

# Fukaya categories of symmetric products and bordered Heegaard-Floer homology

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(work in progress)

# Ozsváth-Szabó invariants as a TQFT?

- closed 4-manifold  $X \implies \Phi(X) \in \mathbb{Z}$
  - closed 3-manifold  $Y \implies HF(Y)$  abelian group  $(\widehat{HF}, HF^+, HF^-)$
  - cobordism  $\partial X = Y_2 - Y_1 \implies \Phi(X) : HF(Y_1) \rightarrow HF(Y_2)$
- 
- surface  $\Sigma \implies$  category  $\mathcal{C}(\Sigma)$ ?
  - 3-manifold with boundary  $\partial Y = \Sigma \implies$  object  $C(Y) \in \mathcal{C}(\Sigma)$ ?  
(e.g.: handlebody)
  - cobordism  $\partial Y = \Sigma_2 - \Sigma_1 \implies$  functor  $C(Y) : \mathcal{C}(\Sigma_1) \rightarrow \mathcal{C}(\Sigma_2)$ ?
  - pairing:  $Y = Y_1 \cup_{\Sigma} Y_2 \implies HF(Y) \simeq \text{hom}_{\mathcal{C}(\Sigma)}(C(Y_1), C(Y_2))$ ?

see also: Perutz, Lekili

# Bordered Heegaard-Floer homology

(R. Lipshitz–P. Ozsváth–D. Thurston)

- $F$  (marked, parameterized) surface  $\implies \mathcal{A}(F)$  differential algebra
- $Y$  3-manifold with  $\partial Y \xrightarrow{\sim} F \implies \widehat{CFA}(Y)$  right  $A_\infty$ -module over  $\mathcal{A}(F)$
- $Y'$  3-manifold with  $\partial Y' \xrightarrow{\sim} -F \implies \widehat{CFD}(Y')$  left dg-module over  $\mathcal{A}(F)$
- cobordisms  $\implies$  bimodules over  $\mathcal{A}(F)$
- **Pairing theorem:**  $\widehat{CF}(Y \cup_F Y') \simeq \widehat{CFA}(Y) \otimes_{\mathcal{A}(F)} \widehat{CFD}(Y')$

Lipshitz-Ozsváth-Thurston define  $\mathcal{A}(F)$  combinatorially, to encode behavior of holomorphic strips upon neck-stretching (SFT).

**Goal:** *Symplectic interpretation* of  $\mathcal{A}(F)$  and  $\widehat{CFA}(Y)$  in terms of **Fukaya categories of  $\text{Sym}^k(F)$** .

# The algebra $\mathcal{A}(F, k)$ (Lipshitz-Ozsváth-Thurston)

Describe  $F$  (genus  $g$ ) by a **pointed matched circle**:

$4g$  points  $1, \dots, 4g$  carrying labels  $1, \dots, 2g, 1, \dots, 2g$

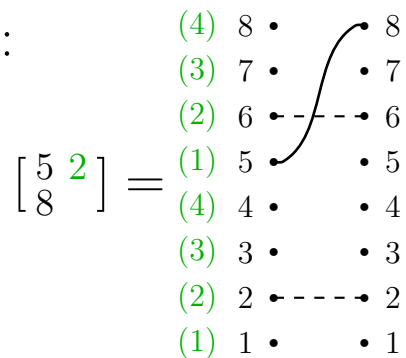
$\mathcal{A}(F, k)$  ( $1 \leq k \leq 2g$ ) generated by  $k$ -tuples of either

- upwards stands (“Reeb chords”) connecting pairs of points  $\begin{bmatrix} i \\ j \end{bmatrix}$  ( $i < j$ )
- pairs of horizontal dotted lines  $\begin{bmatrix} i \end{bmatrix}$

such that the  $k$  source labels (resp. target labels) in  $\{1, \dots, 2g\}$  are all distinct.

View  $\mathcal{A}(F, k)$  as a finite (differential) **category** with objects  $\mathcal{S}_k = k$ -element subsets of  $\{1, \dots, 2g\}$ .

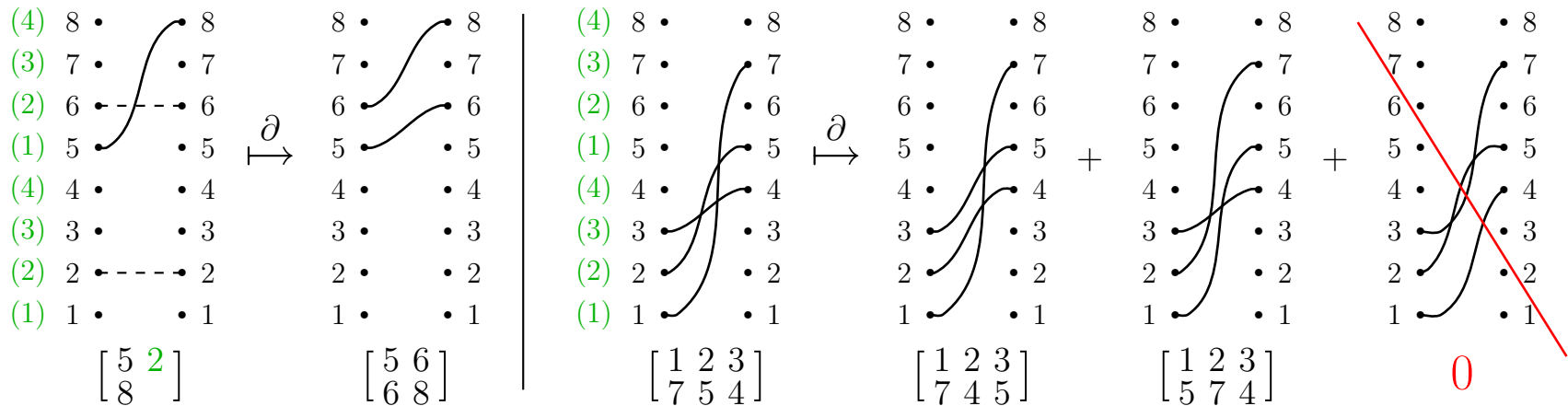
**Example** ( $g = 2, k = 2$ ):



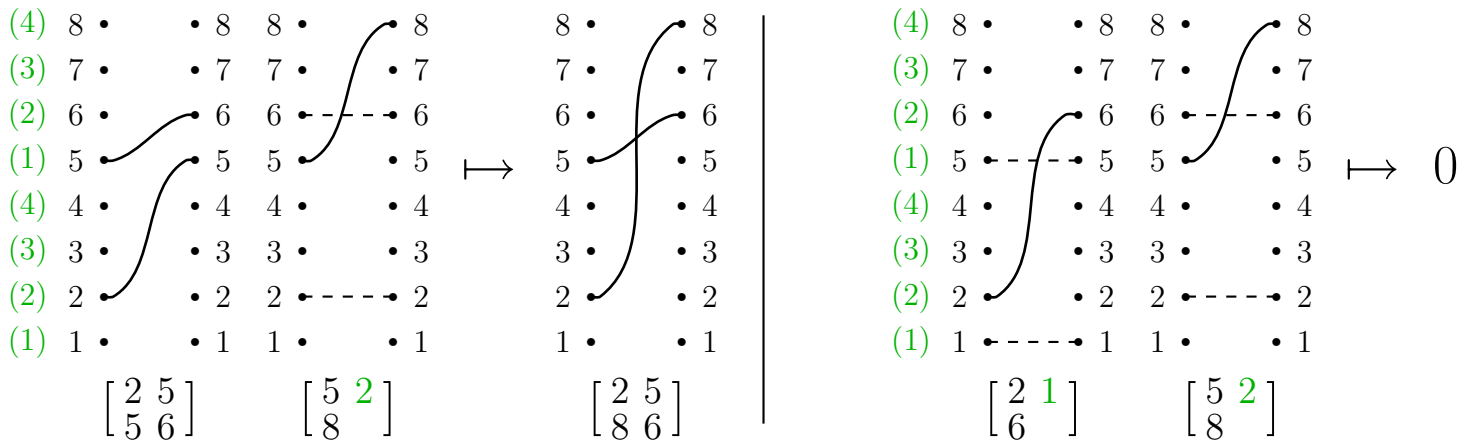
morphism  $\{1, 2\} \rightarrow \{2, 4\}$ .

# The algebra $\mathcal{A}(F, k)$ (continued)

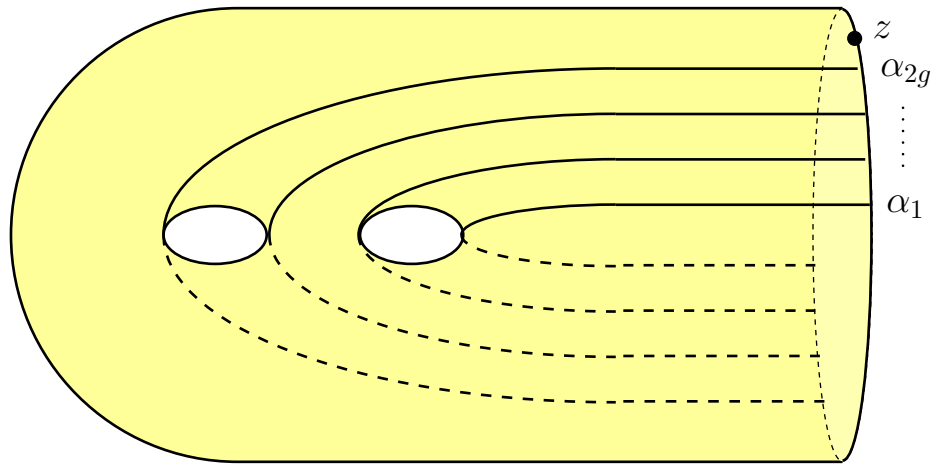
**Differential:** sum all ways of smoothing one crossing (double-crossing  $\sim 0$ ).



**Product:** concatenation (double-crossing  $\sim 0$ )



# $\mathcal{A}(F, k)$ vs. the Fukaya category of $\text{Sym}^k(F)$



**Definition.** For  $s \in \mathcal{S}_k$ , let  $D_s = \prod_{i \in s} \alpha_i \subset \text{Sym}^k(F)$ .

Let  $\mathcal{F}' =$  **relative Fukaya category** of  $(\text{Sym}^k(F), \{z\} \times \text{Sym}^{k-1}(F))$ .  
 (“partially wrapped”)

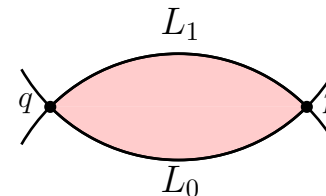
**Theorem 1.**  $\mathcal{A}(F, k) \simeq \bigoplus_{s, s' \in \mathcal{S}_k} \text{hom}_{\mathcal{F}'}(D_s, D_{s'})$ .

# Fukaya categories

$L, L' \subset (M, \omega)$  compact exact Lagr.  $\Rightarrow \text{hom}(L, L') = CF(L, L') = \mathbb{Z}_2^{|L \cap L'|}$

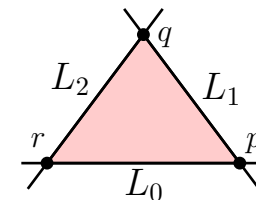
- **Differential**  $\partial : \text{hom}(L_0, L_1) \rightarrow \text{hom}(L_0, L_1)$

$\langle \partial(p), q \rangle$  counts pseudo-holomorphic strips



- **Product**  $m_2 : \text{hom}(L_0, L_1) \otimes \text{hom}(L_1, L_2) \rightarrow \text{hom}(L_0, L_2)$

$\langle m_2(p, q), r \rangle$  counts pseudo-holomorphic triangles



- **Higher products**  $m_k$  ( $A_\infty$  category)

**Partially wrapped case:** (in progress, cf. also Abouzaid, Seidel)

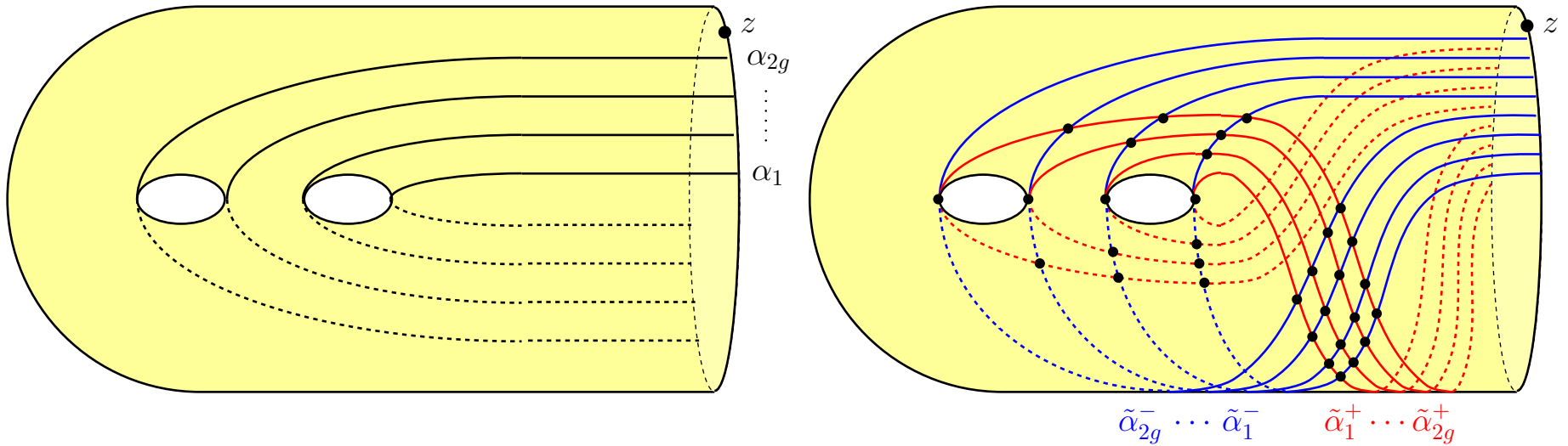
- $\partial M$  contact,  $N \subset \partial M$ ,  $\rho : \partial M \rightarrow [0, 1]$ ,  $\rho^{-1}(0) = N$
  - $H_\rho$  Hamiltonian on  $\hat{M} = M \cup [1, \infty) \times \partial M$ , s.t.  $H_\rho(r, y) = \rho(y) r$  near  $\infty$
- $H_\rho$  “wraps” along Reeb flow of contact hypersurface  $\{r = \rho^{-1}\} \simeq \partial M \setminus N$ , slowing down as one approaches  $N$

$\Rightarrow$  perturb Floer homology by long-time flow of  $H_\rho$ : for  $\partial L, \partial L' \subset \partial M \setminus N$ ,

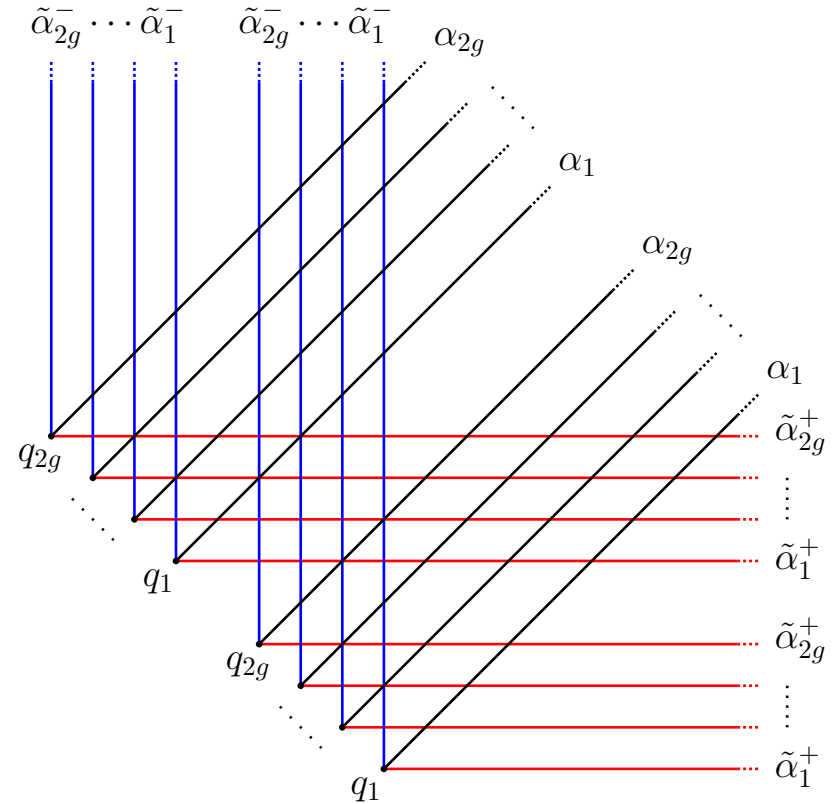
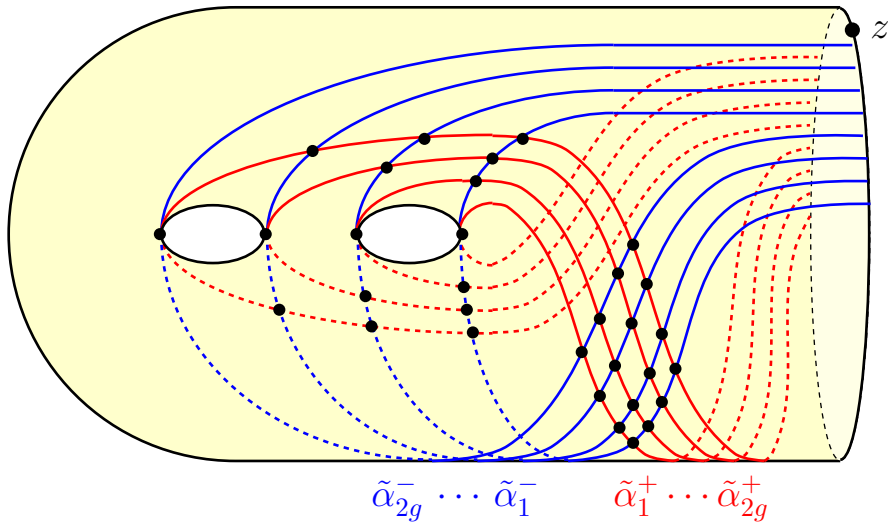
$$\text{hom}_{\mathcal{F}'}(L, L') = \lim_{w \rightarrow +\infty} CF(\phi_{wH_\rho}(L), L').$$

# Partial wrapping in $\text{Sym}^k(F)$

Partial wrapping of  $D_s = \prod_{i \in s} \alpha_i$  rel.  $\{z\} \times \text{Sym}^{k-1}(F)$  gives  $D_s^- \stackrel{\text{Ham}}{\simeq} \prod_{i \in s} \tilde{\alpha}_i^-$ .



# Floer theory for $D_s^\pm \simeq \prod_{i \in s} \tilde{\alpha}_i^\pm$



## Proof of Theorem 1:

- $CF(D_s^-, D_{s'}^+) \cong \text{hom}_{\mathcal{A}(F,k)}(s, s')$  (gen. by  $k$ -tuples of intersections)
- $\partial$  counts empty rectangles (“nice diagram”)
- product  $m_2$  counts unions of triangles (head-to-tail overlap only)
- $m_{k \geq 3} \equiv 0$

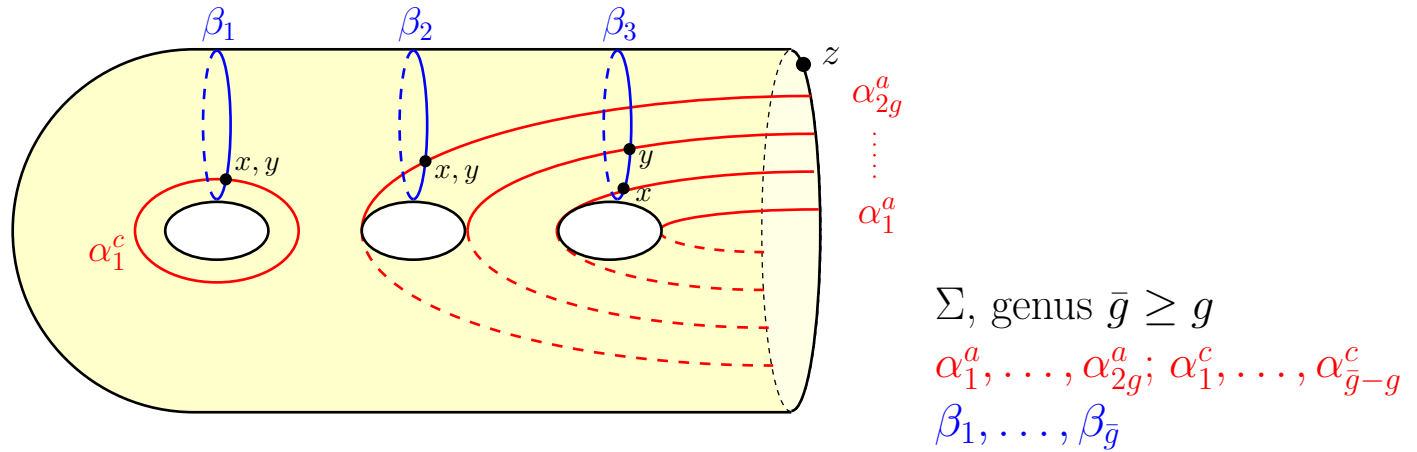
# Generating the relative Fukaya category

**“Theorem” 2.** *The relative Fukaya category  $\mathcal{F}'$  is generated by the  $\binom{2g}{k}$  objects  $D_s$ ,  $s \subseteq \{1, \dots, 2g\}$ ,  $|s| = k$ . Hence,  $\mathcal{F}'\text{-mod} \simeq \mathcal{A}(F, k)\text{-mod}$ .*

**Key:**  $D_s$  are “thimbles” for a Lefschetz fibration  $f_k : \text{Sym}^k(F) \rightarrow \mathbb{C}$ .

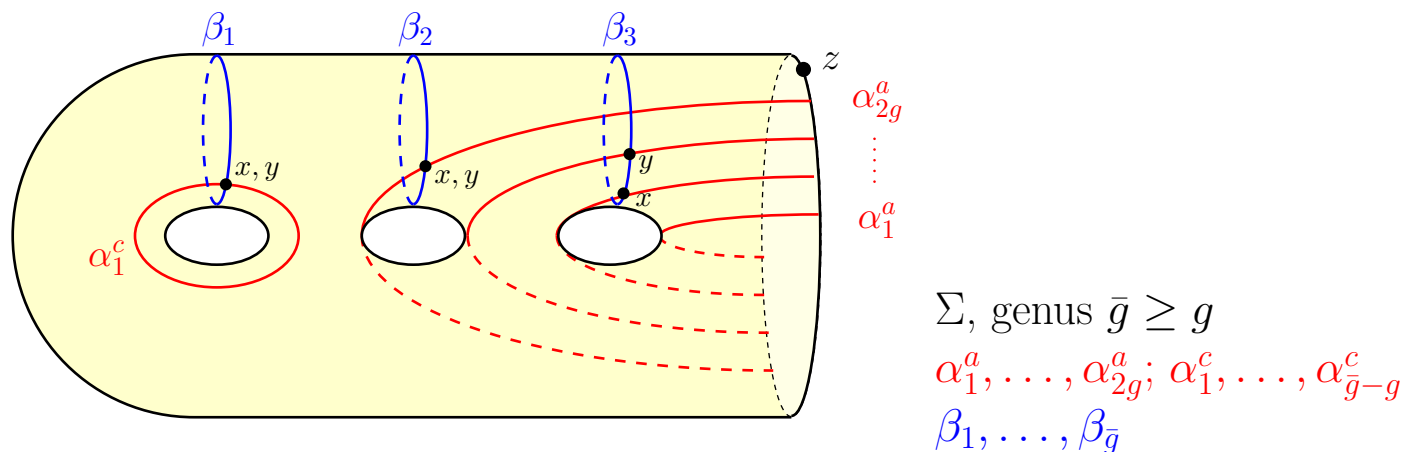
- Start with  $\pi : F \xrightarrow{2:1} \mathbb{C}$  (with  $2g + 1$  branch points).  
Then  $f_k : \{z_1, \dots, z_k\} \mapsto \sum \pi(z_i)$  has  $\binom{2g+1}{k}$  nondegenerate critical points = tuples of *distinct* critical points of  $\pi$ .
- The thimbles (stable manifolds for  $\nabla \text{Re}(f_k)$ ) are products of  $k$  arcs on  $F$  ( $\alpha_1, \dots, \alpha_{2g} +$  one other = thimbles of  $\pi$ ); they generate  $\mathcal{F}'$  [Seidel].
- Can reduce to sub-fibration  $f_k^{-1}(U)$ ,  $U \subset \mathbb{C}$ , with  $\binom{2g}{k}$  thimbles =  $\{D_s\}_{s \in \mathcal{S}_k}$ .

# The $A_\infty$ -module $\widehat{CFA}(Y)$



e.g.,  $x \cdot \begin{bmatrix} 6 & 4 \\ 7 & \end{bmatrix} = y$ .

# The $A_\infty$ -module $\widehat{CFA}(Y)$



**“Theorem” 3.**  $\widehat{CFA}(Y) \simeq \bigoplus_{s \in \mathcal{S}_g} CF(T_\beta, T_\alpha^c \times D_s)$  (right  $\mathcal{A}(F, g)$ -module).

Note:  $\mathcal{A}(F, g)$  and  $\mathcal{F}(\text{Sym}^g F, z \times \text{Sym}^{g-1} F)$  embed into  $\mathcal{F}(\text{Sym}^{\bar{g}} \Sigma, z \times \text{Sym}^{\bar{g}-1} \Sigma)$  via  $T_\alpha^c$

# The pairing theorem

**“Theorem” 4.**  $\widehat{CF}(Y \cup_F Y') \simeq \text{hom}_{\mathcal{A}(F,g)\text{-mod}}(\widehat{CFA}(-Y'), \widehat{CFA}(Y)).$

(Equivalent to Lipshitz-Ozsváth-Thurston’s pairing result)

## Main ingredients:

- Extended Fukaya categories (“quilts”) [Wehrheim-Woodward]:  
view  $T_\beta \circ T_\alpha^c$  as generalized Lagrangian in  $(\text{Sym}^g(F), z \times \text{Sym}^{g-1}(F))$

- Yoneda embedding (+ Theorem 2):

$$\text{hom}_{\mathcal{A}\text{-mod}}(T_{\beta'} \circ T_{\alpha'}^c, T_\beta \circ T_\alpha^c) \simeq CF(T_\beta \circ T_\alpha^c, T_{\beta'} \circ T_{\alpha'}^c)$$

- “ $CF(T_\beta \times T_{\alpha'}^c, T_{\beta'} \times T_\alpha^c) \simeq CF(T_\beta \circ T_\alpha^c, T_{\beta'} \circ T_{\alpha'}^c)$ ” [Lekili-Perutz]

