pairs of nodes, the topology of D_k is a symplectic invariant (if k large).



Topological invariants



Topological data for a branched cover of $\mathbb{CP}^2$:

1) Branch curve: $D \subset \mathbb{CP}^2$ (up to isotopy and node cancellations).

2) Monodromy: $\theta : \pi_1(\mathbb{CP}^2 - D) \twoheadrightarrow S_N$ $\quad (N = \deg f)$ (maps γ_i to transpositions).

D and θ determine (X, ω) up to symplectomorphism.

$\Rightarrow$ In principle it is enough to understand plane curves !

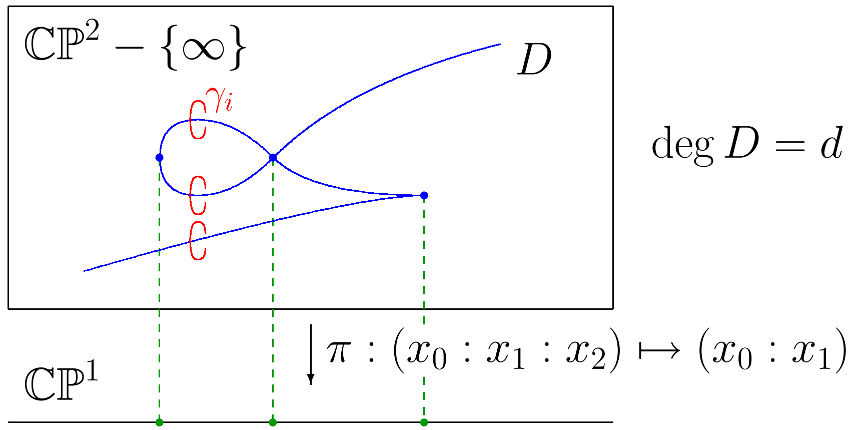
Fact: D is isotopic to a complex curve (up to node cancellations) iff X is Kähler (complex projective).

$\Rightarrow$ study the symplectic isotopy problem.

The topology of plane curves

(Moishezon-Teicher; Auroux-Katzarkov)

Perturbation $\Rightarrow$ D = singular branched cover of $\mathbb{CP}^1$.



Monodromy = $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (braid group)

$\Rightarrow$ D is described by a "braid group factorization" (involving cusps, nodes, tangencies).

The braid factorization characterizes D completely (and gives a combinatorial description of sympl. manifolds)

Problem: can compute for examples, but can't compare.

$\Rightarrow$ more manageable (incomplete) invariant ?

Stabilized fundamental groups

(Auroux-Donaldson-Katzarkov-Yotov 2002)

Question (Zariski...): D sing. plane curve, $\pi_1(\mathbb{CP}^2 - D) = ?$

Moishezon-Teicher: $\pi_1(\mathbb{CP}^2 - D)$ to study complex surfaces.

$\pi_1(\mathbb{CP}^2 - D)$ is related to the braid factorization. (Zariski-van Kampen theorem)

Belief: for high degree branch curves, $\pi_1(\mathbb{CP}^2 - D)$ is determined in a simple manner by the topology of X?

Symplectic stabilization of $\pi_1(\mathbb{CP}^2 - D)$: adding nodes (in manner compatible with $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_N$) introduces commutation relations

$\Rightarrow$ quotient by subgroup $K = \langle [\gamma, \gamma'],\ \gamma, \gamma'$ geom. generators, $\theta(\gamma), \theta(\gamma')$ disjoint$\rangle$.

Theorem. *For $k \gg 0$, $G_k(X, \omega) = \pi_1(\mathbb{CP}^2 - D_k)/K_k$ is a symplectic invariant.*

Stabilized fundamental groups

Fact: $1 \longrightarrow G_k^0 \longrightarrow G_k \xrightarrow{(\theta_k,\delta_k)} S_N \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$

$(N = \deg f_k,\ d = \deg D_k;\ \theta_k = \text{monodromy},\ \delta_k = \text{linking map})$

Theorem. *If* $\pi_1(X) = 1$ *then we have a natural surjection* $\phi_k : \mathrm{Ab}\, G_k^0 \longrightarrow (\mathbb{Z}^2/\Lambda_k)^{N-1}$

$\Lambda_k = \{(k[\omega]\cdot C, K_X \cdot C),\ C \in H_2(X,\mathbb{Z})\}.$

Known examples: (for $k \gg 0$)

- $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ (Moishezon)
- some rational surfaces and K3's (Robb; Teicher et al.)
- Hirzebruch surfaces, double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ (ADKY)

$\Rightarrow$ **Conjectures:** for $k \gg 0$,

1) X alg. surface $\Rightarrow K_k = \{1\}$ and $G_k = \pi_1(\mathbb{CP}^2 - D_k)$.
2) $\pi_1(X) = 1 \Rightarrow \phi_k$ is an isomorphism.
3) $\pi_1(X) = 1 \Rightarrow [G_k^0, G_k^0] =$ quotient of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Isotopy results for plane curves

When is a (singular) symplectic curve in $\mathbb{CP}^2$ (or a complex surface) isotopic to a complex curve?

• Gromov (1985): every smooth symplectic curve of degree 1 or 2 in $\mathbb{CP}^2$ is isotopic to a complex curve.

(Tool: pseudo-holomorphic curves)

• Siebert-Tian (2002): smooth sympl. curves of degree ≤ 17 in $\mathbb{CP}^2$; connected curves of degree ≤ 7 in Hirzebruch surfaces.

• Barraud (2000), Shevchishin (2002): certain simple singular configurations in $\mathbb{CP}^2$.

• Francisco (2004): singular curves of degree $d \leq 9$ with m cusps in $\mathbb{CP}^2$ (if $d \geq 6$, assume $4(d-6) - 1 \leq m < 3d/2$).

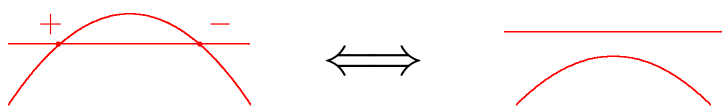
(in classification of branched covers, these are cases without any non-Kähler examples)

Stable isotopy results

D, D' symplectic ("Hurwitz") curves in $\mathbb{CP}^2$ or Hirzebruch surfaces, $[D] = [D']$, same numbers of nodes, cusps, (A_n-sings.):

• Kharlamov-Kulikov (2003) $\Rightarrow$ after adding sufficiently many lines (fibers) to D, D' and smoothing the intersections, D, D' become isotopic.

• A.-Kulikov-Shevchishin (2004): D, D' are isotopic up to creations/cancellations of pairs of nodes.



(in general, not compatible with branched covers!)

For branched covers:

• (2002): X genus 2 Lefschetz fibration $\Rightarrow$ X becomes complex projective after stabilization by fiber sums with rational surfaces along genus 2 curves.

(extends to hyperelliptic Lefschetz fibrations; what about the general case?)

Conjecture: two compact integral symplectic 4-manifolds with same $(c_1^2, c_2, c_1.[\omega], [\omega]^2)$ become symplectomorphic after blow-ups and fiber sums with holomorphic fibrations.

Non-isotopy phenomena

• Fintushel-Stern (1999), Smith (2001): infinitely many non-isotopic smooth connected symplectic curves in certain 4-manifolds (multiples of classes of square zero).

Use braiding constructions on parallel copies; distinguish using topology of branched covers (SW invariants, ...)

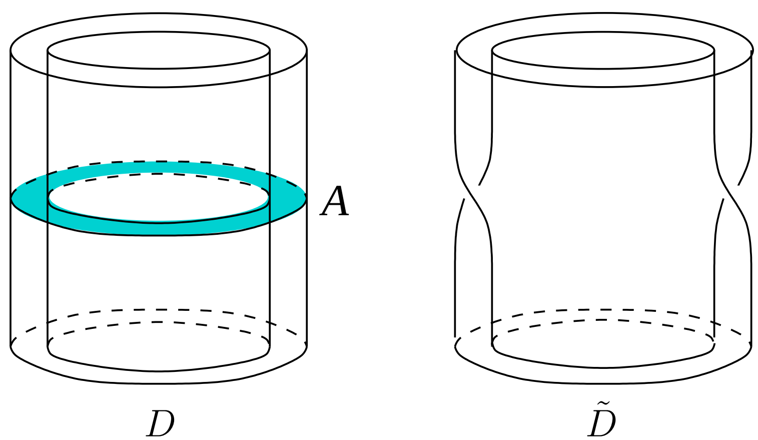
• Etgu-Park, Vidussi (2001-2004)

• Moishezon (1992): infinitely many non-isotopic sing. sympl. curves in $\mathbb{CP}^2$ (fixed number of cusp and node singularities).

Use braid monodromy and π_1 of complement (hard!)

(Auroux-Donaldson-Katzarkov 2002): elementary interpretation?

Moishezon $\Leftrightarrow$ braiding; modifies $c_1(K_X)$ vs. $[\omega_X]$



Given $f : X \to Y$ symplectic covering with branch curve D,

Braiding D / Lagrangian annulus $A \Longleftrightarrow$
Luttinger surgery of X / Lagrangian torus $T \subset f^{-1}(A)$.

(i.e. take out a neighborhood of T and glue it back via a symplectomorphism wrapping the meridian around the torus).

Questions:

• are any two symplectic cuspidal plane curves with same (degree, # nodes, # cusps) equivalent under braiding moves?

• are any two compact integral symplectic 4-manifolds with same $(c_1^2, c_2, c_1.[\omega], [\omega]^2)$ equivalent under Luttinger surgeries?

(Remark: many constructions rely on twisted fiber sums or link surgeries, which reduce to Luttinger surgery)