# Mirror symmetry in the complement of an anticanonical divisor

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August 27, 2007 - IHÉS

arXiv:0706.3207

# Mirror symmetry for Calabi-Yau manifolds

| Symplectic geometry (**A**) | Complex geometry (**B**) |
|---|---|
| $(X, J, \omega, \Omega)$ Calabi-Yau | $(X^\vee, J^\vee, \omega^\vee, \Omega^\vee)$ Calabi-Yau |
| Gromov-Witten invariants | Variations of Hodge structure |
| Lagrangian submanifolds | Analytic cycles |
| Fukaya category | Derived category of coherent sheaves |

## Geometry: Strominger-Yau-Zaslow conjecture

(+Kontsevich-Soibelman, Gross-Siebert, Fukaya, ...)

$X$, $X^\vee$ are dual fibrations by special Lagrangian tori over a base carrying an integral affine structure.*

* Actual examples are hard to come by. SYZ seems to hold only near the "large complex structure limit". There are singularities in codimension 2, and these induce "quantum corrections". Etc...

# Landau-Ginzburg models

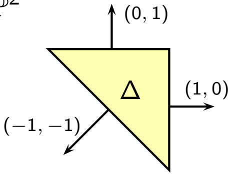
When $c_1(X) \neq 0$, the mirror is a Landau-Ginzburg model $W : M \to \mathbb{C}$ ($M$ noncompact; $W =$ *superpotential*, holomorphic)

Symplectic/complex geometry of $X \Leftrightarrow$ complex/symplectic geometry of *singular fibers* of $W$.

**Question:** how to construct $W : M \to \mathbb{C}$?

If $X$ toric: $M = (\mathbb{C}^*)^n$, $W =$ Laurent polynomial.

$X = \mathbb{CP}^2$




$M = (\mathbb{C}^*)^2$,

$$W = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2} \quad (\Lambda = \textstyle\int_{\mathbb{CP}^1} \omega)$$

(In general, $W = \sum_{F \text{ facet}} e^{-2\pi\alpha(F)}\, z^{\nu(F)}$ where eqn. of $F$ is $\langle \nu(F), \phi \rangle = \alpha(F)$.)

# A rough conjecture

## Conjecture

$(X, \omega, J)$ compact Kähler manifold, $D \subset X$ anticanonical divisor, $\Omega \in \Omega^{n,0}(X \setminus D) \Rightarrow$ can construct a mirror as

- $M =$ moduli space of special Lagrangian tori $L \subset X \setminus D$
  $+$ flat $U(1)$ connections on trivial bundle over $L$
- $W : M \to \mathbb{C}$ counts holomorphic discs of Maslov index 2 in $(X, L)$
  (Fukaya-Oh-Ohta-Ono's $m_0$ obstruction in Floer homology)
- the fiber of $W$ is mirror to $D$.

Conjecture doesn't quite hold as stated. Mainly:

- $W$ presents wall-crossing discontinuities caused by Maslov index 0 discs $\Rightarrow$ need "quantum corrections" to correct these discontinuities.
- According to Hori-Vafa, need to enlarge $M$ by "renormalization".

# Special Lagrangians

$(X, \omega, J)$ compact Kähler manifold, $\dim_{\mathbb{C}} X = n$.
$\sigma \in H^0(K_X^{-1})$, $D = \sigma^{-1}(0)$, $\Omega = \sigma^{-1} \in \Omega^{n,0}(X \setminus D)$.

## Definition

$L^n \subset X \setminus D$ is *special Lagrangian* if $\omega_{|L} = 0$ and $\mathrm{Im}(e^{-i\phi}\Omega)_{|L} = 0$. ($\phi =$cst)

## Proposition

*Special Lagrangian deformations* $= \mathcal{H}^1_\psi(L)$ $(\simeq H^1(L, \mathbb{R}))$, *unobstructed.*

$\mathcal{H}^1_\psi(L) = \{\theta \in \Omega^1(L, \mathbb{R}) \mid d\theta = 0,\ d^*(\psi\theta) = 0\}$ "$\psi$-harmonic" 1-forms
where $\psi = \mathrm{Re}(e^{-i\phi}\Omega)_{|L}/vol(g_{|L}) \in C^\infty(L, \mathbb{R}_+)$.

$v \in C^\infty(NL)$ is SLag iff $-\iota_v\omega = \theta$ and $\iota_v \mathrm{Im}(e^{-i\phi}\Omega) = \psi * \theta$ are closed.

# The geometry of the moduli space

## Definition

$M = \{(L,\nabla) \mid L \subset X \setminus D$ special Lag. torus, $\nabla$ flat $U(1)$ conn. on $\underline{\mathbb{C}} \to L\}$.

## Proposition

- $T_{(L,\nabla)}M = \{(v,\alpha) \in C^\infty(NL) \oplus \Omega^1(L,\mathbb{R}) \mid -\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L) \otimes \mathbb{C}\}$.
- *Complex structure $J^\vee$ on $M$; local holomorphic functions:*
  given $\beta \in H_2(X,L)$, $z_\beta = \exp(-\int_\beta \omega)\,\mathrm{hol}_{\partial\beta}(\nabla) : M \to \mathbb{C}^*$.
- *Compatible Kähler form*
  $\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) = \int_L \alpha_2 \wedge \iota_{v_1}\mathrm{Im}\, e^{-i\phi}\Omega - \alpha_1 \wedge \iota_{v_2}\mathrm{Im}\, e^{-i\phi}\Omega$.
- *Holom. volume form*
  $\Omega^\vee((v_1,\alpha_1),\dots,(v_n,\alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \cdots \wedge (-\iota_{v_n}\omega + i\alpha_n)$.

$\Rightarrow$ Assuming $\psi$-harmonic 1-forms on $L$ have no zeroes, $X$ and $M$ are dual special Lag. torus fibrations in a nbd. of $L$ (the projection is $(L,\nabla) \mapsto L$).

# The superpotential

$\beta \in \pi_2(X, L) \Rightarrow$ moduli space of holom. maps $u : (D^2, \partial D^2) \to (X, L)$ in class $\beta$, of virt. dim. $n - 3 + \mu(\beta)$, where $\mu(\beta) = 2\#(\beta \cap D)$ Maslov index.

## Assumption

$L$ does not bound any nonconstant Maslov index 0 holomorphic discs; Maslov index 2 discs are *regular*.

Then for $\mu(\beta) = 2$, can count holom. discs in class $\beta$ whose boundary passes through a generic given point $p \in L \Rightarrow n_\beta(L) \in \mathbb{Z}$.

## Definition

$$W(L, \nabla) = \sum_{\mu(\beta)=2} n_\beta(L)\, z_\beta, \quad \text{where } z_\beta = \exp(-\textstyle\int_\beta \omega)\, \mathrm{hol}_{\partial\beta}(\nabla).$$

By construction $W : M \to \mathbb{C}$ is holomorphic. (Convergence OK at least if $X$ Fano)

# The toric case (see also Cho-Oh)

$X$ smooth toric variety with moment map $\phi : X \to \mathbb{R}^n$, $\Delta = \phi(X)$.
$D = \phi^{-1}(\partial\Delta)$ toric divisor, $X \setminus D \simeq (\mathbb{C}^*)^n$, $\Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$.

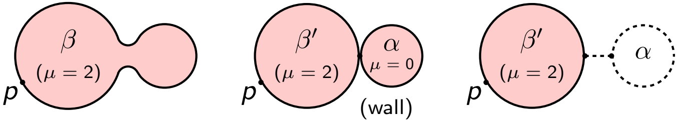
- Toric fibers ($T^n$-orbits) are special Lagrangian.
- $M$ is biholomorphic to $\text{Łog}^{-1}(\text{int}\,\Delta) \subset (\mathbb{C}^*)^n$,
  where $\text{Łog}(z_1, \dots, z_n) = \frac{1}{2\pi}(\log|z_1|, \dots, \log|z_n|)$.
- There are no Maslov index 0 discs; one family of Maslov index 2 discs for each facet $F$ of $\Delta$. Primitive outward normal: $\nu(F) \in \mathbb{Z}^n$.
- $W = \sum_{F \text{ facet}} e^{-2\pi\alpha(F)}\, z^{\nu(F)}$ where eqn. of $F$ is $\langle \nu(F), \phi\rangle = \alpha(F)$.

### Hori-Vafa's "renormalization"

Our mirror is smaller than expected. **Enlarge $M$ by "inflation along $D$":**
Consider $(X, \omega_k)$ where $[\omega_k] = [\omega] + k\, c_1(X)$, $k \to \infty$ ($X$ must be Fano) (in toric case, enlarges $\Delta$ by $k$) and rescale $W$ by factor $e^k$.

# Maslov index 0 discs and wall-crossing

Bubbling of Maslov index 0 discs causes the disc count $n_\beta(L)$ to jump.




Typically, for $n \geq 3$ the disc count depends on $p \in L$ ($\Rightarrow W$ multivalued). For $n = 2$ the disc count is independent of $p \in L$ but jumps where $L$ bounds a Maslov index 0 disc ($\Rightarrow W$ discontinuous).

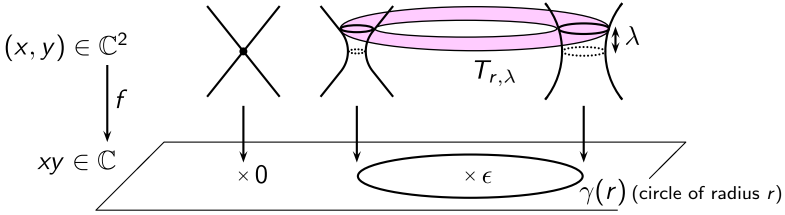
**Proposition (Fukaya-Oh-Ohta-Ono $+\varepsilon$)**

*For $n = 2$, crossing a wall in which $L$ bounds a single Maslov index 0 disc in a class $\alpha$ modifies $W$ by a holomorphic substitution of variables* $z_\beta \mapsto z_\beta\, h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]}$ $\forall \beta \in \pi_2(X, L)$, *where* $h(z_\alpha) = 1 + O(z_\alpha) \in \mathbb{C}[[z_\alpha]]$.

**Conjecture:** the mirror is obtained from $M$ by gluing the various regions delimited by the walls according to these changes of variables.

# Example: $\mathbb{CP}^2$

$X = \mathbb{CP}^2$, $\omega = \omega_{std}$, $\Omega = \dfrac{dx \wedge dy}{xy - \epsilon}$, $D = \{xy = \epsilon\} \cup \{\text{line at } \infty\}$:




$T_{r,\lambda}$ is special Lagrangian; wall-crossing at $r = |\epsilon|$ (when $T_{r,\lambda}$ hits $f^{-1}(0)$).

case $r > |\epsilon|$: standard tori

$$W = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2}$$

case $r < |\epsilon|$: Chekanov tori

$$W = u + \frac{e^{-\Lambda}(1+v)^2}{u^2 v}$$

$u \leftrightarrow$ trivial section
$v \leftrightarrow$ vanishing cycle at 0 ($|v| = \exp(-\lambda)$)

- Geometry of $M$: $v = z_2/z_1$; $u = z_1$ or $z_2$ depending on sign of $\lambda$.
- Quantum corrections (geometry of $W$): $v = z_2/z_1$, $u = z_1 + z_2$.

# Critical values of $W$ and quantum cohomology

$QH^*(X)$ (with $\mathbb{C}$ coefficients) acts on $HF(L,\nabla)$ by quantum cap-product.

**Proposition**

*Assume $L$ does not bound Maslov index 0 holom. discs. If $HF(L,\nabla)\neq 0$, then $W(L,\nabla)$ is an eigenvalue of quantum cup-product by $c_1(X)$.*

(idea: $[D]\cap[L]=W(L,\nabla)\,[L]$).

Combining with Cho-Oh, this gives:

**Theorem** (cf. Kontsevich, ...)

*$X$ smooth toric Fano $\Rightarrow$ all the critical values of $W$ are eigenvalues of $c_1(X)*-:QH^*(X)\to QH^*(X)$.*

(in toric case $HF(L,\nabla)\neq 0 \Leftrightarrow dW=0$; maybe also in general?)

# Relative homological mirror symmetry

- $D \subset X$ carries an induced holom. volume form $\Omega_D = \mathrm{Res}_D(\Omega)$.
- **Conjecture:** near boundary of moduli space, $L \subset$ nbd. of $D$, and $L$ is an $S^1$-bundle over a special Lagrangian in $(D, \Omega_D)$.
- Let $M_D = \{z_\delta = 1\}$ ($\delta$ = class of linking disc): complex hypersurface contained in $\partial M = \{|z_\delta| = 1\}$. Expect: $M_D$ is mirror to $D$.

  (Note: assuming $D$ smooth, in renormalization limit, $M_D \sim$ fiber of $W$ near $\infty$)

**Relative Fukaya category $\mathcal{F}(M, M_D)$**: objects = admissible Lagr. $\mathcal{L} \subset M$ with $\partial\mathcal{L} \subset M_D$ + flat conn. $\nabla$; $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{CF}^*(\mathrm{int}(\mathcal{L}_1), \mathrm{int}(\mathcal{L}_2^+))$
(admissible: $z_\delta \in \mathbb{R}_+$ near $\partial\mathcal{L}$; $\mathcal{L}_2^+$ = perturb $\mathcal{L}_2$ to positive position) [Kontsevich, Seidel]

## Conjecture (relative homological mirror symmetry)

$$\begin{tikzcd}
D^b Coh(X) \arrow[r, "\mathrm{restr}"] \arrow[d, "\simeq"', "HMS"] & D^b Coh(D) \arrow[d, "HMS"', "\simeq"] \\
D^\pi \mathcal{F}(M, M_D) \arrow[r, "\mathrm{restr}", "\mathcal{L} \mapsto \partial\mathcal{L}"'] & D^\pi \mathcal{F}(M_D)
\end{tikzcd}$$