Fukaya categories and bordered Heegaard-Floer homology

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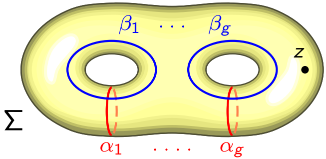
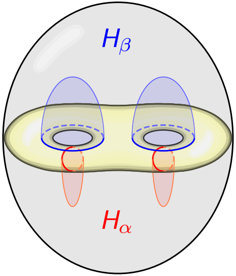
builds on work of: R. Lipshitz, P. Ozsváth, D. Thurston; T. Perutz, Y. Lekili
M. Abouzaid, P. Seidel; S. Ma'u, K. Wehrheim, C. Woodward

Heegaard-Floer homology

Y^3 closed 3-manifold admits a *Heegaard splitting* into two handlebodies $Y = H_\alpha \cup_\Sigma H_\beta$.

This is encoded by a *Heegaard diagram* $(\Sigma, \alpha_1 \dots \alpha_g, \beta_1 \dots \beta_g)$. $(g = genus(\Sigma))$





unordered g-tuples of points on punctured Σ

Let $T_\alpha = \alpha_1 \times \cdots \times \alpha_g$, $T_\beta = \beta_1 \times \cdots \times \beta_g \subset \mathrm{Sym}^g(\Sigma \setminus z)$

Theorem (Ozsváth-Szabó, $\sim$ 2000)

$\widehat{HF}(Y) := HF(T_\beta, T_\alpha)$ is independent of chosen Heegaard diagram.

(Floer homology: complex generated by $T_\alpha \cap T_\beta$ = g-tuples of intersections between α and β curves, differential counts holomorphic curves).

Heegaard-Floer TQFT

Extend Heegaard-Floer to surfaces and 3-manifolds with boundary?

2 answers: Lipshitz-Ozsváth-Thurston '08 (explicit, computable)
vs. Lekili-Perutz '10 (geometric, can be extended to $HF^{\pm}$)

- Y^3 closed $\rightsquigarrow$ $\widehat{HF}(Y)$ abelian group
- W^4 cobordism ($\partial W = Y_2 - Y_1$) $\rightsquigarrow$ $\widehat{F}_W : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$
- Σ surface (punctured, decorated) $\rightsquigarrow$ category $\mathcal{C}(\Sigma)$ (Γ_Σ acts faithfully)
  (modules over) finite dg-algebra $\mathcal{A}(\Sigma)$
  (extended, balanced) Fukaya category $\mathcal{F}^\#(\mathrm{Sym}^g(\Sigma))$
- Y^3 with boundary $\partial Y = \Sigma$ $\rightsquigarrow$ object $C(Y) \in \mathcal{C}(\Sigma)$?
  $\widehat{CFA}(Y)$ right A_∞ $\mathcal{A}(\Sigma)$-module (also: $\widehat{CFD}(Y)$ left dg-module)
  $\mathbf{T}_Y$ (generalized) Lagrangian submanifold of $\mathrm{Sym}^g(\Sigma)$
- cobordism $\partial Y = \Sigma_2 - \Sigma_1$ $\rightsquigarrow$ functor $\mathcal{C}(\Sigma_1) \to \mathcal{C}(\Sigma_2)$
  from bimodule $\widehat{CFDA}(Y)$, (generalized) Lagr. correspondence $\mathbf{T}_Y$
- $\widehat{HF}(Y_1 \cup_\Sigma Y_2) = \hom_{\text{mod-}\mathcal{A}}(\widehat{CFA}(-Y_2), \widehat{CFA}(Y_1)) = HF(\mathbf{T}_{Y_1}, \mathbf{T}_{-Y_2})$

Goal: relate these two approaches

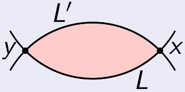
Plan

- Background: Floer homology, Fukaya categories, correspondences
- The Lekili-Perutz approach: correspondences from cobordisms
- The Lipshitz-Ozsváth-Thurston strands algebra
- The partially wrapped Fukaya category of $\mathrm{Sym}^k(\Sigma)$
- Modules and bimodules from bordered 3-manifolds

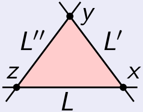
Floer homology, Fukaya categories and correspondences

Σ Riemann surface, $M = \mathrm{Sym}^k(\Sigma)$ monotone symplectic manifold
$\rightsquigarrow$ Fukaya category $\mathcal{F}(M)$: objects = Lagrangian submanifolds* (closed) (monotone, balanced)

- $\hom(L, L') = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2\, x$
- differential $\partial : CF(L, L') \to CF(L, L')$
  coeff. of y in ∂x counts holom. strips



- composition $CF(L, L') \otimes CF(L', L'') \to CF(L, L'')$
  coeff. of z in $x \cdot y$ counts holom. triangles

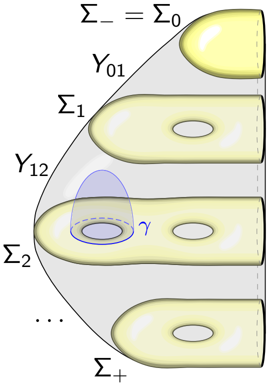


- more (A_∞-category)

(for product Lagrangians, holom. curves in $\mathrm{Sym}^k(\Sigma)$ can be seen on Σ)

- Lagrangian correspondences $M_1 \xrightarrow{L} M_2$ = Lagrangian submanifolds $L \subset (M_1 \times M_2, -\omega_1 \oplus \omega_2)$ generalize symplectomorphisms.
- "generalized Lagrangians" = formal images of Lagrangians under sequences of correspondences; Floer theory extends well.
  $\rightsquigarrow$ extended Fukaya cat. $\mathcal{F}^{\#}(M)$ (Ma'u-Wehrheim-Woodward).

Lekili-Perutz: correspondences from cobordisms



Perutz: Elementary cobordism $Y_{12} : \Sigma_1 \rightsquigarrow \Sigma_2$
$\Longrightarrow$ Lagrangian correspondence
$$\mathbf{T}_{12} \subset \mathrm{Sym}^k(\Sigma_1) \times \mathrm{Sym}^{k+1}(\Sigma_2) \ (k \geq 0)$$
(roughly: k points on $\Sigma_1 \mapsto$ "same" k points on Σ_2 plus one point anywhere on γ)

Lekili-Perutz: decompose Y^3 into sequence of elementary cobordisms $Y_{i,i+1}$, compose all $\mathbf{T}_{i,i+1}$ to get a generalized correspondence $\mathbf{T}_Y$.
$\mathbf{T}_Y : \mathrm{Sym}^{k_-}(\Sigma_-) \to \mathrm{Sym}^{k_+}(\Sigma_+) \quad (\partial Y = \Sigma_+ - \Sigma_-)$

Theorem (Lekili-Perutz)

$\mathbf{T}_Y$ *is independent of decomposition of* Y *into elementary cobordisms.*

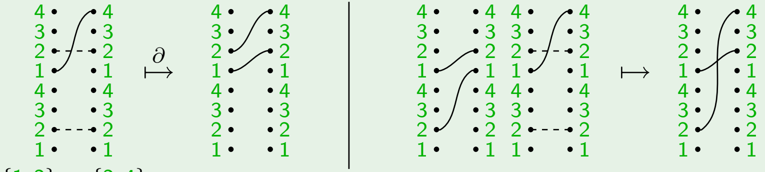
- View Y^3 (*sutured:* $\partial Y = \Sigma_+ \cup \Sigma_-$) as cobordism of surfaces w. boundary
- For a handlebody (as cobordism $D^2 \rightsquigarrow \Sigma_g$), $\mathbf{T}_Y \simeq$ product torus
- Y^3 closed, $Y \setminus B^3 : D^2 \rightsquigarrow D^2$, then $\mathbf{T}_Y \simeq \widehat{HF}(Y) \in \mathcal{F}^\#(pt) = \mathrm{Vect}$

The Lipshitz-Ozsváth-Thurston strands algebra $\mathcal{A}(\Sigma, k)$

Describe Σ by a pointed matched circle: segment with $4g$ points carrying labels $1, \dots, 2g, 1, \dots, 2g$ (= how to build $\Sigma = D^2 \cup 2g$ 1-handles)

$\mathcal{A}(\Sigma, k)$ is generated (over $\mathbb{Z}_2$) by k-tuples of $\{$upward strands, pairs of horizontal dotted lines$\}$ s.t. the k source labels (resp. target labels) in $\{1, \dots, 2g\}$ are all distinct.

Example ($g = k = 2$)



$\{1, 2\} \mapsto \{2, 4\}$

- Differential: sum all ways of smoothing one crossing.
- Product: concatenation (end points must match).
- Treat  as  +  and set  $= 0$.

The extended Fukaya category vs. $\mathcal{A}(\Sigma, k)$

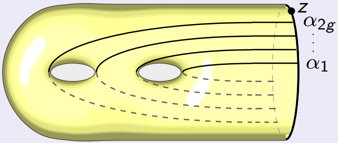
Theorem

$\mathcal{F}^{\#}(\mathrm{Sym}^k(\Sigma))$ embeds fully faithfully into mod-$\mathcal{A}(\Sigma, k)$ (A_∞-modules)

Main tool: partially wrapped Fukaya cat. $\mathcal{F}^{\#}(\mathrm{Sym}^k(\Sigma), z)$ $(z \in \partial\Sigma)$

- Enlarge $\mathcal{F}^{\#}$: allow noncompact objects = products of k disjoint properly embedded arcs; Floer theory perturbed by Hamiltonian flow.
- Roughly: $\hom(L_0, L_1) := CF(\tilde{L}_0, \tilde{L}_1)$, isotoping arcs so that end points of $\tilde{L}_0$ lie *above* those of $\tilde{L}_1$ in $\partial\Sigma \setminus \{z\}$ (without crossing z)
- Similarly, product is defined by perturbing so that $\tilde{L}_0 > \tilde{L}_1 > \tilde{L}_2$.

(after Abouzaid-Seidel)

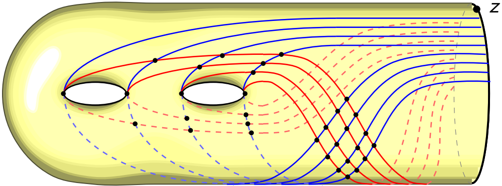


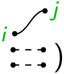
Let $D_s = \prod_{i \in s} \alpha_i$ $(s \subseteq \{1 \ldots 2g\}, |s| = k)$. Then:

1. $\bigoplus_{s,t} \hom(D_s, D_t) \simeq \mathcal{A}(\Sigma, k)$
2. the objects D_s generate $\mathcal{F}^{\#}(\mathrm{Sym}^k(\Sigma), z)$

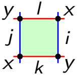
$\bigoplus \hom(D_s, D_t) \simeq \mathcal{A}(\Sigma, k)$

By def. of $\mathcal{F}^{\#}(\mathrm{Sym}^k(\Sigma), z)$, $\hom(D_s, D_t) = CF(\tilde{D}_s^+, \tilde{D}_t^-)$ $\left(\tilde{D}_s^{\pm} = \prod_{i \in s} \tilde{\alpha}_i^{\pm}\right)$

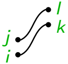


Dictionary: points of $\tilde{\alpha}_i^+ \cap \tilde{\alpha}_j^-$ $\longleftrightarrow$ strands 

(intersections on central axis $\longleftrightarrow$) } generators = k-tuples

- Differential: y appears in ∂x iff 

$\longleftrightarrow$ $x =$ 

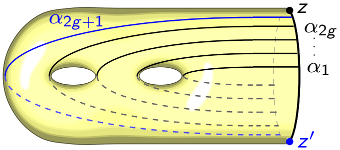
and $y =$ 

- Similarly for product (triple diagram); all diagrams are "nice"

More generally: $Z \subset \partial\Sigma$ finite, $\alpha_i \subset \Sigma$ disjoint arcs s.t. each component of $\Sigma \setminus \bigcup \alpha_i$ contains ≥ 1 point of Z. Let $D_s = \prod_{i \in s} \alpha_i \in \mathcal{F}^{\#}(\mathrm{Sym}^k\Sigma, Z)$. Then $\bigoplus \hom(D_s, D_t)$ is a combinatorially explicit, LOT-type, dg-algebra.

$\{D_s = \prod_{i\in s} \alpha_i\}_{s\subseteq\{1\dots 2g\}}$ generate $\mathcal{F}^{\#}(\mathrm{Sym}^k(\Sigma), z)$

- $\pi : \Sigma \xrightarrow{2:1} \mathbb{C}$ induces a Lefschetz fibration $f_k : \mathrm{Sym}^k(\Sigma) \to \mathbb{C}$ with $\binom{2g+1}{k}$ critical points. Its thimbles = products of α_i $(1 \leq i \leq 2g+1)$ generate $\mathcal{F}(f_k) \simeq \mathcal{F}(\mathrm{Sym}^k\Sigma, \{z, z'\})$ (Seidel)



- These $\binom{2g+1}{k}$ objects also generate $\mathcal{F}^{\#}(\mathrm{Sym}^k\Sigma, z)$.

Uses: acceleration functor $\mathcal{F}(\mathrm{Sym}^k\Sigma, \{z, z'\}) \to \mathcal{F}(\mathrm{Sym}^k\Sigma, z)$ (Abouzaid-Seidel)

- $\alpha_{i_1} \times \cdots \times \alpha_{2g+1} \simeq$ twisted complex built from $\{\alpha_{i_1} \times \cdots \times \alpha_j\}_{j=1}^{2g}$

Uses: arc slides are mapping cones

More generally: $Z \subset \partial\Sigma$ finite, $\alpha_i \subset \Sigma$ disjoint arcs s.t. each component of $\Sigma \setminus \bigcup \alpha_i$ is a disc containing ≤ 1 point of Z. Then the products $D_s = \prod_{i\in s} \alpha_i$ generate $\mathcal{F}^{\#}(\mathrm{Sym}^k\Sigma, Z)$.

Yoneda embedding and A_∞-modules

Recall: Y^3, $\partial Y = \Sigma \cup D^2 \Rightarrow$ gen. Lagr. $\mathbf{T}_Y \in \mathcal{F}^\#(\mathrm{Sym}^g \Sigma)$ (Lekili-Perutz)

- Yoneda embedding: $\mathbf{T}_Y \mapsto \mathcal{Y}(\mathbf{T}_Y) = \bigoplus_s \hom(\mathbf{T}_Y, D_s)$

  right A_∞-module over $\bigoplus_{s,t} \hom(D_s, D_t) \simeq \mathcal{A}(\Sigma, g)$.
- In fact, $\mathcal{Y}(\mathbf{T}_Y) \simeq \widehat{CFA}(Y)$ (bordered Heegaard-Floer module)
- Pairing theorem: if $Y = Y_1 \cup Y_2$, $\partial Y_1 = -\partial Y_2 = \Sigma \cup D^2$, then

  $\widehat{CF}(Y) \simeq \hom_{\mathcal{F}^\#}(\mathbf{T}_{Y_1}, \mathbf{T}_{-Y_2}) \simeq \hom_{\text{mod-}\mathcal{A}}(\mathcal{Y}(\mathbf{T}_{-Y_2}), \mathcal{Y}(\mathbf{T}_{Y_1}))$.
- also: (using $\mathcal{A}(-\Sigma, g) \simeq \mathcal{A}(\Sigma, g)^{op}$)

  $\widehat{CF}(Y) \simeq \mathbf{T}_{Y_1} \circ \mathbf{T}_{Y_2} \simeq \mathcal{Y}(\mathbf{T}_{Y_1}) \otimes_{\mathcal{A}} \mathcal{Y}(\mathbf{T}_{Y_2})$.

More generally, if $\partial Y = \Sigma_+ \cup -\Sigma_-$ (sutured manifold), the generalized corresp. $\mathbf{T}_Y \in \mathcal{F}^\#(-\mathrm{Sym}^{k_-}\Sigma_- \times \mathrm{Sym}^{k_+}\Sigma_+)$ yields an A_∞-bimodule

$$\mathcal{Y}(\mathbf{T}_Y) = \bigoplus_{s,t} \hom(D_{-,s}, \mathbf{T}_Y, D_{+,t}) \in \mathcal{A}(\Sigma_-, k_-)\text{-mod-}\mathcal{A}(\Sigma_+, k_+)$$

(cf. Ma'u-Wehrheim-Woodward). $\mathcal{Y}(\mathbf{T}_Y) \simeq \widehat{CFDA}(Y)$? (same properties)

Future directions

- $HF^{\pm}$ for bordered 3-manifolds? (in computable form)
  Want: combinatorial model for (filtered, balanced) $\mathcal{F}^{\#}$ of closed symmetric product?
- 4-manifold invariants; use this technology to relate Perutz invariants of broken Lefschetz fibrations to Ozsváth-Szabó?
- similar constructions in Khovanov homology (after Seidel-Smith)? (understand Lefschetz fibrations on Hilbert schemes of conic bundles and their Fukaya categories)