SYMPLECTIC 4-MANIFOLDS, SINGULAR PLANE CURVES, AND ISOTOPY PROBLEMS

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ABSTRACT. We give an overview of various recent results concerning the topology of symplectic 4-manifolds and singular plane curves, using branched covers and isotopy problems as a unifying theme. While this paper does not contain any new results, we hope that it can serve as an introduction to the subject, and will stimulate interest in some of the open questions mentioned in the final section.

1. Introduction

An important problem in 4-manifold topology is to understand which manifolds carry symplectic structures (i.e., closed non-degenerate 2-forms), and to develop invariants that can distinguish symplectic manifolds. Additionally, one would like to understand to what extent the category of symplectic manifolds is richer than that of Kähler (or complex projective) manifolds. Similar questions may be asked about singular curves inside, e.g., the complex projective plane. The two types of questions are related to each other via symplectic branched covers.

A branched cover of a symplectic 4-manifold with a (possibly singular) symplectic branch curve carries a natural symplectic structure. Conversely, using approximately holomorphic techniques it can be shown that every compact symplectic 4-manifold is a branched cover of the complex projective plane, with a branch curve presenting nodes (of both orientations) and complex cusps as its only singularities (cf. §3). The topology of the 4-manifold and that of the branch curve are closely related to each other; for example, using braid monodromy techniques to study the branch curve, one can reduce the classification of symplectic 4-manifolds to a (hard) question about factorizations in the braid group (cf. §4). Conversely, in some examples the topology of the branch curve complement (in particular its fundamental group) admits a simple description in terms of the total space of the covering (cf. §5).

In the language of branch curves, the failure of most symplectic manifolds to admit integrable complex structures translates into the failure of

From lectures given at the 2004 Clay Mathematics Institute Summer School on Floer Homology, Gauge Theory, and Low Dimensional Topology at the Alfréd Rényi Institute, Budapest, Hungary (`www.claymath.org/programs/summer_school/2004/`).

Partially supported by NSF grant DMS-0244844.

most symplectic branch curves to be isotopic to complex curves. While the symplectic isotopy problem has a negative answer for plane curves with cusp and node singularities, it is interesting to investigate this failure more precisely. Various partial results have been obtained recently about situations where isotopy holds (for smooth curves; for curves of low degree), and about isotopy up to stabilization or regular homotopy (cf. §6). On the other hand, many known examples of non-isotopic curves can be understood in terms of twisting along Lagrangian annuli (or equivalently, Luttinger surgery of the branched covers), leading to some intriguing open questions about the topology of symplectic 4-manifolds versus that of Kähler surfaces.

2. Background

In this section we review various classical facts about symplectic manifolds; the reader unfamiliar with the subject is referred to the book [19] for a systematic treatment of the material.

Recall that a *symplectic form* on a smooth manifold is a 2-form ω such that $d\omega = 0$ and $\omega \wedge \cdots \wedge \omega$ is a volume form. The prototype of a symplectic form is the 2-form $\omega_0 = \sum dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$. In fact, one of the most classical results in symplectic topology, Darboux's theorem, asserts that every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$: hence, unlike Riemannian metrics, symplectic structures have no local invariants.

Since we are interested primarily in compact examples, let us mention compact oriented surfaces (taking ω to be an arbitrary area form), and the complex projective space $\mathbb{CP}^n$ (equipped with the Fubini-Study Kähler form). More generally, since any submanifold to which ω restricts non-degenerately inherits a symplectic structure, all complex projective manifolds are symplectic. However, the symplectic category is strictly larger than the complex projective category, as first evidenced by Thurston in 1976 [36]. In 1994 Gompf obtained the following spectacular result using the *symplectic sum* construction [14]:

Theorem 1 (Gompf)**.** *Given any finitely presented group G, there exists a compact symplectic 4-manifold (X, ω) such that $\pi_1(X) \simeq G$.*

Hence, a general symplectic manifold cannot be expected to carry a complex structure; however, we can equip it with a compatible *almost-complex* structure, i.e. there exists $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathrm{Id}$ and $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric. Hence, at any given point $x \in X$ the tangent space (T_xX, ω, J) can be identified with $(\mathbb{C}^n, \omega_0, i)$, but there is no control over the manner in which J varies from one point to another (J is not *integrable*). In particular, the $\bar\partial$ operator associated to J does not satisfy $\bar\partial^2 = 0$, and hence there are no local holomorphic coordinates.

An important problem in 4-manifold topology is to understand the hierarchy formed by the three main classes of compact oriented 4-manifolds: (1) complex projective, (2) symplectic, and (3) smooth. Each class is a proper subset of the next one, and many obstructions and examples are known, but we are still very far from understanding what exactly causes a smooth 4-manifold to admit a symplectic structure, or a symplectic 4-manifold to admit an integrable complex structure.

One of the main motivations to study symplectic 4-manifolds is that they retain some (but not all) features of complex projective manifolds: for example the structure of their Seiberg-Witten invariants, which in both cases are non-zero and count certain embedded curves [31, 32]. At the same time, every compact oriented smooth 4-manifold with $b_2^+ \geq 1$ admits a "near-symplectic" structure, i.e. a closed 2-form which vanishes along a union of circles and is symplectic over the complement of its zero set [13, 16]; and it appears that some structural properties of symplectic manifolds carry over to the world of smooth 4-manifolds (see e.g. [33, 5]).

Many new developments have contributed to improve our understanding of symplectic 4-manifolds over the past ten years (while results are much scarcer in higher dimensions). Perhaps the most important source of new results has been the study of pseudo-holomorphic curves in their various incarnations: Gromov-Witten invariants, Floer homology, ... (for an overview of the subject see [20]). At the same time, gauge theory (mostly Seiberg-Witten theory, but also more recently Ozsvath-Szabo theory) has made it possible to identify various *obstructions* to the existence of symplectic structures in dimension 4 (cf. e.g. [31, 32]). On the other hand, various new constructions, such as link surgery [11], symplectic sum [14], and symplectic rational blowdown [30] have made it possible to exhibit interesting families of non-Kähler symplectic 4-manifolds. In a slightly different direction, approximately holomorphic geometry (first introduced by Donaldson in [9]) has made it possible to obtain various structure results, showing that symplectic 4-manifolds can be realized as symplectic Lefschetz pencils [10] or as branched covers of $\mathbb{CP}^2$ [2]. In the rest of this paper we will focus on this latter approach, and discuss the topology of *symplectic branched covers* in dimension 4.

3. Symplectic branched covers

Let X and Y be compact oriented 4-manifolds, and assume that Y carries a symplectic form ω_Y.

Definition 2. *A smooth map $f : X \to Y$ is a* symplectic branched covering *if given any point $p \in X$ there exist neighborhoods $U \ni p$, $V \ni f(p)$, and local coordinate charts $\phi : U \to \mathbb{C}^2$ (orientation-preserving) and $\psi : V \to \mathbb{C}^2$ (adapted to ω_Y, i.e. such that ω_Y restricts positively to any complex line in $\mathbb{C}^2$), in which f is given by one of:*

(i) $(x, y) \mapsto (x, y)$ *(local diffeomorphism),*
(ii) $(x, y) \mapsto (x^2, y)$ *(simple branching),*
(iii) $(x, y) \mapsto (x^3 - xy, y)$ *(ordinary cusp).*

These local models are the same as for the singularities of a generic holomorphic map from $\mathbb{C}^2$ to itself, except that the requirements on the local coordinate charts have been substantially weakened. The *ramification curve* $R = \{p \in X,\ \det(df) = 0\}$ is a smooth submanifold of X, and its image $D = f(R)$ is the *branch curve*, described in the local models by the equations $z_1 = 0$ for $(x, y) \mapsto (x^2, y)$ and $27z_1^2 = 4z_2^3$ for $(x, y) \mapsto (x^3 - xy, y)$. The conditions imposed on the local coordinate charts imply that D is a *symplectic curve* in Y (i.e., $\omega_{Y|TD} > 0$ at every point of D). Moreover the restriction of f to R is an immersion everywhere except at the cusps. Hence, besides the ordinary complex cusps imposed by the local model, the only generic singularities of D are transverse double points ("nodes"), which may occur with either the complex orientation or the anti-complex orientation.

We have the following result [2]:

Proposition 3. *Given a symplectic branched covering $f : X \to Y$, the manifold X inherits a natural symplectic structure ω_X, canonical up to isotopy, in the cohomology class $[\omega_X] = f^*[\omega_Y]$.*

The symplectic form ω_X is constructed by adding to $f^*\omega_Y$ a small multiple of an exact form α with the property that, at every point of R, the restriction of α to $\mathrm{Ker}(df)$ is positive. Uniqueness up to isotopy follows from the convexity of the space of such exact 2-forms and Moser's theorem.

Conversely, we can realize every compact symplectic 4-manifold as a symplectic branched cover of $\mathbb{CP}^2$ [2], at least if we assume *integrality*, i.e. if we require that $[\omega] \in H^2(X, \mathbb{Z})$, which does not place any additional restrictions on the diffeomorphism type of X:

Theorem 4. *Given an integral compact symplectic 4-manifold (X^4, ω) and an integer $k \gg 0$, there exists a symplectic branched covering $f_k : X \to \mathbb{CP}^2$, canonical up to isotopy if k is sufficiently large.*

Moreover, the natural symplectic structure induced on X by the Fubini-Study Kähler form and f_k (as given by Proposition 3) agrees with ω up to isotopy and scaling (multiplication by k).

The main tool in the construction of the maps f_k is *approximately holomorphic geometry* [9, 10, 2]. Equip X with a compatible almost-complex structure, and consider a complex line bundle $L \to X$ such that $c_1(L) = [\omega]$: then for $k \gg 0$ the line bundle $L^{\otimes k}$ admits many approximately holomorphic sections, i.e. sections such that $\sup |\bar\partial s| \ll \sup |\partial s|$. Generically, a triple of such sections (s_0, s_1, s_2) has no common zeroes, and determines a projective map $f : p \mapsto [s_0(p) : s_1(p) : s_2(p)]$. Theorem 4 is then proved by constructing triples of sections which satisfy suitable transversality estimates, ensuring that the structure of f near its critical locus is the expected one [2]. (In the complex case it would be enough to pick three generic holomorphic sections, but in the approximately holomorphic context one needs to work harder and obtain uniform transversality estimates on the derivatives of f.)

Because for large k the maps f_k are canonical up to isotopy through symplectic branched covers, the topology of f_k and of its branch curve D_k can be used to define invariants of the symplectic manifold (X,ω). The only generic singularities of the plane curve D_k are nodes (transverse double points) of either orientation and complex cusps, but in a generic one-parameter family of branched covers pairs of nodes with opposite orientations may be cancelled or created. However, recalling that a node of D_k corresponds to the occurrence of two simple branch points in a same fiber of f_k, the creation of a pair of nodes can only occcur in a manner compatible with the branched covering structure, i.e. involving disjoint sheets of the covering. Hence, for large k the sequence of branch curves D_k is, up to isotopy (equisingular deformation among symplectic curves), cancellations and admissible creations of pairs of nodes, an invariant of (X,ω).

The ramification curve of f_k is just a smooth connected symplectic curve representing the homology class Poincaré dual to $3k[\omega]-c_1(TX)$, but the branch curve D_k becomes more and more complicated as k increases: in terms of the symplectic volume and Chern numbers of X, its degree (or homology class) d_k, genus g_k, and number of cusps κ_k are given by

$$d_k = 3k^2\,[\omega]^2 - k\,c_1\cdot[\omega], \qquad 2g_k-2 = 9k^2\,[\omega]^2 - 9k\,c_1\cdot[\omega] + 2c_1^2,$$
$$\kappa_k = 12k^2\,[\omega]^2 - 9k\,c_1\cdot[\omega] + 2c_1^2 - c_2.$$

It is also worth mentioning that, to this date, there is no evidence suggesting that negative nodes actually do occur in these high degree branch curves; our inability to rule our their presence might well be a shortcoming of the approximately holomorphic techniques, rather than an intrinsic feature of symplectic 4-manifolds. So in the following sections we will occasionally consider the more conventional problem of understanding isotopy classes of curves presenting only positive nodes and cusps, although most of the discussion applies equally well to curves with negative nodes.

Assuming that the topology of the branch curve is understood (we will discuss how to achieve this in the next section), one still needs to consider the branched covering f itself. The structure of f is determined by its *monodromy morphism* $\theta:\pi_1(\mathbb{CP}^2-D)\to S_N$, where N is the degree of the covering f. Fixing a base point $p_0\in\mathbb{CP}^2-D$, the image by θ of a loop γ in the complement of D is the permutation of the fiber $f^{-1}(p_0)$ induced by the monodromy of f along γ. (Since viewing this permutation as an element of S_N depends on the choice of an identification between $f^{-1}(p_0)$ and $\{1,\dots,N\}$, the morphism θ is only well-defined up to conjugation by an element of S_N.) By Proposition 3, the isotopy class of the branch curve D and the monodromy morphism θ determine completely the symplectic 4-manifold (X,ω) up to symplectomorphism.

Consider a loop γ which bounds a small topological disc intersecting D transversely once: such a loop plays a role similar to the meridian of a knot, and is called a *geometric generator* of $\pi_1(\mathbb{CP}^2-D)$. Then $\theta(\gamma)$ is a transposition (because of the local model near a simple branch point). Since the image of θ is generated by transpositions and acts transitively on the fiber (assuming X to be connected), θ is a surjective group homomorphism. Moreover, the smoothness of X above the singular points of D imposes certain compatibility conditions on θ. Therefore, not every singular plane curve can be the branch curve of a smooth covering; moreover, the morphism θ, if it exists, is often unique (up to conjugation in S_N). In the case of algebraic curves, this uniqueness property, which holds except for a finite list of well-known counterexamples, is known as Chisini's conjecture, and was essentially proved by Kulikov a few years ago [18].

The upshot of the above discussion is that, in order to understand symplectic 4-manifolds, it is in principle enough to understand singular plane curves. Moreover, if the branch curve of a symplectic covering $f : X \to \mathbb{CP}^2$ happens to be a complex curve, then the integrable complex structure of $\mathbb{CP}^2$ can be lifted to an integrable complex structure on X, compatible with the symplectic structure; this implies that X is a complex projective surface. So, considering the branched coverings constructed in Theorem 4, we have:

Corollary 5. *For $k \gg 0$ the branch curve $D_k \subset \mathbb{CP}^2$ is isotopic to a complex curve (up to node cancellations) if and only if X is a complex projective surface.*

This motivates the study of the *symplectic isotopy problem*, which we will discuss in §6. For now we focus on the use of braid monodromy invariants to study the topology of singular plane curves. In the present context, the goal of this approach is to reduce the classification of symplectic 4-manifolds to a purely algebraic problem, in a manner vaguely reminiscent of the role played by Kirby calculus in the classification of smooth 4-manifolds; as we shall see below, representing symplectic 4-manifolds as branched covers of $\mathbb{CP}^2$ naturally leads one to study the calculus of factorizations in braid groups.

4. The topology of singular plane curves

The topology of singular algebraic plane curves has been studied extensively since Zariski. One of the main tools is the notion of *braid monodromy* of a plane curve, which has been used in particular by Moishezon and Teicher in many papers since the early 1980s in order to study branch curves of generic projections of complex projective surfaces (see [34] for a detailed overview). Braid monodromy techniques can be applied to the more general case of *Hurwitz curves* in ruled surfaces, i.e. curves which behave in a generic manner with respect to the ruling. In the case of $\mathbb{CP}^2$, we consider the projection $\pi : \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$ given by $(x:y:z) \mapsto (x:y)$.

Definition 6. *A curve $D \subset \mathbb{CP}^2$ (not passing through $(0{:}0{:}1)$) is a Hurwitz curve (or braided curve) if D is positively transverse to the fibers of π everywhere except at finitely many points where D is smooth and non-degenerately tangent to the fibers.*

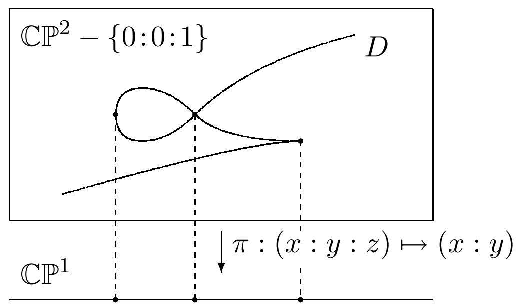

FIGURE 1. A Hurwitz curve in $\mathbb{CP}^2$

The projection π makes D a singular branched cover of $\mathbb{CP}^1$, of degree $d = \deg D = [D] \cdot [\mathbb{CP}^1]$. Each fiber of π is a complex line $\ell \simeq \mathbb{C} \subset \mathbb{CP}^2$, and if ℓ does not pass through any of the singular points of D nor any of its vertical tangencies, then $\ell \cap D$ consists of d distinct points. We can trivialize the fibration π over an affine subset $\mathbb{C} \subset \mathbb{CP}^1$, and define the *braid monodromy morphism*

$$\rho : \pi_1(\mathbb{C} - \mathrm{crit}(\pi_{|D})) \to B_d.$$

Here B_d is the Artin braid group on d strings (the fundamental group of the configuration space $\mathrm{Conf}_d(\mathbb{C})$ of d distinct points in $\mathbb{C}$), and for any loop γ the braid $\rho(\gamma)$ describes the motion of the d points of $\ell \cap D$ inside the fibers of π as one moves along the loop γ.

Equivalently, choosing an ordered system of arcs generating the free group $\pi_1(\mathbb{C} - \mathrm{crit}(\pi_{|D}))$, one can express the braid monodromy of D by a *factorization*

$$\Delta^2 = \prod_i \rho_i$$

of the central element Δ^2 (representing a full rotation by 2π) in B_d, where each factor ρ_i is the monodromy around one of the special points (cusps, nodes, tangencies) of D.

A same Hurwitz curve can be described by different factorizations of Δ^2 in B_d: switching to a different ordered system of generators of $\pi_1(\mathbb{C} - \mathrm{crit}(\pi_{|D}))$ affects the collection of factors $\langle \rho_1, \dots, \rho_r \rangle$ by a sequence of *Hurwitz moves*, i.e. operations of the form

$$\langle \rho_1, \cdots, \rho_i, \rho_{i+1}, \cdots, \rho_r \rangle \longleftrightarrow \langle \rho_1, \cdots, (\rho_i \rho_{i+1} \rho_i^{-1}), \rho_i, \cdots, \rho_r \rangle;$$

and changing the identification between the reference fiber $(\ell, \ell \cap D)$ of π and the base point in $\mathrm{Conf}_d(\mathbb{C})$ affects braid monodromy by a *global conjugation*

$$\langle \rho_1, \cdots, \rho_r \rangle \longleftrightarrow \langle b^{-1} \rho_1 b, \cdots, b^{-1} \rho_r b \rangle.$$

For Hurwitz curves whose only singularities are cusps and nodes (of either orientation), or more generally curves with A_n (and $\overline{A}_n$) singularities, the braid monodromy factorization determines the isotopy type completely (see for example [17]). Hence, determining whether two given Hurwitz curves are isotopic among Hurwitz curves is equivalent to determining whether two given factorizations of Δ^2 coincide up to Hurwitz moves and global conjugation.

It is easy to see that any Hurwitz curve in $\mathbb{CP}^2$ can be made symplectic by an isotopy through Hurwitz curves: namely, the image of any Hurwitz curve by the rescaling map $(x : y : z) \mapsto (x : y : \lambda z)$ is a Hurwitz curve, and symplectic for $|\lambda| \ll 1$. On the other hand, a refinement of Theorem 4 makes it possible to assume without loss of generality that the branch curves $D_k \subset \mathbb{CP}^2$ are Hurwitz curves [7]. So, from now on we can specifically consider symplectic coverings with Hurwitz branch curves. In this setting, braid monodromy gives a purely combinatorial description of the topology of compact (integral) symplectic 4-manifolds.

The braid monodromy of the branch curves D_k given by Theorem 4 can be computed explicitly for various families of complex projective surfaces (non-Kähler examples are currently beyond reach). In fact, in the complex case the branched coverings f_k are isotopic to generic projections of projective embeddings. Accordingly, most of these computations rely purely on methods from algebraic geometry, using the degeneration techniques extensively developed by Moishezon and Teicher (see [1, 21, 22, 24, 26, 34, 35] and references within); but approximately holomorphic methods can be used to simplify the calculations and bring a whole new range of examples within reach [6]. This includes some complex surfaces of general type which are mutually homeomorphic and have identical Seiberg-Witten invariants but of which it is unknown whether they are symplectomorphic or even diffeomorphic (the *Horikawa surfaces*).

However, the main obstacle standing in the way of this approach to the topology of symplectic 4-manifolds is the intractability of the so-called "Hurwitz problem" for braid monodromy factorizations: namely, there is no algorithm to decide whether two given braid monodromy factorizations are identical up to Hurwitz moves. Therefore, since we are unable to compare braid monodromy factorizations, we have to extract the information contained in them by indirect means, via the introduction of more manageable (but less powerful) invariants.

5. Fundamental groups of branch curve complements

The idea of studying algebraic plane curves by determining the fundamental groups of their complements is a very classical one, which goes back to Zariski and Van Kampen. More recently, Moishezon and Teicher have shown that fundamental groups of branch curve complements can be used as a major tool to further our understanding of complex projective surfaces (cf. e.g. [21, 25, 34]). By analogy with the situation for knots in S^3, one expects the topology of the complement to carry a lot of information about the curve; however in this case the fundamental group does not determine the isotopy type. For an algebraic curve in $\mathbb{CP}^2$, or more generally for a Hurwitz curve, the fundamental group of the complement is determined in an explicit manner by the braid monodromy factorization, via the Zariski-Van Kampen theorem. Hence, calculations of fundamental groups of complements usually rely on braid monodromy techniques.

A close examination of the available data suggests that, contrarily to what has often been claimed, in the specific case of generic projections of complex surfaces projectively embedded by sections of a sufficiently ample linear system (i.e. taking $k \gg 0$ in Theorem 4), the fundamental group of the branch curve complement may be determined in an elementary manner by the topology of the surface (see below).

In the symplectic setting, the fundamental group of the complement of the branch curve D of a covering $f : X \to \mathbb{CP}^2$ is affected by node creation or cancellation operations. Indeed, adding pairs of nodes (in a manner compatible with the monodromy morphism $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_N$) introduces additional commutation relations between geometric generators of the fundamental group. Hence, it is necessary to consider a suitable "symplectic stabilization" of $\pi_1(\mathbb{CP}^2 - D)$ [6]:

Definition 7. *Let K be the normal subgroup of $\pi_1(\mathbb{CP}^2 - D)$ generated by the commutators $[\gamma, \gamma']$ for all pairs γ, γ' of geometric generators such that $\theta(\gamma)$ and $\theta(\gamma')$ are disjoint commuting transpositions. Then the symplectic stabilization of $\pi_1(\mathbb{CP}^2 - D)$ is the quotient $\bar{G} = \pi_1(\mathbb{CP}^2 - D)/K$.*

Considering the branch curves D_k of the coverings given by Theorem 4, we have the following result [6]:

Theorem 8 (A.-Donaldson-Katzarkov-Yotov)**.** *For $k \gg 0$, the stabilized group $\bar{G}_k(X, \omega) = \pi_1(\mathbb{CP}^2 - D_k)/K_k$ is an invariant of the symplectic manifold (X^4, ω).*

The fundamental group of the complement of a plane branch curve $D \subset \mathbb{CP}^2$ comes naturally equipped with two morphisms: the symmetric group valued monodromy homomorphism θ discussed above, and the abelianization map $\delta : \pi_1(\mathbb{CP}^2 - D) \to H_1(\mathbb{CP}^2 - D, \mathbb{Z})$. Since we only consider irreducible branch curves, we have $H_1(\mathbb{CP}^2 - D, \mathbb{Z}) \simeq \mathbb{Z}_d$, where $d = \deg D$, and δ counts the linking number (mod d) with the curve D. The morphisms θ and δ are surjective, but the image of $(\theta, \delta) : \pi_1(\mathbb{CP}^2 - D) \to S_N \times \mathbb{Z}_d$ is the index 2 subgroup consisting of all pairs (σ, p) such that the permutation σ and the integer p have the same parity (note that d is always even). The subgroup K introduced in Definition 7 lies in the kernel of (θ, δ); therefore, setting $G^0 = \mathrm{Ker}(\theta, \delta)/K$, we have an exact sequence

$$1 \longrightarrow G^0 \longrightarrow \bar{G} \xrightarrow{(\theta,\delta)} S_N \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

Moreover, assume that the symplectic 4-manifold X is simply connected, and denote by $L = f^*[\mathbb{CP}^1]$ the pullback of the hyperplane class and by $K_X = -c_1(TX)$ the canonical class. Then we have the following result [6]:

Theorem 9 (A.-Donaldson-Katzarkov-Yotov)**.** *If $\pi_1(X) = 1$ then there is a natural surjective homomorphism $\phi : \mathrm{Ab}(G^0) \twoheadrightarrow (\mathbb{Z}^2/\Lambda)^{N-1}$, where $\Lambda = \{(L \cdot C, K_X \cdot C),\ C \in H_2(X,\mathbb{Z})\} \subset \mathbb{Z}^2$.*

The fundamental groups of the branch curve complements have been computed for generic polynomial maps to $\mathbb{CP}^2$ on various algebraic surfaces, using braid monodromy techniques (cf. §4) and the Zariski-Van Kampen theorem. Since in the symplectic setting Theorem 4 gives uniqueness up to isotopy only for $k \gg 0$, we restrict ourselves to those examples for which the fundamental groups have been computed for $\mathbb{CP}^2$-valued maps of arbitrarily large degree.

The first such calculations were carried out by Moishezon and Teicher, for $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ [22], and Hirzebruch surfaces ([24], see also [6]); the answer is also known for some specific linear systems on rational surfaces and K3 surfaces realized as complete intersections (by work of Robb [26], see also related papers by Teicher et al). Additionally, the symplectic stabilizations of the fundamental groups have been computed for all double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along connected smooth algebraic curves [6], which includes an infinite family of surfaces of general type.

In all these examples it turns out that, if one considers projections of sufficiently large degree (i.e., assuming $k \geq 3$ for $\mathbb{CP}^2$ and $k \geq 2$ for the other examples), the structure of G^0 is very simple, and obeys the following conjecture:

Conjecture 10. *Assume that X is a simply connected algebraic surface and $k \gg 0$. Then:* (1) *the symplectic stabilization operation is trivial, i.e. $K = \{1\}$ and $\bar{G} = \pi_1(\mathbb{CP}^2 - D)$;* (2) *the homomorphism $\phi : \mathrm{Ab}(G^0) \to (\mathbb{Z}^2/\Lambda)^{N-1}$ is an isomorphism; and* (3) *the commutator subgroup $[G^0, G^0]$ is a quotient of $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

6. The symplectic isotopy problem

The symplectic isotopy problem asks under which conditions (assumptions on degree, genus, types and numbers of singular points) it is true that any symplectic curve in $\mathbb{CP}^2$ (or more generally in a complex surface) is symplectically isotopic to a complex curve (by isotopy, we mean a continuous family of symplectic curves with the same singularities).

The first result in this direction is due to Gromov, who proved that every smooth symplectic curve of degree 1 or 2 in $\mathbb{CP}^2$ is isotopic to a complex curve [15]. The argument relies on a careful study of the deformation problem for pseudo-holomorphic curves: starting from an almost-complex structure J for which the given curve C is pseudo-holomorphic, and considering a family of almost-complex structures $(J_t)_{t\in[0,1]}$ interpolating between J and the standard complex structure, one can prove the existence of smooth J_t-holomorphic curves C_t realizing an isotopy between C and a complex curve.

The isotopy property is expected to hold for smooth and nodal curves in all degrees, and also for curves with sufficiently few cusps. For smooth curves, successive improvements of Gromov's result have been obtained by Sikorav (for degree 3), Shevchishin (for degree ≤ 6), and more recently Siebert and Tian [28]:

Theorem 11 (Siebert-Tian)**.** *Every smooth symplectic curve of degree* ≤ 17 *in* $\mathbb{CP}^2$ *is symplectically isotopic to a complex curve.*

Some results have been obtained by Barraud and Shevchishin for nodal curves of low genus. For example, the following result holds [27]:

Theorem 12 (Shevchishin)**.** *Every irreducible nodal symplectic curve of genus* $g \leq 4$ *in* $\mathbb{CP}^2$ *is symplectically isotopic to a complex curve.*

Moreover, work in progress by S. Francisco is expected to lead to an isotopy result for curves of low degree with node and cusp singularities (subject to specific constraints on the number of cusps).

If one aims to classify symplectic 4-manifolds by enumerating all branched covers of $\mathbb{CP}^2$ according to the degree and number of singularities of the branch curve, then the above cases are those for which the classification is the simplest and does not include any non-Kähler examples. On the other hand, Corollary 5 implies that the isotopy property cannot hold for all curves with node and cusp singularities; in fact, explicit counterexamples have been constructed by Moishezon [23] (see below).

Even when the isotopy property fails, the classification of singular plane curves becomes much simpler if one considers an equivalence relation weaker than isotopy, such as *regular homotopy*, or *stable isotopy*. Namely, let D_1, D_2 be two Hurwitz curves (see Definition 6) in $\mathbb{CP}^2$ (or more generally in a rational ruled surface), with node and cusp singularities (or more generally singularities of type A_n). Assume that D_1 and D_2 represent the same homology class, and that they have the same numbers of singular points of each type. Then we have the following results [8, 17]:

Theorem 13 (A.-Kulikov-Shevchishin)**.** *Under the above assumptions,* D_1 *and* D_2 *are* regular homotopic *among Hurwitz curves, i.e. they are isotopic up to creations and cancellations of pairs of nodes.*

Theorem 14 (Kharlamov-Kulikov)**.** *Under the above assumptions, let* D'_i ($i \in \{1,2\}$) *be the curve obtained by adding to* D_i *a union of* n *generic lines (or fibers of the ruling) intersecting* D_i *transversely at smooth points, and smoothing out all the resulting intersections. Then for all large enough values of* n *the Hurwitz curves* D'_1 *and* D'_2 *are isotopic.*

Unfortunately, Theorem 13 does not seem to have any implications for the topology of symplectic 4-manifolds, because the node creation operations appearing in the regular homotopy need not be admissible: even if both D_1 and D_2 are branch curves of symplectic coverings, the homotopy may involve plane curves for which the branched cover is not smooth. For similar reasons, the applicability of Theorem 14 to branch curves is limited to the case of double covers, i.e. symplectic 4-manifolds which admit *hyperelliptic* Lefschetz fibrations. In particular, for genus 2 Lefschetz fibrations we have the following result [3]:

Theorem 15. *If the symplectic 4-manifold X admits a genus 2 Lefschetz fibration, then X becomes complex projective after stabilization by fiber sums with rational surfaces along genus 2 curves.*

It follows from Theorem 14 that this result extends to all Lefschetz fibrations with monodromy contained in the hyperelliptic mapping class group. However, few symplectic 4-manifolds admit such fibrations, and in general the following question remains open:

Question 16. *Let X_1, X_2 be two integral compact symplectic 4-manifolds with the same $(c_1^2,\, c_2,\, c_1\cdot[\omega],\, [\omega]^2)$. Do X_1 and X_2 become symplectomorphic after sufficiently many fiber sums with the same complex projective surfaces (chosen among a finite collection of model holomorphic fibrations)?*

This question can be thought of as the symplectic analogue of the classical result of Wall which asserts that any two simply connected smooth 4-manifolds with the same intersection form become diffeomorphic after repeatedly performing connected sums with $S^2 \times S^2$ [37].

A closer look at the known examples of non-isotopic singular plane curves suggests that an even stronger statement might hold.

It was first observed in 1999 by Fintushel and Stern [12] that many symplectic 4-manifolds contain infinite families of non-isotopic smooth connected symplectic curves representing the same homology class (see also [29]). The simplest examples are obtained by "braiding" parallel copies of the fiber in an elliptic surface, and are distinguished by comparing the Seiberg-Witten invariants of the corresponding double branched covers. Other examples have been constructed by Smith, Etgü and Park, and Vidussi. However, for singular plane curves the first examples were obtained by Moishezon more than ten years ago [23]:

Theorem 17 (Moishezon)**.** *For all $p \geq 2$, there exist infinitely many pairwise non-isotopic singular symplectic curves of degree $9p(p-1)$ in $\mathbb{CP}^2$ with $27(p-1)(4p-5)$ cusps and $\frac{27}{2}(p-1)(p-2)(3p^2+3p-8)$ nodes, not isotopic to any complex curve.*

Moishezon's approach is purely algebraic (using braid monodromy factorizations), and very technical; the curves that he constructs are distinguished by the fundamental groups of their complements [23]. However a much simpler geometric description of this construction can be given in terms of braiding operations, which makes it possible to distinguish the curves just by comparing the canonical classes of the associated branched covers [4].

Given a symplectic covering $f : X \to Y$ with branch curve D, and given a Lagrangian annulus A with interior in $Y \setminus D$ and boundary contained in D, we can *braid* the curve D along the annulus A by performing the local

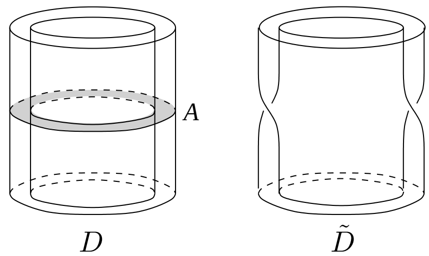

FIGURE 2. The braiding construction

operation depicted on Figure 2. Namely, we cut out a neighborhood U of A, and glue it back via a non-trivial diffeomorphism which interchanges two of the connected components of $D \cap \partial U$, in such a way that the product of S^1 with the trivial braid is replaced by the product of S^1 with a half-twist (see [4] for details).

Braiding the curve D along the Lagrangian annulus A affects the branched cover X by a *Luttinger surgery* along a smooth embedded Lagrangian torus T which is one of the connected components of $f^{-1}(A)$ [4]. This operation consists of cutting out from X a tubular neighborhood of T, foliated by parallel Lagrangian tori, and gluing it back via a symplectomorphism wrapping the meridian around the torus (in the direction of the preimage of an arc joining the two boundaries of A), while the longitudes are not affected.

The starting point of Moishezon's construction is the complex curve D_0 obtained by considering $3p(p-1)$ smooth cubics in a pencil, removing balls around the 9 points where these cubics intersect, and inserting into each location the branch curve of a generic degree p polynomial map from $\mathbb{CP}^2$ to itself. By repeatedly braiding D_0 along a well-chosen Lagrangian annulus, one obtains symplectic curves D_j, $j \in \mathbb{Z}$. Moishezon's calculations show that, whereas for the initial curve the fundamental group of the complement $\pi_1(\mathbb{CP}^2 - D_0)$ is infinite, the groups $\pi_1(\mathbb{CP}^2 - D_j)$ are finite for all $j \neq 0$, and of different orders [23]. On the other hand, it is fairly easy to check that, as expected from Theorem 9, this change in fundamental groups can be detected by considering the canonical class of the p^2-fold covering X_j of $\mathbb{CP}^2$ branched along D_j. Namely, the canonical class of X_0 is proportional to the cohomology class of the symplectic form induced by the branched covering: $c_1(K_{X_0}) = \lambda[\omega_{X_0}]$, where $\lambda = \frac{6p-9}{p}$. On the other hand, $c_1(K_{X_j}) = \lambda[\omega_{X_j}] + \mu\, j\, [T]^{PD}$, where $\mu = \frac{2p-3}{p} \neq 0$, and the homology class $[T]$ of the Lagrangian torus T is not a torsion element in $H_2(X_j, \mathbb{Z})$ [4].

Many constructions of non-Kähler symplectic 4-manifolds can be thought of in terms of twisted fiber sum operations, or Fintushel-Stern surgery along fibered links. However the key component in each of these constructions can be understood as a particular instance of Luttinger surgery; so it makes sense to ask to what extent Luttinger surgery may be responsible for the greater variety of symplectic 4-manifolds compared to complex surfaces. More precisely, we may ask the following questions:

Question 18. *Let D_1, D_2 be two symplectic curves with nodes and cusps in $\mathbb{CP}^2$, of the same degree and with the same numbers of nodes and cusps. Is it always possible to obtain D_2 from D_1 by a sequence of braiding operations along Lagrangian annuli?*

Question 19. *Let X_1, X_2 be two integral compact symplectic 4-manifolds with the same $(c_1^2,\, c_2,\, c_1\cdot[\omega],\, [\omega]^2)$. Is it always possible to obtain X_2 from X_1 by a sequence of Luttinger surgeries?*

This question is the symplectic analogue of a question asked by Ron Stern about smooth 4-manifolds, namely whether any two simply connected smooth 4-manifolds with the same Euler characteristic and signature differ from each other by a sequence of logarithmic transformations. However, here we do not require the manifolds to be simply connected, we do not even require them to have the same fundamental group.

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