THÈSE D'HABILITATION DE L'UNIVERSITÉ PARIS XI

Spécialité : MATHÉMATIQUES

présentée par

Denis AUROUX

Sujet de la thèse :

TECHNIQUES APPROXIMATIVEMENT HOLOMORPHES ET INVARIANTS DE MONODROMIE EN TOPOLOGIE SYMPLECTIQUE.

Remerciements

Je tiens tout d'abord à remercier ceux qui par leurs conseils, leurs encouragements et leur disponibilité ont eu une influence décisive sur le début de ma carrière de mathématicien : en premier lieu Jean Pierre Bourguignon et Misha Gromov, mais aussi Pierre Pansu, François Laudenbach, Jean-Claude Sikorav et Claude Viterbo.

Je remercie également tous ceux qui par de nombreuses discussions enrichissantes ont contribué à la réalisation de ces travaux : tout d'abord Ludmil Katzarkov pour une collaboration fructueuse au fil des années, ainsi que Simon Donaldson, Cliff Taubes, Gang Tian, Tom Mrowka, Ivan Smith, Paul Seidel, Bob Gompf, Emmanuel Giroux, Paul Biran, Fran Presas et bien d'autres encore.

Enfin je tiens à remercier tous les membres du Centre de Mathématiques de l'École Polytechnique et du Département de Mathématiques du M.I.T. pour les conditions de travail et l'environnement scientifique exceptionnels dont j'ai bénéficié.

Adresse de l'auteur : Denis AUROUX
Centre de Mathématiques, École Polytechnique
91128 Palaiseau Cedex, France.

e-mail : auroux@math.polytechnique.fr

Table des matières

PRÉSENTATION DES TRAVAUX

1. Introduction

Les travaux décrits dans ce texte portent sur l'étude de la topologie des variétés symplectiques compactes à l'aide de techniques de géométrie approximativement holomorphe. L'objectif est, par analogie avec la géométrie algébrique complexe, de construire des "systèmes linéaires" sur les variétés symplectiques compactes, puis d'utiliser ces objets pour définir de nouveaux invariants topologiques. Ces invariants, très différents de ceux obtenus par des méthodes de théorie de jauge ou de comptage de courbes pseudo-holomorphes, laissent espérer une meilleure compréhension de la topologie des variétés symplectiques et notamment des différences entre variétés symplectiques et variétés kählériennes.

1.1. **Rappels et généralités.** Rappelons qu'une *forme symplectique* sur une variété C^∞ est une 2-forme ω fermée ($d\omega = 0$) et non dégénérée ($\omega^n = \mathrm{vol} > 0$). Contrairement au cas riemannien où la courbure est un invariant local, toutes les variétés symplectiques sont localement symplectomorphes à $\mathbb{R}^{2n}$ muni de la forme standard $\omega_0 = \sum dx_i \wedge dy_i$ (théorème de Darboux). Le problème de la classification des variétés symplectiques est donc avant tout de nature topologique.

Les surfaces de Riemann $(\Sigma, \mathrm{vol}_\Sigma)$ sont des variétés symplectiques; de façon plus générale, toute variété kählérienne est symplectique, ce qui inclut toutes les variétés projectives complexes. Toutefois la catégorie symplectique est beaucoup plus vaste que celle des variétés complexes : ainsi Gompf a montré en 1994 que tout groupe de présentation finie peut être réalisé comme le groupe fondamental d'une variété symplectique compacte de dimension 4 [Go1], alors que pour une variété kählérienne le premier nombre de Betti est toujours pair.

A défaut d'être complexe, toute variété symplectique admet une structure *presque complexe* compatible, i.e. un endomorphisme $J \in \mathrm{End}(TX)$ vérifiant $J^2 = -\mathrm{Id}$ et tel que $g(u, v) := \omega(u, Jv)$ est une métrique riemannienne. En tout point, (X, ω, J) est semblable à $(\mathbb{C}^n, \omega_0, i)$, mais J n'est pas nécessairement *intégrable* : ainsi $\nabla J \neq 0$, $\bar\partial^2 \neq 0$, et le crochet de Lie de deux champs de vecteurs de type $(1, 0)$ n'est pas nécessairement de type $(1, 0)$. Sur une variété symplectique, il n'y a donc en général pas de fonctions holomorphes, même localement, et en particulier pas de coordonnées locales holomorphes.

Les problèmes auxquels s'attaque la topologie symplectique sont des questions telles que : quelles variétés lisses admettent des structures symplectiques ? peut-on classifier les structures symplectiques sur une variété lisse donnée ? Il faut noter qu'un résultat classique de Moser indique que, si la classe de cohomologie $[\omega] \in H^2(X, \mathbb{R})$ est fixée, alors les déformations de la structure symplectique sont triviales. Les motivations pour l'étude des variétés symplectiques sont aussi bien d'ordre physique (mécanique classique ; théorie des cordes ; ...) que géométrique.

Certaines propriétés des variétés complexes s'étendent au cas symplectique, mais c'est loin d'être la règle générale. C'est en dimension 4 que la situation est la mieux connue, notamment grâce aux travaux de Taubes sur la structure des invariants de Seiberg-Witten des variétés symplectiques et leur relation avec les invariants de Gromov-Witten [Ta]. En revanche, lorsque $\dim X \geq 6$, il y a très peu de résultats, et par exemple aucune obstruction non triviale (en-dehors des conditions cohomologiques évidentes liées au fait que ω est non dégénérée) n'est connue à l'existence d'une structure symplectique sur une variété donnée.

1.2. **Géométrie approximativement holomorphe.** L'idée de base introduite au milieu des années 90 par Donaldson, et que j'ai développée par la suite, est la suivante : en présence d'une structure presque complexe, le défaut d'intégrabilité est une obstruction à l'existence d'objets holomorphes (sections de fibrés, systèmes linéaires), mais on peut travailler de façon similaire avec des objets approximativement holomorphes.

Soit (X, ω) une variété symplectique compacte de dimension $2n$. On supposera tout du long que $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$; cette condition d'intégralité ne restreint pas le type topologique de X, car toute forme symplectique peut être perturbée jusqu'à rendre sa classe de cohomologie rationnelle, puis entière après multiplication par un facteur constant. Soit J une structure presque complexe compatible avec ω, et soit $g(.,.) = \omega(., J.)$ la métrique riemannienne correspondante.

On considère un fibré en droites complexes L sur X tel que $c_1(L) = \frac{1}{2\pi}[\omega]$, muni d'une métrique hermitienne et d'une connexion hermitienne ∇^L dont la forme de courbure est $F(\nabla^L) = -i\omega$. La structure presque complexe induit une décomposition de la connexion : $\nabla^L = \partial^L + \bar{\partial}^L$, avec $\partial^L s(v) = \frac{1}{2}(\nabla^L s(v) - i\nabla^L s(Jv))$ et $\bar{\partial}^L s(v) = \frac{1}{2}(\nabla^L s(v) + i\nabla^L s(Jv))$.

Si la structure presque complexe J est intégrable, i.e. si X est une variété complexe kählérienne, alors le fibré L est un fibré en droites holomorphe ample, c'est-à-dire que pour k suffisamment grand le fibré $L^{\otimes k}$ admet de nombreuses sections holomorphes. Dans ce cas, la variété X se plonge dans un espace projectif (par un théorème de Kodaira); des sections hyperplanes génériques fournissent des hypersurfaces lisses de X (par le théorème de Bertini), et plus généralement le système linéaire formé par les sections de $L^{\otimes k}$ permet de construire diverses structures : pinceaux de Lefschetz, ...

Lorsque la variété X est seulement symplectique, si le défaut d'intégrabilité de J empêche l'existence de sections holomorphes, il est cependant possible de trouver un modèle local *approximativement holomorphe* : un voisinage d'un point $x \in X$ muni de la forme symplectique ω et de la structure presque complexe J s'identifie à un voisinage de l'origine dans $\mathbb{C}^n$ muni de la forme symplectique standard ω_0 et d'une structure presque complexe de la forme $i + O(|z|)$. Dans ce modèle local, le fibré $L^{\otimes k}$ muni de la connexion $\nabla = (\nabla^L)^{\otimes k}$ de courbure $-ik\omega$ peut s'identifier au fibré trivial $\underline{\mathbb{C}}$ muni de la connexion $d + \frac{k}{4}\sum(z_j\, d\bar{z}_j - \bar{z}_j\, dz_j)$. La section de $L^{\otimes k}$ définie localement par $s_{k,x}(z) = \exp(-\frac{1}{4}k|z|^2)$ est alors approximativement holomorphe [Do1]. Plus précisément :

Définition 1.1. *Une suite de sections s_k de $L^{\otimes k}$ est dite* approximativement holomorphe *si, pour la métrique redimensionnée $g_k = kg$, et en normalisant les sections s_k de sorte que $\|s_k\|_{C^r, g_k} \sim C$, on a une inégalité de la forme $\|\bar{\partial} s_k\|_{C^{r-1}, g_k} < C' k^{-1/2}$, où C et C' sont des constantes indépendantes de k.*

Le changement de métrique, qui dilate les distances d'un facteur $\sqrt{k}$, est nécessaire pour l'obtention d'estimées uniformes du fait de la courbure de plus en plus grande du fibré $L^{\otimes k}$. L'idée intuitive est que, pour k grand, les sections du fibré $L^{\otimes k}$ de courbure $-ik\omega$ voient la géométrie de X à petite échelle (de l'ordre de $1/\sqrt{k}$), ce qui rend la structure presque complexe J presque intégrable et permet d'approcher de mieux en mieux la condition d'holomorphie $\bar\partial s = 0$.

Il est à noter que, la condition ci-dessus étant ouverte, il n'est pas possible de définir un "espace de sections approximativement holomorphes" de $L^{\otimes k}$ de façon simple (cf. les travaux de Borthwick et Uribe [BU] ou de Shiffman et Zelditch pour d'autres approches du problème).

Les techniques approximativement holomorphes ont permis d'obtenir de nombreux résultats d'existence et de structure pour les variétés symplectiques compactes, à commencer par la construction de sous-variétés symplectiques obtenue par Donaldson vers 1995 [Do1]. Il s'agit essentiellement, une fois obtenues de nombreuses sections approximativement holomorphes, d'en trouver certaines dont le comportement géométrique est aussi générique que possible. Plusieurs résultats de ce type sont décrits dans la suite de ce texte, ainsi que leurs applications à l'obtention de nouveaux invariants topologiques permettant de caractériser les variétés symplectiques.

Pour terminer cette introduction, il est à mentionner que plusieurs des résultats de topologie symplectique obtenus à l'aide des techniques approximativement holomorphes ont été transposés au cadre de la géométrie de contact (en dimension impaire) par Presas et ses collaborateurs (cf. par exemple [IMP]), le résultat le plus spectaculaire étant un théorème d'existence de structures de livres ouverts sur les variétés de contact compactes récemment obtenu par Giroux et Mohsen [GM].

2. Constructions de sous-variétés symplectiques

En conservant les notations introduites au §1.2, le premier résultat obtenu à l'aide des méthodes approximativement holomorphes est le théorème d'existence de sous-variétés symplectiques suivant, dû à Donaldson [Do1] :

Théorème 2.1 (Donaldson). *Pour $k \gg 0$, $L^{\otimes k}$ admet des sections approximativement holomorphes s_k dont les lieux d'annulation W_k sont des hypersurfaces symplectiques lisses.*

Ce résultat part de l'observation que, si la section s_k s'annule transversalement et si l'on a $|\bar\partial s_k(x)| \ll |\partial s_k(x)|$ en tout point de $W_k = s_k^{-1}(0)$, alors la sous-variété W_k est symplectique (i.e., $\omega_{|W_k}$ est non dégénérée, ce qui implique que $(W_k, \omega_{|W_k})$ est symplectique), et même approximativement J-holomorphe (i.e. $J(TW_k)$ est proche de TW_k). Le point crucial est donc d'obtenir une borne inférieure en tout point de W_k pour ∂s_k, pour compenser le défaut d'holomorphie.

Définition 2.2. *On dit que des sections s_k de $L^{\otimes k}$ sont* uniformément transverses à 0 *s'il existe une constante $\eta > 0$ (indépendante de k) telle que, en tout point de X tel que $|s_k(x)| < \eta$, on a $|\partial s_k(x)|_{g_k} > \eta$.*

Pour des sections approximativement holomorphes, l'obtention d'une telle estimée uniforme de transversalité suffit à obtenir le Théorème 2.1. La construction de telles sections comporte deux grandes étapes : la première est un résultat de transversalité effectif local pour des fonctions à valeurs complexes, et fait appel à un résultat de Yomdin sur la complexité des ensembles semi-algébriques réels; la seconde étape est un procédé de globalisation original, qui permet par perturbations successives des sections s_k d'obtenir des propriétés de transversalité uniforme sur des ouverts de plus en plus grands jusqu'à recouvrir X [Do1].

La première étape (résultat de transversalité effectif local) a récemment fait l'objet d'une simplification notable [Au5], qui permet de formuler l'argument de façon nettement plus courte et sans faire appel aux travaux de Yomdin. L'observation élémentaire qui sous-tend cette simplification est la suivante : une fonction holomorphe $f : \mathbb{C}^n \to \mathbb{C}$ s'annule transversalement (i.e., 0 est une valeur régulière) si et seulement si son 1-jet (f, df) ne s'annule nulle part. Dès lors, en travaillant sur les 1-jets plutôt que sur les fonctions, et en s'autorisant des perturbations affines plutôt que constantes, il devient nettement plus facile d'obtenir un résultat de transversalité uniforme de type Sard pour les fonctions approximativement holomorphes.

Les sous-variétés symplectiques construites par Donaldson possèdent plusieurs propriétés spécifiques qui les rapprochent davantage des sous-variétés complexes que des sous-variétés symplectiques générales. Tout d'abord, elles vérifient le théorème de l'hyperplan de Lefschetz : jusqu'en dimension moitié, les groupes d'homologie et d'homotopie de W_k sont identiques à ceux de X [Do1]. De façon plus importante, elles vérifient une propriété d'unicité asymptotique décrite dans [Au1] :

Théorème 2.3 ([Au1]). *Pour k suffisamment grand fixé, les sous-variétés W_k obtenues à l'aide du Théorème 2.1 pour des constantes de transversalité données sont, à isotopie symplectique près, indépendantes de tous les choix effectués (y compris celui de la structure presque complexe J).*

Ce résultat d'unicité, qui repose sur une extension de l'argument de Donaldson au cas de familles de sections dépendant d'un paramètre réel, est intéressant car il permet d'envisager l'utilisation d'invariants topologiques ou symplectiques associés aux sous-variétés W_k pour caractériser la variété X. De plus, il donne une information sur le problème d'isotopie symplectique, qui consiste à déterminer si les courbes symplectiques connexes représentant une classe d'homologie donnée dans une variété symplectique de dimension 4 sont toutes mutuellement isotopes ou non : les sous-variétés construites par Donaldson ne peuvent pas fournir de contre-exemples à ce problème dont la réponse générale est négative (cf. par exemple [FS]), mais pour lequel aucun contre-exemple n'est connu pour les courbes lisses connexes dont le fibré normal est de degré positif.

Le comportement géométrique des sous-variétés symplectiques obtenues à l'aide des techniques approximativement holomorphes est remarquable, en particulier vis-à-vis des sous-variétés lagrangiennes (ou plus généralement isotropes). Ainsi, de façon analogue à ce qui se produit en géométrie kählérienne, les sous-variétés isotropes vérifient une propriété de *convexité rationnelle*, ce qui se manifeste notamment par la possibilité de choisir les hypersurfaces symplectiques W_k de telle sorte qu'elles évitent une sous-variété isotrope donnée tout en passant par un point arbitrairement choisi du complémentaire.

Le résultat suivant a également été obtenu de façon indépendante par Damien Gayet et Jean-Paul Mohsen, et a fait l'objet de la publication commune [AGM] :

Théorème 2.4 ([AGM]). *Soit $\mathcal{L} \subset X$ une sous-variété isotrope compacte, et soit $N = |\mathrm{Tor}\, H_1(\mathcal{L}, \mathbb{Z})|$. Alors, pour tout entier k multiple de N suffisamment grand, il* *existe des sections approximativement holomorphes de $L^{\otimes k}$ dont les lieux d'annulation W_k sont des sous-variétés symplectiques lisses disjointes de $\mathcal{L}$, passant par un point quelconque de $X - \mathcal{L}$.*

Le Théorème 2.4 présente un intérêt particulier en relation avec le résultat de structure obtenu par Biran [Bi], qui permet de décomposer la variété X en d'une part un fibré en disques sur W_k, et d'autre part un complexe cellulaire isotrope. En principe, il devrait également permettre d'obtenir des obstructions à l'existence de certains plongements lagrangiens (cf. notamment les travaux de Nemirovski).

3. Systèmes linéaires approximativement holomorphes

Si les sous-variétés symplectiques dont la construction et les propriétés ont été abordées au §2 offrent déjà de vastes perspectives, les applications les plus intéressantes des techniques aproximativement holomorphes en topologie symplectique font intervenir des *systèmes linéaires* engendrés par deux sections ou plus, qui permettent de munir les variétés symplectiques compactes de diverses structures topologiques extrêmement riches telles que pinceaux de Lefschetz, applications projectives, etc...

3.1. **Pinceaux de Lefschetz symplectiques.** Si l'on considère non plus une, mais deux sections de $L^{\otimes k}$, Donaldson a montré qu'il est possible de munir la variété X de structures de *pinceaux de Lefschetz symplectiques* [Do2, Do3] : un couple de sections approximativement holomorphes (s_k^0, s_k^1) de $L^{\otimes k}$ convenablement choisies définit une famille d'hypersurfaces $\Sigma_{k,\alpha} = \{x \in X,\ s_k^0(x) - \alpha s_k^1(x) = 0\}$, $\alpha \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Les sous-variétés $\Sigma_{k,\alpha}$ sont symplectiques, et elles sont toutes lisses excepté un nombre fini d'entre elles qui présentent une singularité isolée (un point double ordinaire) ; elles s'intersectent le long des *points base* du pinceau, qui forment une sous-variété symplectique lisse $Z_k = \{s_k^0 = s_k^1 = 0\}$ de codimension 4. Ainsi, lorsque $\dim X = 4$ (le cas le plus étudié), les fibres sont des surfaces compactes orientées dont certaines présentent une singularité nodale, et Z_k est constitué d'un nombre fini de points.

On peut également définir l'application projective $f_k = [s_k^0 : s_k^1] : X - Z_k \to \mathbb{CP}^1$, dont les points critiques correspondent aux singularités des fibres $\Sigma_{k,\alpha} = f_k^{-1}(\alpha) \cup Z_k$. La fonction f_k est une fonction de Morse complexe, c'est-à-dire qu'au voisinage d'un point critique on a un modèle local $f_k(z) = z_1^2 + \cdots + z_n^2$ en coordonnées approximativement holomorphes.

L'argument de Donaldson repose à nouveau sur des perturbations successives des sections s_k^0 et s_k^1 afin d'obtenir des propriétés de transversalité uniforme, non seulement pour les sections (s_k^0, s_k^1) mais aussi pour la dérivée ∂f_k [Do3]. Il est à noter que la partie la plus technique de l'argument peut être simplifiée de la même façon que pour la construction de sous-variétés [Au5].

Donaldson montre également que, pour $k \gg 0$ fixé, les pinceaux de Lefschetz obtenus sont tous identiques à isotopie près, indépendamment des choix de construction.

3.2. **Revêtements ramifiés de $\mathbb{CP}^2$ et applications projectives.** On considère maintenant des systèmes linéaires engendrés par trois sections approximativement holomorphes (s_k^0, s_k^1, s_k^2) de $L^{\otimes k}$: pour $k \gg 0$, il est à nouveau possible d'obtenir un comportement générique pour l'application projective $f_k = (s_k^0 : s_k^1 : s_k^2)$ (à valeurs dans $\mathbb{CP}^2$) associée au système linéaire.

Si la variété X est de dimension 4, le système linéaire n'a pas de points base, et l'application f_k est un *revêtement ramifié* :

Théorème 3.1 ([Au2]). *Pour k suffisamment grand, trois sections approximativement holomorphes convenablement choisies de $L^{\otimes k}$ au-dessus de X^4 déterminent un revêtement ramifié $f_k : X^4 \to \mathbb{CP}^2$ à modèles locaux génériques : pour tout point $x \in X$, il existe des coordonnées approximativement holomorphes locales au voisinage de x et de $f_k(x)$ dans lesquelles f_k s'identifie à l'un des trois modèles locaux $(u,v) \mapsto (u,v)$ (difféomorphisme local), $(u,v) \mapsto (u^2,v)$ (ramification simple), ou $(u,v) \mapsto (u^3 - uv, v)$ (cusp). De plus, pour $k \gg 0$ les revêtements ainsi construits sont uniquement déterminés à isotopie près.*

Les trois modèles locaux qui apparaissent dans l'énoncé du théorème sont les mêmes que pour une application holomorphe générique en dimension complexe 2. Toutefois ils sont ici réalisés dans des coordonnées locales qui ne sont pas holomorphes. La propriété importante des systèmes de coordonnées considérés est la suivante : si l'on transporte la structure symplectique via le système de coordonnées, on obtient une forme symplectique sur $\mathbb{C}^2$ dont la restriction à une droite complexe quelconque est toujours positive.

Le lieu des points critiques de f_k est une courbe symplectique lisse (connexe) $R_k \subset X$. En revanche, la courbe symplectique $D_k = f_k(R_k) \subset \mathbb{CP}^2$ ("courbe de ramification", ou "courbe discriminante") n'est immergée qu'en dehors des points où le troisième modèle local s'applique. En ces points, la courbe D_k présente un *cusp complexe* ($27x^2 = 4y^3$). Outre les cusps, la courbe D_k présente également génériquement des *points doubles*, qui n'apparaissent pas dans les modèles locaux car ils correspondent à des ramifications en deux points distincts de la même fibre de f_k ; bien que D_k soit approximativement holomorphe, les deux orientations sont a priori envisageables pour ses points doubles, contrairement au cas complexe.

Réciproquement, un revêtement de $\mathbb{CP}^2$ ramifié le long d'une courbe symplectique singulière admet toujours une structure symplectique naturelle (canonique à isotopie près), obtenue par relèvement de la forme de Kähler de $\mathbb{CP}^2$ et perturbation le long de la courbe de ramification.

Le résultat d'unicité du Théorème 3.1 implique que, pour $k \gg 0$, il est possible de définir des invariants de la variété symplectique (X, ω) à partir de la monodromie du revêtement et de la topologie de la courbe $D_k \subset \mathbb{CP}^2$; toutefois, la courbe D_k n'est déterminée qu'à création ou annulation de paires de points doubles admissibles près.

Lorsque $\dim X > 4$, le lieu des points base $Z_k = \{s_k^0 = s_k^1 = s_k^2 = 0\}$ n'est plus vide, et l'application projective f_k n'est plus définie partout ; les points base forment génériquement une sous-variété symplectique lisse de X, de codimension réelle 6. A cette différence près, le résultat précédent s'étend en dimension supérieure :

Théorème 3.2 ([Au3]). *Pour k suffisamment grand, trois sections approximativement holomorphes convenablement choisies du fibré $L^{\otimes k}$ déterminent une application projective $f_k : X^{2n} - Z_k \to \mathbb{CP}^2$ à modèles locaux génériques, de façon canonique à isotopie près.*

Près d'un point de Z_k, un modèle local pour f_k est $(z_1, \dots, z_n) \mapsto [z_1 : z_2 : z_3]$. Hors de Z_k, les trois modèles locaux génériques deviennent respectivement :

(*i*) $(z_1, \dots, z_n) \mapsto (z_1, z_2)$;
(*ii*) $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$;
(*iii*) $(z_1, \dots, z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$.

Le lieu des points critiques $R_k \subset X$ est de nouveau une courbe symplectique lisse (connexe), tandis que son image $D_k = f_k(R_k) \subset \mathbb{CP}^2$ est encore une courbe symplectique singulière, dont les seules singularités génériques sont des cusps complexes ainsi que des points doubles (le signe de l'intersection pouvant être soit positif soit négatif). Les fibres de f_k sont des sous-variétés symplectiques de codimension réelle 4, s'intersectant le long de Z_k; les fibres au-dessus des points de $\mathbb{CP}^2 - D_k$ sont lisses, tandis que celles au-dessus d'un point générique de D_k présentent un point double ordinaire, et celles au-dessus d'un cusp de D_k présentent une singularité de type A_2.

Comme en dimension 4, le résultat d'unicité à isotopie près implique que, modulo d'éventuelles créations ou annulations de paires de points doubles d'orientations opposées dans la courbe D_k, la topologie de la fibration f_k peut être utilisée pour définir des invariants de la variété symplectique (X, ω) (voir §4).

3.3. **Fibrés de jets et transversalité uniforme.** Les constructions décrites ci-dessus reposent sur l'analyse minutieuse des diverses situations locales possibles pour f_k et sur des arguments de transversalité permettant d'assurer l'existence de sections de $L^{\otimes k}$ se comportant de façon générique. Il s'agit donc de recenser les divers cas particuliers, génériques ou non, susceptibles de se produire ; chacun correspond à l'annulation d'une certaine quantité exprimable en fonction des sections s_k^0, s_k^1, s_k^2 et de leurs dérivées.

Afin de faciliter ce type de constructions, et pour pouvoir étendre les résultats à des systèmes linéaires déterminés par plus de trois sections ou à des situations plus générales encore, il est nécessaire de développer une version approximativement holomorphe de la théorie des singularités. L'ingrédient essentiel de cette approche est un théorème de transversalité uniforme pour les jets de sections approximativement holomorphes ([Au4], Théorème 1.1).

Etant données des sections approximativement holomorphes s_k de fibrés très positifs E_k (par exemple $E_k = \mathbb{C}^m \otimes L^{\otimes k}$) sur une variété symplectique X, il est possible de considérer leurs r-jets, $j^r s_k = (s_k, \partial s_k, (\partial\partial s_k)_{\mathrm{sym}}, \dots, (\partial^r s_k)_{\mathrm{sym}})$, qui sont des sections de *fibrés de jets* $\mathcal{J}^r E_k = \bigoplus_{j=0}^r (T^*X^{(1,0)})^{\otimes j}_{\mathrm{sym}} \otimes E_k$. Les fibrés $\mathcal{J}^r E_k$ peuvent naturellement être stratifiés par des sous-variétés approximativement holomorphes, correspondant aux divers comportements locaux possibles à l'ordre r pour les sections s_k. Le comportement génériquement attendu correspond au cas où le jet $j^r s_k$ est transverse aux sous-variétés de la stratification. Le résultat est le suivant :

Théorème 3.3 ([Au4]). *Etant données des stratifications $\mathcal{S}_k$ des fibrés de jets $\mathcal{J}^r E_k$ par des sous-variétés approximativement holomorphes (en nombre fini, régulières au sens de Whitney, uniformément transverses aux fibres, et de courbure bornée indépendamment de k), pour k suffisamment grand les fibrés E_k admettent des sections approximativement holomorphes s_k dont les r-jets sont uniformément transverses aux stratifications $\mathcal{S}_k$. De plus ces sections peuvent être choisies arbitrairement proches de sections données.*

En l'appliquant à des stratifications convenablement choisies, ce théorème fournit la partie principale de l'argument requis pour construire des m-uplets de sections approximativement holomorphes de $L^{\otimes k}$ (et donc des applications projectives à valeurs dans $\mathbb{CP}^{m-1}$) présentant un comportement générique. De plus, une version à un paramètre réel du Théorème 3.3 permet d'obtenir des résultats d'unicité asymptotique à isotopie près pour ces sections génériques ([Au4], Théorème 3.2).

4. Invariants de monodromie

La topologie des structures introduites ci-dessus (pinceaux de Lefschetz, revêtements ramifiés, ...) peut être étudiée à l'aide de la notion de *monodromie*, dans le but de définir des invariants de variétés symplectiques.

De façon générale, la monodromie d'une fibration lisse $f : X \to Y$ de fibre F est décrite par un morphisme $\phi : \pi_1(Y) \to \pi_0\mathrm{Diff}(F)$, obtenu en considérant le difféomorphisme (défini seulement à isotopie près en l'absence du choix d'une connexion sur la fibration) induit sur la fibre générique par un déplacement le long d'un lacet dans la base Y ; en présence de structures supplémentaires sur les fibres (sections distinguées, structures symplectiques, ...) on pourra aussi considérer des classes d'isotopie de difféomorphismes relatifs, de symplectomorphismes, etc... Lorsque f possède des points critiques, i.e. en présence de fibres singulières, la monodromie est définie en se restreignant à la préimage de $Y_0 = Y - \mathrm{crit}(f)$, la monodromie autour des fibres singulières étant alors d'une importance toute particulière pour l'étude de la topologie de l'application f.

4.1. **Monodromie de fibrations de Lefschetz.** On considère à nouveau une structure de pinceau de Lefschetz symplectique sur X, donnée par des hypersurfaces $\Sigma_{k,\alpha}$ s'intersectant transversalement le long de la sous-variété des points base Z_k ; les hypersurfaces $\Sigma_{k,\alpha}$ sont les surfaces de niveaux d'une fonction de Morse complexe $f_k : X - Z_k \to \mathbb{CP}^1$, et sont lisses excepté un nombre fini d'entre elles qui contiennent un point double ordinaire comme seule singularité. Ce point double est obtenu à partir d'une fibre générique par contraction d'une sphère lagrangienne appelée *cycle évanescent*.

En considérant la variété $\hat{X}$ formée par éclatement de X le long de Z_k, on obtient une *fibration de Lefschetz* $\hat{f}_k : \hat{X} \to \mathbb{CP}^1$, dont on peut étudier la monodromie autour des fibres singulières (correspondant aux valeurs critiques de f_k). Cette monodromie prend ses valeurs dans le *mapping class group* symplectique,

$$\mathrm{Map}^\omega(\Sigma_k, Z_k) = \pi_0(\{\phi \in \mathrm{Symp}(\Sigma_k, \omega),\ \phi_{|V(Z_k)} = \mathrm{Id}\}),$$

où Σ_k est une fibre générique de $\hat{f}_k$ (correspondant au choix d'un point de référence $\alpha \in \mathbb{CP}^1$), et $V(Z_k)$ est un voisinage de Z_k dans la fibre Σ_k. On obtient ainsi un morphisme de monodromie $\psi_k : \pi_1(\mathbb{C} - \mathrm{crit}\, f_k) \to \mathrm{Map}^\omega(\Sigma_k, Z_k)$. La monodromie autour d'une fibre singulière est un *twist de Dehn* (positif) le long du cycle évanescent (une sphère lagrangienne plongée $S^{n-1} \subset \Sigma_k - Z_k$).

Dans le cas où $\dim X = 4$, les fibres sont des surfaces compactes et Z_k est un ensemble fini de points ; le groupe $\mathrm{Map}^\omega(\Sigma_k, Z_k)$ s'identifie donc au "mapping class group" $\mathrm{Map}_{g,N}$ d'une surface de Riemann de genre $g = g(\Sigma_k)$ à $N = \mathrm{card}\, Z_k$ composantes de bord, et la monodromie autour d'une fibre singulière (une surface de Riemann possédant un point double à croisement normal) est un twist de Dehn le long d'une courbe fermée simple.

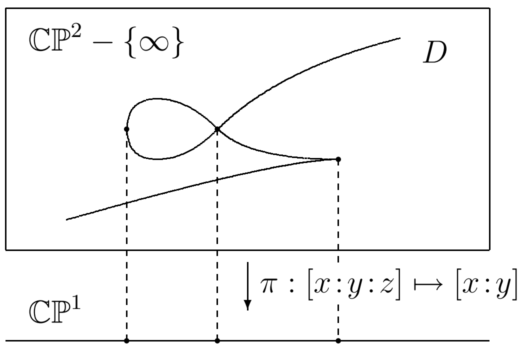
Le résultat d'unicité asymptotique de Donaldson implique que, pour k suffisamment grand, la monodromie des pinceaux de Lefschetz construits à partir de sections approximativement holomorphes de $L^{\otimes k}$ est un invariant symplectique de (X, ω). Inversement, Gompf a montré que la donnée du morphisme de monodromie détermine entièrement la variété X munie de sa structure symplectique [Go2] ; de plus, en dimension 4 l'espace total d'une "fibration de Lefschetz topologique" au-dessus de $\mathbb{CP}^1$ admet toujours une structure symplectique [GS].

Les propriétés géométriques et topologiques des pinceaux et fibrations de Lefschetz ont fait l'objet de nombreuses études ces dernières années, particulièrement en dimension 4 ; cf. par exemple [ABKP], [Sm1], [EK]. Donaldson et Smith ont montré qu'une variante des invariants de Gromov-Witten peut être exprimée en termes de fibrations en produits symétriques associées à un pinceau de Lefschetz, ce qui leur a permis de redémontrer sans faire appel à la théorie de Seiberg-Witten des résultats de Taubes sur l'existence de courbes symplectiques dans certaines classes d'homologie [DS, Sm2]. Par ailleurs, Seidel a introduit une version combinatoire de l'homologie de Floer lagrangienne pour les pinceaux de Lefschetz [Se], ce qui lui a permis d'obtenir une description simplifiée, effectivement calculable, de certaines catégories de Fukaya (A_∞-catégories dont les objets sont des sous-variétés lagrangiennes et dont les morphismes sont donnés par l'homologie de Floer), et de vérifier ainsi les conjectures de symétrie miroir de Kontsevich pour des exemples simples.

Tous ces travaux montrent clairement l'intérêt que présente le calcul de la monodromie des pinceaux de Lefschetz symplectiques construits par Donaldson. Malheureusement, ce calcul est très délicat en pratique ; de fait, la méthode de calcul la plus efficace consiste souvent à faire intervenir un système linéaire comportant une troisième section (c'est-à-dire une application à valeurs dans $\mathbb{CP}^2$), et à calculer les invariants de monodromie qui lui sont associés avant d'en déduire ceux du pinceau de Lefschetz correspondant.

4.2. **Monodromie de revêtements ramifiés de $\mathbb{CP}^2$.** Les données topologiques qui caractérisent un revêtement ramifié $f_k : X^4 \to \mathbb{CP}^2$ sont d'une part la courbe de ramification $D_k \subset \mathbb{CP}^2$ (à isotopie et annulation de paires de points doubles près), et d'autre part un morphisme de monodromie $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_N$ décrivant l'agencement des $N = \deg f_k$ feuillets du revêtement au-dessus de $\mathbb{CP}^2 - D_k$.

L'étude d'une courbe singulière $D \subset \mathbb{CP}^2$ peut se faire en utilisant les techniques de groupes de tresses introduites en géométrie algébrique complexe par Moishezon et Teicher [Mo1, Te1] : l'idée est de choisir une projection linéaire $\pi : \mathbb{CP}^2 - \{\text{pt}\} \to \mathbb{CP}^1$, par exemple $\pi([x:y:z]) = [x:y]$, de telle sorte que la courbe D soit en position



générale par rapport aux fibres de π. La restriction $\pi_{|D}$ est alors un revêtement ramifié singulier de degré $d = \deg D$, dont les points particuliers sont d'une part les singularités de D (points doubles et cusps) et d'autre part des points de tangence verticale où la courbe D devient tangente aux fibres de π.

Hormis celles qui contiennent des points particuliers de D, les fibres de π sont des droites qui intersectent la courbe D en d points distincts. Si l'on choisit un point de référence dans $\mathbb{CP}^1$ (et la fibre correspondante $\ell \simeq \mathbb{C} \subset \mathbb{CP}^2$ de π), et si l'on se restreint à un ouvert affine afin de pouvoir trivialiser la fibration π, la topologie du revêtement ramifié $\pi_{|D}$ peut être décrite par un morphisme $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$, où B_d est le groupe des tresses à d brins : la tresse $\rho(\gamma)$ correspond au mouvement des d points de $\ell \cap D$ à l'intérieur des fibres de π lors d'un déplacement le long du lacet γ. De façon équivalente, si l'on choisit un système de lacets qui engendrent le groupe libre $\pi_1(\mathbb{C} - \{\text{pts}\})$, le morphisme ρ peut être décrit par une *factorisation* dans le groupe de tresses B_d, faisant intervenir la monodromie autour de chacun des points particuliers de D (laquelle, pour chaque type de point, est toujours conjuguée à un modèle local standard).

Le morphisme ρ et la factorisation correspondante dépendent de choix de trivialisation, qui les affectent par conjugaison simultanée (changement de trivialisation de la fibre ℓ de π) ou par opérations de Hurwitz (changement de générateurs de $\pi_1(\mathbb{C} - \{\text{pts}\})$). Il y a équivalence complète entre la donnée d'un morphisme $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ à ces opérations algébriques près et la donnée d'une courbe singulière symplectique plane D de degré d compatible avec la projection π à isotopie symplectique (parmi les courbes singulières compatibles avec la projection π) près. En revanche, la courbe D n'est isotope à une courbe complexe que pour certains choix particuliers du morphisme ρ.

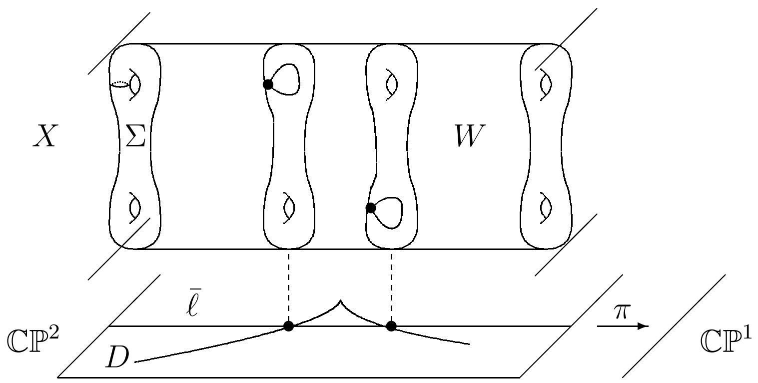
Contrairement au cas complexe, dans le cas symplectique il n'est pas évident a priori que la courbe de ramification D_k puisse posséder les propriétés attendues vis-à-vis de la projection linéaire π ; cela requiert en fait une amélioration du Théorème 3.1 afin de contrôler le comportement de D_k près des points particuliers (tangences verticales, points doubles et cusps) [AK, Au3]. Par ailleurs, il faut tenir compte des créations ou annulations de paires de points doubles admissibles dans D_k, qui affectent le morphisme $\rho_k : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ par insertion ou suppression de paires de facteurs. Le résultat d'unicité du Théorème 3.1 implique alors le résultat suivant, obtenu en collaboration avec L. Katzarkov [AK] :

Théorème 4.1 ([AK]). *Pour k fixé suffisamment grand, les morphismes de monodromie (ρ_k, θ_k) associés aux revêtements ramifiés approximativement holomorphes $f_k : X \to \mathbb{CP}^2$ définis par trois sections de $L^{\otimes k}$ sont, à conjugaisons, opérations de Hurwitz et insertions ou suppressions près, des invariants de la variété symplectique (X^4, ω). De plus, ces invariants sont* complets, *en ce sens que la donnée de ρ_k et de θ_k permet de reconstruire (X^4, ω) à symplectomorphisme près.*

Il est intéressant de mentionner que les pinceaux de Lefschetz symplectiques construits par Donaldson peuvent être réobtenus très facilement à partir des revêtements ramifiés f_k, simplement en considérant les applications composées $\pi \circ f_k$ à valeurs dans $\mathbb{CP}^1$. Autrement dit, les fibres $\Sigma_{k,\alpha}$ du pinceau sont les préimages par f_k des fibres de π, les fibres singulières du pinceau correspondant aux points de tangence verticale de D_k. En fait, les morphismes de monodromie ψ_k des pinceaux de Lefschetz peuvent être construits de façon explicite à partir de θ_k et ρ_k : par restriction à la droite $\bar{\ell} = \ell \cup \{\infty\}$, le morphisme θ_k à valeurs dans S_N décrit la topologie d'une fibre du pinceau en tant que revêtement ramifié de $\mathbb{CP}^1$ à N feuillets et d points de ramification, ce qui permet de définir un *morphisme de relèvement* $(\theta_k)_*$ d'un sous-groupe de B_d à valeurs dans $\mathrm{Map}(\Sigma_k, Z_k) = \mathrm{Map}_{g,N}$. On a alors $\psi_k = (\theta_k)_* \circ \rho_k$ (cf. [AK], §5.2). Cette relation permet de tirer parti des diverses techniques disponibles pour le calcul de la monodromie de revêtements ramifiés [Mo2, Te1, ADKY], afin d'obtenir des formules explicites décrivant la monodromie de pinceaux de Lefschetz dans des cas qui ne sont pas accessibles au calcul direct (voir notamment [AK2]).

4.3. **Monodromie d'applications projectives.** Lorsque $\dim X > 4$, la topologie d'une application projective $f_k : X - Z_k \to \mathbb{CP}^2$ et la courbe discriminante $D_k \subset \mathbb{CP}^2$ peuvent être décrites de la même façon que pour un revêtement ramifié ; la seule différence est que le morphisme θ_k décrivant la monodromie de la fibration au-dessus du complémentaire de D_k prend désormais ses valeurs dans le "mapping class group" symplectique $\mathrm{Map}^\omega(\Sigma_k, Z_k)$ de la fibre générique de f_k. Le Théorème 4.1 demeure vrai dans ce cadre [Au3]. Toutefois, ces invariants sont difficilement exploitables, notamment lorsque $\dim X \geq 8$, car le groupe $\mathrm{Map}^\omega(\Sigma_k, Z_k)$ est très mal connu.

Cependant, il existe une procédure de réduction dimensionnelle qui permet d'éviter cet écueil. En effet, la restriction de f_k à la droite $\bar{\ell} \subset \mathbb{CP}^2$ définit un pinceau de Lefschetz sur une hypersurface symplectique $W_k \subset X$, de fibre générique Σ_k et de monodromie θ_k.



Cette structure peut être enrichie par ajout d'une section supplémentaire de $L^{\otimes k}$ de façon à obtenir une application de W_k dans $\mathbb{CP}^2$, qui peut de nouveau être caractérisée par des invariants de monodromie, et ainsi de suite jusqu'en petite dimension. Au final, étant donnée une variété symplectique (X^{2n}, ω) et un entier $k \gg 0$, on obtient $n-1$ courbes singulières $D_k^{(n)}, D_k^{(n-1)}, \dots, D_k^{(2)} \subset \mathbb{CP}^2$, décrites par autant de morphismes à valeurs dans des groupes de tresses, et un morphisme $\theta_k^{(2)}$ de $\pi_1(\mathbb{CP}^2 - D_k^{(2)})$ dans un groupe symétrique. Ces invariants suffisent à reconstruire de proche en proche les différentes sous-variétés de X qui interviennent dans le processus de réduction, et en fin de compte ils déterminent la variété (X, ω) à symplectomorphisme près [Au3] :

Théorème 4.2 ([Au3]). *Pour k fixé suffisamment grand, les morphismes de monodromie $\rho_k^{(n)}, \dots, \rho_k^{(2)}$ à valeurs dans des groupes de tresses et $\theta_k^{(2)}$ à valeurs dans un groupe symétrique associés à des systèmes linéaires de sections de $L^{\otimes k}$ sont, à conjugaison, opérations de Hurwitz et insertions ou suppressions près, des invariants* *de la variété symplectique* (X^{2n}, ω). *De plus, ces invariants sont* complets *: ils permettent de reconstruire* (X^{2n}, ω) *à symplectomorphisme près.*

Cette stratégie d'étude semble beaucoup plus prometteuse que celle consistant à considérer directement des applications projectives à valeurs dans des espaces $\mathbb{CP}^m$ pour $m \geq 3$. En effet, même si les méthodes de monodromie continuent en principe à s'appliquer dans ce cadre, la courbe $D_k \subset \mathbb{CP}^2$ est alors remplacée par une hypersurface dans $\mathbb{CP}^m$, susceptible de présenter des singularités très compliquées et donc délicate à étudier.

4.4. **Techniques de calcul.** En principe, les Théorèmes 4.1 et 4.2 ramènent la classification des variétés symplectiques compactes à des questions purement combinatoires concernant les groupes de tresses et les groupes symétriques, et la topologie symplectique semble se réduire en grande partie à l'étude de certaines courbes planes singulières, ou de façon équivalente de certains mots dans des groupes de tresses.

Le calcul explicite de ces invariants de monodromie est difficile dans le cas général, mais est rendu possible pour un grand nombre de surfaces complexes par l'utilisation de techniques de "dégénération" et de perturbations approximativement holomorphes. Ainsi, les invariants définis par le Théorème 4.1 sont calculables explicitement pour divers exemples tels que $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ [Mo2], quelques intersections complètes (surfaces de Del Pezzo ou K3) [Ro], la surface d'Hirzebruch $\mathbb{F}_1$, et tous les revêtements doubles de $\mathbb{CP}^1 \times \mathbb{CP}^1$ (parmi lesquels une famille infinie de surfaces de type général) [ADKY].

La technique de dégénération, développée par Moishezon et Teicher [Mo2, Te1], consiste à partir d'un plongement projectif de la surface complexe X et à déformer l'image de ce plongement en une configuration singulière X_0 constituée d'une union de plans s'intersectant le long de droites. La courbe discriminante d'une projection de X_0 sur $\mathbb{CP}^2$ est donc une union de droites ; la façon dont une désingularisation de X_0 affecte cette courbe peut être étudiée explicitement, en considérant un certain nombre de modèles locaux standard au voisinage des divers points de X_0 où trois plans et plus s'intersectent. Ceci permet de traiter de nombreux exemples en petit degré, mais pour le cas $k \gg 0$ qui nous intéresse (systèmes linéaires très positifs) seules des surfaces très simples peuvent être abordées.

Pour aller au-delà, il est plus efficace de s'affranchir de la rigidité des applications algébriques, et de s'autoriser des perturbations approximativement holomorphes dont la flexibilité supérieure permet de choisir des modèles locaux plus accessibles au calcul. Il devient ainsi possible de calculer directement les invariants de monodromie pour tous les systèmes linéaires de la forme $f^*O(p,q)$ sur des revêtements doubles de $\mathbb{CP}^1 \times \mathbb{CP}^1$ ramifiés le long de courbes algébriques lisses connexes de degré arbitraire [ADKY]. Il devient également possible d'obtenir une formule générale de "stabilisation", qui décrit explicitement les invariants de monodromie associés au système linéaire $L^{\otimes 2k}$ en fonction de ceux associés au système linéaire $L^{\otimes k}$ (lorsque $k \gg 0$), pour les revêtements ramifiés de $\mathbb{CP}^2$ comme pour les pinceaux de Lefschetz en dimension 4 [AK2].

Toutefois, malgré ces succès, un obstacle sérieux limite l'utilisation des invariants de monodromie en pratique : il n'est pas possible de les exploiter efficacement pour différencier deux variétés symplectiques homéomorphes en toute généralité, car il n'existe pas d'algorithme pour comparer deux mots dans un groupe de tresses (ou dans un mapping class group) à opérations de Hurwitz près. Cet obstacle théorique oblige à se tourner vers des invariants moins complets mais plus maniables.

5. Groupes fondamentaux de complémentaires de courbes planes

Le groupe fondamental du complémentaire est un invariant qu'il est très naturel d'associer à une courbe plane singulière $D \subset \mathbb{CP}^2$, notamment dans le cas d'une courbe de ramification. Son étude pour divers types de courbes algébriques est un sujet classique depuis les travaux de Zariski, et a été beaucoup développée dans les années 80 et 90 grâce notamment aux travaux de Moishezon et Teicher [Mo1, Mo2, Te1]. La relation avec les invariants de monodromie est directe : grâce au théorème de Zariski-van Kampen, le morphisme de monodromie $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ fournit une présentation explicite de $\pi_1(\mathbb{CP}^2 - D)$. Toutefois, comme l'introduction ou l'élimination de paires de points doubles affecte ce groupe fondamental, il ne peut être directement utilisé comme invariant pour un revêtement ramifié symplectique, et doit être remplacé par un quotient convenable, le *groupe fondamental stabilisé*, qui a été introduit et étudié dans un travail en commun avec S. Donaldson, L. Katzarkov et M. Yotov [ADKY].

En reprenant les notations du §4.2, l'inclusion $i : \ell - (\ell \cap D_k) \to \mathbb{CP}^2 - D_k$ de la fibre de référence de la projection linéaire π induit un morphisme surjectif sur les groupes fondamentaux; les images des générateurs standard du groupe libre et leurs conjugués forment un sous-ensemble $\Gamma_k \subset \pi_1(\mathbb{CP}^2 - D_k)$ dont les éléments sont appelés *générateurs géométriques*. Les images des générateurs géométriques par le morphisme θ_k sont des transpositions dans S_N. La création d'une paire de points doubles dans la courbe D_k revient à quotienter $\pi_1(\mathbb{CP}^2 - D_k)$ par une relation de la forme $[\gamma_1, \gamma_2] \sim 1$, où $\gamma_1, \gamma_2 \in \Gamma_k$ sont tels que $\theta_k(\gamma_1)$ et $\theta_k(\gamma_2)$ sont des transpositions disjointes. On note K_k le sous-groupe normal de $\pi_1(\mathbb{CP}^2 - D_k)$ engendré par tous ces commutateurs $[\gamma_1, \gamma_2]$.

Théorème 5.1 ([ADKY]). *Pour $k \gg 0$ fixé, le* groupe fondamental stabilisé $\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)/K_k$ *est un invariant de la variété symplectique* (X^4, ω).

Cet invariant peut être calculé explicitement pour les divers exemples où les invariants de monodromie sont calculables ($\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, intersections complètes Del Pezzo et K3, surface d'Hirzebruch $\mathbb{F}_1$, revêtements doubles de $\mathbb{CP}^1 \times \mathbb{CP}^1$). Ces divers exemples ont servi de point de départ à diverses observations et conjectures concernant les groupes fondamentaux de complémentaires de courbes de ramification [ADKY].

Tout d'abord, il est à noter que, dans tous les exemples connus, pour k suffisamment grand l'opération de stabilisation devient triviale : $K_k = \{1\}$, c'est-à-dire que les générateurs géométriques associés à des transpositions disjointes commutent déjà dans $\pi_1(\mathbb{CP}^2 - D_k)$. Par exemple, lorsque $X = \mathbb{CP}^2$ on a $\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)$ pour tout $k \geq 3$. Le quotient par K_k ne semble donc pas engendrer de perte d'information, du moins pour $k \gg 0$ (la situation pour de petites valeurs de k pouvant être très différente).

Le principal résultat de structure est le suivant :

Théorème 5.2 ([ADKY]). *Il existe une suite exacte naturelle*

$$1 \longrightarrow G_k^0 \longrightarrow \bar{G}_k \longrightarrow S_n \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$

où $n = \deg f_k$ *et* $d = \deg D_k$*. De plus, si* X *est simplement connexe, alors il existe un morphisme surjectif naturel* $\phi_k : G_k^0 \twoheadrightarrow (\mathbb{Z}^2/\Lambda_k)^{n-1}$*, où*

$$\Lambda_k = \{(c_1(K_X) \cdot \alpha, c_1(L^{\otimes k}) \cdot \alpha),\ \alpha \in H_2(X, \mathbb{Z})\}$$

est un sous-groupe de $\mathbb{Z}^2$ *entièrement déterminé par les propriétés numériques de la classe symplectique et de la classe canonique.*

Dans cet énoncé, les deux composantes du morphisme $\bar{G}_k \to S_n \times \mathbb{Z}_d$ sont respectivement données par la monodromie du revêtement ramifié, $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_n$, et le morphisme d'abélianisation $\delta_k : \pi_1(\mathbb{CP}^2 - D_k) \to H_1(\mathbb{CP}^2 - D_k, \mathbb{Z}) \simeq \mathbb{Z}_d$. Pour sa part, le morphisme ϕ_k est défini en considérant les n relèvements à X d'un lacet fermé γ élément de G_k^0, ou plus précisément leurs classes d'homologie (dont la somme est nulle) dans le complémentaire d'une section hyperplane et de la courbe de ramification dans X.

En outre, les exemples connus incitent à formuler la conjecture suivante, beaucoup plus forte, sur la structure des sous-groupes G_k^0 :

Conjecture 5.3 ([ADKY]). *Si* X *est simplement connexe, alors pour* k *suffisamment grand le morphisme* ϕ_k *induit un isomorphisme au niveau de l'abélianisé, c'est-à-dire que* $\mathrm{Ab}\, G_k^0 \simeq (\mathbb{Z}^2/\Lambda_k)^{n-1}$*, tandis que* $\mathrm{Ker}\, \phi_k = [G_k^0, G_k^0]$ *est un quotient de* $\mathbb{Z}_2 \times \mathbb{Z}_2$*.*

6. Variétés symplectiques et variétés kählériennes

Il est bien connu depuis les années 70 que la topologie des variétés symplectiques compactes de dimension 4 offre une diversité beaucoup plus grande que celle des variétés kählériennes. D'autre part, il a été découvert plus récemment (dans les années 90) que les courbes symplectiques (lisses ou singulières) dans une variété donnée peuvent aussi présenter une plus vaste palette de possibilités que les courbes complexes, auxquelles elles ne sont pas toujours isotopes. Le Théorème 3.1 jette un pont entre ces deux phénomènes : en effet, un revêtement de $\mathbb{CP}^2$ (ou plus généralement d'une surface complexe) ramifié le long d'une courbe complexe hérite automatiquement d'une structure complexe, ce qui signifie que, à partir d'une variété symplectique n'admettant pas de structure kählérienne, le Théorème 3.1 fournit toujours des courbes de ramification qui ne sont isotopes à aucune courbe complexe dans $\mathbb{CP}^2$. L'étude de ces phénomènes d'isotopie et de non-isotopie présente donc un intérêt majeur pour la compréhension de la topologie des variétés symplectiques de dimension 4. Des résultats ont été obtenus dans deux directions.

6.1. **Phénomènes d'isotopie et de stabilisation.** Le problème d'isotopie symplectique consiste à déterminer si, dans une variété donnée, toutes les sous-variétés symplectiques réalisant une classe d'homologie donnée sont isotopes à des sous-variétés complexes. Le premier résultat positif est dû à Gromov, qui a montré à l'aide de son résultat de compacité pour les courbes pseudo-holomorphes que, dans $\mathbb{CP}^2$, une courbe symplectique lisse de degré 1 ou 2 est toujours isotope à une courbe complexe. Des améliorations successives de cette technique ont permis de traiter le cas de courbes lisses de degré supérieur dans $\mathbb{CP}^2$ ou dans $\mathbb{CP}^1 \times \mathbb{CP}^1$; le meilleur résultat connu actuellement est dû à Siebert et Tian, et permet de traiter le cas des courbes lisses de $\mathbb{CP}^2$ jusqu'en degré 17 [ST]. Des résultats sur certaines configurations singulières très simples sont également connus.

Au niveau des variétés symplectiques de dimension 4, la conséquence plus immédiate du phénomène d'isotopie symplectique est l'holomorphicité de certaines fibrations de Lefschetz. Ainsi, la classification complète des fibrations de Lefschetz elliptiques (qui sont toutes holomorphes) est un résultat classique de Moishezon. Les fibrations de Lefschetz de genre 2 sont toutes hyperelliptiques, ce qui permet de les réaliser comme des revêtements doubles de surfaces réglées, ramifiés le long de courbes symplectiques intersectant la fibre en 6 points (la courbe de ramification est lisse si toutes les fibres singulières sont irréductibles ; en présence de fibres réductibles des transformations birationnelles sont nécessaires). Ceci a permis à Siebert et Tian de démontrer, grâce à leur résultat d'isotopie symplectique pour les courbes lisses dans $\mathbb{CP}^1 \times \mathbb{CP}^1$ et dans la surface d'Hirzebruch $\mathbb{F}_1$, que toutes les fibrations de Lefschetz de genre 2 à fibres irréductibles et à monodromie transitive (i.e. dont la courbe de ramification est connexe) sont isomorphes à des fibrations holomorphes [ST].

Néanmoins, dans le cas où on autorise la présence de fibres singulières réductibles, cette propriété disparaît, et il existe des fibrations de Lefschetz symplectiques de genre 2 dont l'espace total n'est difféomorphe à aucune surface complexe [OS]. Il est alors naturel de se demander si une propriété plus faible demeure vraie, lorsque l'on autorise la stabilisation par sommes connexes fibrées avec certaines fibrations holomorphes. Dans le cas du genre 2, il suffit en fait de considérer la stabilisation par une seule fibration, la fibration holomorphe f_0 présentant 20 fibres singulières irréductibles et dont l'espace total est une surface rationnelle. Le résultat obtenu est le suivant :

Théorème 6.1 ([Au6]). *Soit $f : X \to S^2$ une fibration de Lefschetz de genre 2. Alors la somme fibrée de f avec un nombre suffisant de copies de la fibration holomorphe f_0 est isomorphe à une fibration holomorphe.*

De plus, cette somme fibrée $f \# n\, f_0$ ($n \gg 0$) est entièrement déterminée par son nombre total de fibres singulières et par le nombre de fibres singulières réductibles de chaque type (deux composantes de genre 1, ou composantes de genres 0 et 2).

Ce résultat conduit à formuler la question suivante : étant donné une fibration de Lefschetz symplectique (sans restriction sur le genre des fibres), est-il toujours possible, par sommes fibrées successives avec des fibrations holomorphes choisies parmi une liste finie de possibilités, de se ramener à une fibration holomorphe (éventuellement elle-même décomposable en un certain nombre de blocs simples) ? Une réponse à cette question améliorerait grandement notre compréhension de la structure des fibrations de Lefschetz symplectiques.

6.2. **Phénomènes de non-isotopie, tressage et chirurgie de Luttinger.** A l'inverse des cas d'isotopie décrits ci-dessus, la réponse au problème d'isotopie symplectique semble en général être négative. Les premiers contre-exemples connus dans le cas de courbes symplectiques lisses connexes sont dûs à Fintushel et Stern [FS], qui ont construit par un procédé de *tressage* des familles infinies de courbes symplectiques deux à deux non isotopes, réalisant une même classe d'homologie (multiple de la fibre) dans des surfaces elliptiques, et à Smith, qui a utilisé la même construction en genre supérieur. Toutefois, ces deux constructions sont précédées par un résultat de Moishezon [Mo3], qui a établi dès le début des années 90 un résultat qui donne l'existence dans $\mathbb{CP}^2$ de familles infinies de courbes symplectiques de degré fixé et possédant des cusps et points doubles en nombre fixé, deux à deux non isotopes ; une reformulation du résultat permet de voir qu'il s'agit encore une fois d'une construction de tressage. Dans un travail en commun avec Donaldson et Katzarkov, ce résultat de Moishezon a été réétudié et mis en relation avec une construction de chirurgie le long d'un tore lagrangien dans une variété symplectique de dimension 4, appelée *chirurgie de Luttinger* [ADK]. Ceci a permis de simplifier notablement l'argument de Moishezon, qui se basait sur de longs et délicats calculs de groupes fondamentaux de complémentaires de courbes, tout en le reliant à diverses constructions développées en topologie de la dimension 4.

Etant donnés un tore lagrangien T plongé dans une variété symplectique (X^4, ω), un lacet plongé homotopiquement non trivial $\gamma \subset T$ et un entier k, la chirurgie de Luttinger est une opération qui consiste à découper dans X un voisinage tubulaire de T, feuilleté par des tores lagrangiens parallèles à T, et à le recoller de telle sorte que le nouveau méridien diffère de l'ancien par k tours le long du lacet γ (tandis que les longitudes ne sont pas affectées), ce qui fournit une nouvelle variété symplectique $(\tilde{X}, \tilde{\omega})$. Cette construction relativement peu étudiée, qui permet notamment de passer d'un produit $T^2 \times \Sigma$ à n'importe quel fibré en surfaces au-dessus de T^2, ou encore de transformer une somme fibrée classique en somme "twistée", permet de décrire de façon unifiée de nombreux exemples de variétés symplectiques de dimension 4 construits au cours de ces dernières années.

La construction de tressage de courbes symplectiques, quant à elle, part d'une courbe symplectique éventuellement singulière $\Sigma \subset (Y^4, \omega_Y)$ et de deux cylindres symplectiques plongés dans Σ pouvant être joints par un anneau lagrangien contenu dans le complémentaire de Σ, et consiste à effectuer k demi-torsions entre ces cylindres pour obtenir une nouvelle courbe symplectique $\tilde{\Sigma}$ dans Y. Dans le cas où Σ est la courbe de ramification d'un revêtement ramifié symplectique $f : X \to Y$, on a le résultat suivant :

Proposition 6.2 ([ADK]). *Le revêtement ramifié de Y le long de la courbe symplectique $\tilde{\Sigma}$ obtenue par tressage de Σ le long d'un anneau lagrangien $A \subset Y - \Sigma$ est naturellement symplectomorphe à la variété $\tilde{X}$ construite à partir du revêtement ramifié X par chirurgie de Luttinger le long d'un tore lagrangien $T \subset X$ obtenu par relèvement de A.*

Ainsi, une fois construite une famille infinie de courbes symplectiques grâce au procédé de tressage, il ne reste plus qu'à trouver des invariants qui distinguent les revêtements ramifiés correspondants pour conclure à la non-isotopie des courbes construites. Dans la construction de Fintushel et Stern, ce rôle est joué par les invariants de Seiberg-Witten, dont le comportement est bien compris pour les fibrations elliptiques et leurs chirurgies.

Dans le cas des exemples de Moishezon, une construction de tressage permet, à partir de courbes complexes $\Sigma_{p,0} \subset \mathbb{CP}^2$ ($p \geq 2$) de degré $d_p = 9p(p-1)$ et comptant $\kappa_p = 27(p-1)(4p-5)$ cusps et $\nu_p = 27(p-1)(p-2)(3p^2+3p-8)/2$ points doubles, de construire des courbes symplectiques $\Sigma_{p,k} \subset \mathbb{CP}^2$ pour tout $k \in \mathbb{Z}$, de même degré et avec les mêmes singularités. Grâce à la Proposition 6.2, ces courbes peuvent être vues comme les courbes de ramification de revêtements ramifiés symplectiques, dont les espaces totaux $X_{p,k}$ diffèrent par des chirurgies de Luttinger le long d'un tore lagrangien $T \subset X_{p,0}$. L'effet de ces chirurgies sur le fibré canonique et sur la forme symplectique peut être décrit explicitement, ce qui permet de distinguer les variétés $X_{p,k}$: la classe canonique de $(X_{p,k}, \omega_{p,k})$ est donnée par $p\,c_1(K_{p,k}) = (6p-9)[\omega_{p,k}] + (2p-3)k\,PD([T])$. De plus, $[T] \in H_2(X_{p,k}, \mathbb{Z})$ n'est pas une classe de torsion, et si $p \not\equiv 0 \mod 3$ ou $k \equiv 0 \mod 3$ alors c'est une classe primitive [ADK]. Ceci implique que, parmi les courbes $\Sigma_{p,k}$, il en existe une infinité qui sont deux à deux non isotopes.

Il est à noter que l'argument utilisé par Moishezon pour distinguer les courbes $\Sigma_{p,k}$, qui repose sur le difficile calcul des groupes fondamentaux $\pi_1(\mathbb{CP}^2 - \Sigma_{p,k})$ [Mo3], est mis en relation avec celui présenté ici par le biais de la Conjecture 5.3, dont on peut conclure *a posteriori* qu'elle est satisfaite par les revêtements ramifiés $X_{p,k}$.

Le fait que la plupart des exemples connus de courbes non-isotopes se réduisent à des constructions de tressage, et qu'un très grand nombre de constructions de variétés symplectiques par chirurgie puissent se reformuler en termes de chirurgies de Luttinger, incite à poser la question du rôle de cette construction dans la différenciation des variétés symplectiques par rapport aux variétés kählériennes. Ainsi, une réponse à la question de l'existence de variétés symplectiques compactes de dimension 4 ne pouvant pas être obtenues à partir de surfaces complexes par chirurgies de Luttinger successives et déformations ferait certainement progresser notre compréhension de la topologie symplectique en dimension 4.

Bibliographie

[Au1] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[Au2] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$, Inventiones Math. **139** (2000), 551–602.

[Au3] D. Auroux, *Symplectic maps to projective spaces and symplectic invariants*, Proc. 7th Gökova Geometry-Topology Conference (2000), Turkish J. Math. **25** (2001), 1–42 (`math.GT/0007130`).

[Au4] D. Auroux, *Estimated transversality in symplectic geometry and projective maps*, "Symplectic Geometry and Mirror Symmetry", Proc. 4th KIAS International Conference, Seoul (2000), World Sci., Singapore, 2001, pp. 1–30 (`math.SG/0010052`).

[Au5] D. Auroux, *A remark about Donaldson's construction of symplectic submanifolds*, à paraître dans J. Symplectic Geom. (`math.DG/0204286`).

[Au6] D. Auroux, *Fiber sums of genus 2 Lefschetz fibrations*, à paraître dans Turkish J. Math. (`math.GT/0204285`).

[ADK] D. Auroux, S.K. Donaldson, L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, à paraître dans Math. Ann. (`math.GT/0206005`).

[ADKY] D. Auroux, S.K. Donaldson, L. Katzarkov, M. Yotov, *Fundamental groups of complements of plane curves and symplectic invariants*, preprint (`math.GT/0203183`).

[AGM] D. Auroux, D. Gayet, J.-P. Mohsen, *Symplectic hypersurfaces in the complement of an isotropic submanifold*, Math. Ann. **321** (2001), 739–754.

[AK] D. Auroux, L. Katzarkov, *Branched coverings of* $\mathbb{CP}^2$ *and invariants of symplectic 4-manifolds*, Inventiones Math. **142** (2000), 631–673.

[AK2] D. Auroux, L. Katzarkov, *The degree doubling formula for braid monodromies and Lefschetz pencils*, en préparation.

[ABKP] J. Amorós, F. Bogomolov, L. Katzarkov, T. Pantev, *Symplectic Lefschetz fibrations with arbitrary fundamental groups*, J. Differential Geom. **54** (2000), 489–545.

[Bi] P. Biran, *Lagrangian barriers and symplectic embeddings*, Geom. Funct. Anal. **11** (2001), 407–464.

[BU] D. Borthwick, A. Uribe, *Nearly Kählerian embeddings of symplectic manifolds*, Asian J. Math. **4** (2000), 599–620.

[Do1] S. K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. **44** (1996), 666–705.

[Do2] S.K. Donaldson, *Lefschetz fibrations in symplectic geometry*, Documenta Math., Extra Volume Int. Congress Math. 1998 (Berlin), II, 309–314.

[Do3] S.K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. **53** (1999), 205–236.

[DS] S. Donaldson, I. Smith, *Lefschetz pencils and the canonical class for symplectic 4-manifolds*, preprint (`math.SG/0012067`).

[EK] H. Endo, D. Kotschick, *Bounded cohomology and non-uniform perfection of mapping class groups*, Inventiones Math. **144** (2001), 169–175.

[FS] R. Fintushel, R. Stern, *Symplectic surfaces in a fixed homology class*, J. Differential Geom. **52** (1999), 203–222.

[GM] E. Giroux, J.-P. Mohsen, en préparation.

[Go1] R. Gompf, *A new construction of symplectic manifolds*, Ann. Math. **142** (1995), 527–595.

[Go2] R. Gompf, *A topological characterization of symplectic manifolds*, preprint (`math.SG/0210103`).

[GS] R. Gompf, A.I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Math. **20**, Amer. Math. Soc., Providence, 1999.

[IMP] A. Ibort, D. Martinez-Torres, F. Presas, *On the construction of contact submanifolds with prescribed topology*, J. Differential Geom. **56** (2000), 235–283.

[Mo1] B. Moishezon, *Stable branch curves and braid monodromies*, Algebraic Geometry (Chicago, 1980), Lecture Notes in Math. **862**, Springer, Heidelberg, 1981, pp. 107–192.

[Mo2] B. Moishezon, *On cuspidal branch curves*, J. Algebraic Geom. **2** (1993), 309–384.

[Mo3] B. Moishezon, *The arithmetic of braids and a statement of Chisini*, Geometric Topology (Haifa, 1992), Contemp. Math. **164**, Amer. Math. Soc., Providence, 1994, pp. 151–175.

[OS] B. Ozbagci, A. Stipsicz, *Noncomplex smooth 4-manifolds with genus 2 Lefschetz fibrations*, Proc. Amer. Math. Soc. **128** (2000), 3125–3128.

[Ro] A. Robb, *On branch curves of algebraic surfaces*, Singularities and Complex Geometry (Beijing, 1994), Amer. Math. Soc./Int. Press Stud. Adv. Math. **5**, Amer. Math. Soc., Providence, 1997, pp. 193–221.

[Se] P. Seidel, *Vanishing cycles and mutation*, Proc. 3rd European Congress of Mathematics (Barcelona, 2000), Vol. II, Progr. Math. **202**, Birkhäuser, Basel, 2001, pp. 65–85 (`math.SG/0007115`).

[ST] B. Siebert, G. Tian, *On the holomorphicity of genus two Lefschetz fibrations*, preprint.

[Si] J.C. Sikorav, *Construction de sous-variétés symplectiques*, Séminaire Bourbaki n° 844, Astérisque **252** (1998), 231–253.

[Sm1] I. Smith, *Lefschetz pencils and divisors in moduli space*, Geom. Topol. **5** (2001), 579–608.

[Sm2] I. Smith, *Serre-Taubes duality for pseudoholomorphic curves*, preprint (`math.SG/0011223`).

[Ta] C.H. Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2** (1995), 221–238.

[Te1] M. Teicher, *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math., **62** (part 1), Amer. Math. Soc., Providence, 1997, pp. 127–150.

[Te2] M. Teicher, *New invariants for surfaces*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math. **231**, Amer. Math. Soc., Providence, 1999, pp. 271–281 (`math.AG/9902152`).

ASYMPTOTICALLY HOLOMORPHIC FAMILIES OF SYMPLECTIC SUBMANIFOLDS

DENIS AUROUX

Abstract. We construct a wide range of symplectic submanifolds in a compact symplectic manifold as the zero sets of asymptotically holomorphic sections of vector bundles obtained by tensoring an arbitrary vector bundle by large powers of the complex line bundle whose first Chern class is the symplectic form. We also show that, asymptotically, all sequences of submanifolds constructed from a given vector bundle are isotopic. Furthermore, we prove a result analogous to the Lefschetz hyperplane theorem for the constructed submanifolds.

1. Introduction

In a recent paper [1], Donaldson has exhibited an elementary construction of symplectic submanifolds of codimension 2 in a compact symplectic manifold, where the submanifolds are seen as the zero sets of asymptotically holomorphic sections of well-chosen line bundles. In this paper, we extend this construction to higher rank bundles as well as one-parameter families, and obtain as a consequence an important isotopy result.

In all the following, (X, ω) will be a compact symplectic manifold of dimension $2n$, such that the cohomology class $[\frac{\omega}{2\pi}]$ is integral. A compatible almost-complex structure J and the corresponding riemannian metric g are fixed. Let L be the complex line bundle on X whose first Chern class is $c_1(L) = [\frac{\omega}{2\pi}]$. Fix a hermitian structure on L, and let ∇^L be a hermitian connection on L whose curvature 2-form is equal to $-i\omega$ (it is clear that such a connection always exists).

We will consider families of sections of bundles of the form $E \otimes L^k$ on X, defined for all large values of an integer parameter k, where E is any hermitian vector bundle over X. The connection ∇^L induces a connection of curvature $-ik\omega$ on L^k, and together with any given hermitian connection ∇^E on E this yields a hermitian connection on $E \otimes L^k$ for any k. We are interested in sections which satisfy the following two properties :

Definition 1. *A sequence of sections s_k of $E \otimes L^k$ (for large k) is said to be asymptotically holomorphic with respect to the given connections and almost-complex structure if the following bounds hold :*

$$|s_k| = O(1), \qquad |\nabla s_k| = O(k^{1/2}), \qquad |\bar\partial s_k| = O(1),$$
$$|\nabla\nabla s_k| = O(k), \qquad |\nabla\bar\partial s_k| = O(k^{1/2}).$$

Since X is compact, up to a change by a constant factor in the estimates, the notion of asymptotic holomorphicity does not actually depend on the chosen hermitian structures and on the chosen connection ∇^E. On the contrary, the connection ∇^L is essentially determined by the symplectic form ω, and the positivity property of its curvature is the fundamental ingredient that makes the construction possible.

Definition 2. *A section s of a vector bundle $E \otimes L^k$ is said to be η-transverse to 0 if whenever $|s(x)| < \eta$, the covariant derivative $\nabla s(x) : T_xX \to (E \otimes L^k)_x$ is surjective and admits a right inverse whose norm is smaller than $\eta^{-1}.k^{-1/2}$. A family of sections is transverse to 0 if there exists an $\eta > 0$ such that η-transversality to 0 holds for all large values of k.*

In the case of line bundles, η-transversality to 0 simply means that the covariant derivative of the section is larger than $\eta k^{1/2}$ wherever the section is smaller than η. Also note that transversality to 0 is an *open* property : if s is η-transverse to 0, then any section σ such that $|s - \sigma| < \epsilon$ and $|\nabla s - \nabla \sigma| < k^{1/2}\epsilon$ is automatically $(\eta - \epsilon)$-transverse to 0. The following holds clearly, independently of the choice of the connections on the vector bundles :

Proposition 1. *Let s_k be sections of the vector bundles $E \otimes L^k$ which are simultaneously asymptotically holomorphic and transverse to 0. Then for all large enough k, the zero sets W_k of s_k are embedded symplectic submanifolds in X. Furthermore, the submanifolds W_k are asymptotically J-holomorphic, i.e. $J(TW_k)$ is within $O(k^{-1/2})$ of TW_k.*

The result obtained by Donaldson [1] can be expressed as follows :

Theorem 1. *For all large k there exist sections of the line bundles L^k which are transverse to 0 and asymptotically holomorphic (with respect to connections of curvature $-ik\omega$ on L^k).*

Our main result is the following (the extension to almost-complex structures that depend on t was suggested by the referee) :

Theorem 2. *Let E be a complex vector bundle of rank r over X, and let a parameter space T be either $\{0\}$ or $[0, 1]$. Let $(J_t)_{t \in T}$ be a family of almost-complex structures on X compatible with ω. Fix a constant $\epsilon > 0$, and let $(s_{t,k})_{t\in T, k\geq K}$ be a sequence of families of asymptotically J_t-holomorphic sections of $E \otimes L^k$ defined for all large k, such that the sections $s_{t,k}$ and their derivatives depend continuously on t.*

Then there exist constants $\tilde{K} \geq K$ and $\eta > 0$ (depending only on ϵ, the geometry of X and the bounds on the derivatives of $s_{t,k}$), and a sequence $(\sigma_{t,k})_{t \in T, k \geq \tilde{K}}$ of families of asymptotically J_t-holomorphic sections of $E \otimes L^k$ defined for all $k \geq \tilde{K}$, such that

(a) the sections $\sigma_{t,k}$ and their derivatives depend continuously on t,
(b) for all $t \in T$, $|\sigma_{t,k} - s_{t,k}| < \epsilon$ and $|\nabla \sigma_{t,k} - \nabla s_{t,k}| < k^{1/2}\epsilon$,
(c) for all $t \in T$, $\sigma_{t,k}$ is η-transverse to 0.

Note that, since we allow the almost-complex structure on X to depend on t, great care must be taken as to the choice of the metric on X used for the estimates on derivatives. The most reasonable choice, and the one which will be made in the proof, is to always use the same metric, independently of t (so, there is no relation between g, ω and J_t). However, it is clear from the statement of the theorem that, since the spaces X and T are compact, any change in the choice of metric can be absorbed by simply changing the constants $\tilde{K}$ and η, and so the result holds in all generality.

Theorem 2 has many consequences. Among them, we mention the following extension of Donaldson's result to higher rank bundles :

Corollary 1. *For any complex vector bundle E over X and for all large k, there exist asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0, and thus whose zero sets are embedded symplectic submanifolds in X. Furthermore given a sequence of asymptotically holomorphic sections of $E \otimes L^k$ and a constant $\epsilon > 0$, we can require that the transverse sections lie within ϵ in C^0 sense (and $k^{1/2}\epsilon$ in C^1 sense) of the given sections.*

Therefore, the homology classes that one can realize by this construction include all classes whose Poincaré dual is of the form $[\frac{k\omega}{2\pi}]^r + c_1.[\frac{k\omega}{2\pi}]^{r-1} + \ldots + c_r$, with $c_1, \ldots, c_r$ the Chern classes of any complex vector bundle and k any sufficiently large integer.

An important result that one can obtain on the sequences of submanifolds constructed using Corollary 1 is the following isotopy result derived from the case where $T = [0,1]$ in Theorem 2 and which had been conjectured by Donaldson in the case of line bundles :

Corollary 2. *Let E be any complex vector bundle over X, and let $s_{0,k}$ and $s_{1,k}$ be two sequences of sections of $E \otimes L^k$. Assume that these sections are asymptotically holomorphic with respect to almost-complex structures J_0 and J_1 respectively, and that they are ϵ-transverse to 0. Then for all large k the zero sets of $s_{0,k}$ and $s_{1,k}$ are isotopic through asymptotically holomorphic symplectic submanifolds. Moreover, this isotopy can be realized through symplectomorphisms of X.*

This result follows from Theorem 2 by defining sections $s_{t,k}$ and almost-complex structures J_t that interpolate between $(s_{0,k}, J_0)$ and $(s_{1,k}, J_1)$ in the following way : for $t \in [0, \frac{1}{3}]$, let $s_{t,k} = (1-3t)s_{0,k}$ and $J_t = J_0$; for $t \in [\frac{1}{3}, \frac{2}{3}]$, let $s_{t,k} = 0$ and take J_t to be a path of compatible almost-complex structures from J_0 to J_1 (this is possible since the space of compatible almost-complex structures is connected) ; and for $t \in [\frac{2}{3}, 1]$, let $s_{t,k} = (3t-2)s_{1,k}$ and $J_t = J_1$. One can then apply Theorem 2 and obtain for all large k and for all $t \in [0,1]$ sections $\sigma_{t,k}$ that differ from $s_{t,k}$ by at most $\epsilon/2$ and are η-transverse to 0 for some η. Since transversality to 0 is an open property, the submanifolds cut out by $\sigma_{0,k}$ and $\sigma_{1,k}$ are clearly isotopic to those cut out by $s_{0,k}$ and $s_{1,k}$. Moreover, the family $\sigma_{t,k}$ gives an isotopy between the zero sets of $\sigma_{0,k}$ and $\sigma_{1,k}$. So the constructed submanifolds are isotopic. The proof that this isotopy can be realized through symplectomorphisms of X will be given in Section 4.

As a first step in the characterization of the topology of the constructed submanifolds, we also prove the following statement, extending the result obtained by Donaldson in the case of the line bundles L^k :

Proposition 2. *Let E be a vector bundle of rank r over X, and let W_k be a sequence of symplectic submanifolds of X constructed as the zero sets of asymptotically holomorphic sections s_k of $E \otimes L^k$ which are transverse to 0, for all large k. Then when k is sufficiently large, the inclusion $i : W_k \to X$ induces an isomorphism on homotopy groups π_p for $p < n - r$, and a surjection on π_{n-r}. The same property also holds for homology groups.*

Section 2 contains the statement and proof of the local result on which the whole construction relies. Section 3 deals with the proof of a semi-global statement, using a globalization process to obtain results on large subsets of X from the local picture.

The proofs of Theorem 2 and Corollary 2 are then completed in Section 4. Section 5 contains miscellaneous results on the topology and geometry of the obtained submanifolds, including Proposition 2.

Acknowledgments. The author wishes to thank Professor Mikhael Gromov (IHES) for valuable suggestions and guidance throughout the elaboration of this paper, and Professor Jean-Pierre Bourguignon (Ecole Polytechnique) for his support.

2. The local result

The proof of Theorem 2 relies on a local transversality result for approximatively holomorphic functions, which we state and prove immediately.

Proposition 3. *There exists an integer p depending only on the dimension n, with the following property : let δ be a constant with $0 < \delta < \frac{1}{2}$, and let $\sigma = \delta.\log(\delta^{-1})^{-p}$. Let $(f_t)_{t\in T}$ be a family of complex-valued functions over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$, depending continuously on the parameter $t \in T$ and satisfying for all t the following bounds over B^+ :*

$$|f_t| \leq 1, \qquad |\bar\partial f_t| \leq \sigma, \qquad |\nabla\bar\partial f_t| \leq \sigma.$$

Then there exists a family of complex numbers $w_t \in \mathbb{C}$, depending continuously on t, such that for all $t \in T$, $|w_t| \leq \delta$, and $f_t - w_t$ has a first derivative larger than σ at any point of the interior ball B of radius 1 where its norm is smaller than σ.

Proposition 3 extends a similar result proved in detail in [1], which corresponds to the case where $T = \{0\}$. The proof of Proposition 3 is based on the same ideas as Donaldson's proof, which is in turn based on considerations from real algebraic geometry following the method of Yomdin [7][3], with the only difference that we must get everything to depend continuously on t. Note that this statement is false for more general parameter spaces T than $\{0\}$ and $[0,1]$, since for example when T is the unit disc in $\mathbb{C}$ and $f_t(z) = t$, one looks for a continuous map $t \mapsto w_t$ of the disc to itself without a fixed point, in contradiction with Brouwer's theorem.

The idea is to deal with polynomial functions g_t approximating f_t, for which a general result on the complexity of real semi-algebraic sets gives constraints on the near-critical levels. This part of the proof is similar to that given in [1], so we skip the details. To obtain polynomial functions, we approximate f_t first by a continuous family of holomorphic functions $\tilde f_t$ differing from f_t by at most a fixed multiple of σ in C^1 sense, using that $\bar\partial f_t$ is small. The polynomials g_t are then obtained by truncating the Taylor series expansion of $\tilde f_t$ to a given degree. It can be shown that by this method one can obtain polynomial functions g_t of degree d less than a constant times $\log(\sigma^{-1})$, such that g_t differs from f_t by at most $c.\sigma$ in C^1 sense, where c is a fixed constant (see [1]). This approximation process does not hold on the whole ball where f_t is defined, which is why we needed f_t to be defined on B^+ to get a result over the slightly smaller ball B (see Lemmas 27 and 28 of [1]).

For a given complex-valued function h over B, call $Y_{h,\epsilon}$ the set of all points in B where the derivative of h has norm less than ϵ, and call $Z_{h,\epsilon}$ the ϵ-tubular neighborhood of $h(Y_{h,\epsilon})$. What we wish to construct is a path w_t avoiding by at least σ all near-critical levels of f_t, i.e. consisting of values that lie outside of $Z_{f_t,\sigma}$. Since g_t is within $c.\sigma$ of f_t, it is clear that $Z_{f_t,\sigma}$ is contained in $Z_t = Z_{g_t,(c+1)\sigma}$. However a general result on the complexity of real semi-algebraic sets yields constraints on the set $Y_{g_t,(c+1)\sigma}$. The precise statement which one applies to the real polynomial $|dg_t|^2$ is the following (Proposition 25 of [1]) :

Lemma 1. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be a polynomial function of degree d, and let $S(\theta) \subset \mathbb{R}^m$ be the subset $S(\theta) = \{x \in \mathbb{R}^m : |x| \leq 1, F(x) \leq 1 + \theta\}$. Then for arbitrarily small $\theta > 0$ there exist fixed constants C and ν depending only on the dimension m such that $S(0)$ may be decomposed into pieces $S(0) = S_1 \cup S_2 \cdots \cup S_A$, where $A \leq Cd^\nu$, in such a way that any pair of points in the same piece S_r can be joined by a path in $S(\theta)$ of length less than Cd^ν.*

So, as described in [1], given any fixed t, the set $Y_{g_t,(c+1)\sigma}$ of near-critical points of the polynomial function g_t of degree d can be subdivided into at most $P(d)$ subsets, where P is a fixed polynomial, in such a way that two points lying in the same subset can be joined by a path of length at most $P(d)$ inside $Y_{g_t,2(c+1)\sigma}$. It follows that the image by g_t of $Y_{g_t,(c+1)\sigma}$ is contained in the union of $P(d)$ discs of radius at most $2(c+1)\sigma P(d)$, so that the set Z_t of values which we wish to avoid is contained in the union Z_t^+ of $P(d)$ discs of radius $\sigma Q(d)$, where $Q = 3(c+1)P$ is a fixed polynomial and $d = O(\log \sigma^{-1})$.

If one assumes δ to be larger than $\sigma Q(d)P(d)^{1/2}$, it follows immediately from this constraint on Z_t that Z_t cannot fill the disc D of all complex numbers of norm at most δ : this immediately proves the case $T = \{0\}$. However, when $T = [0,1]$, we also need w_t to depend continuously on t. For this purpose, we show that if δ is large enough, $D - Z_t^+$, when decomposed into connected components, splits into several small components and only *one* large component.

Indeed, given a component C of $D - Z_t^+$, the simplest situation is that it does not meet the boundary of D. Then its boundary is a curve consisting of pieces of the boundaries of the balls making up Z_t^+, so its length is at most $2\pi P(d)Q(d)\sigma$, and it follows that C has diameter less than $\pi P(d)Q(d)\sigma$. Considering two components C_1 and C_2 which meet the boundary of D at points z_1 and z_2, we can consider an arc γ joining the boundary of D to itself that separates C_1 from C_2 and is contained in the boundary of Z_t^+. Assuming that δ is larger than e.g. $100P(d)Q(d)\sigma$, since the length of γ is at most $2\pi P(d)Q(d)\sigma$, it must stay close to either z_1 or z_2 in order to separate them : γ must remain within a distance of at most $10P(d)Q(d)\sigma$ from one of them. It follows that there exists $i \in \{1,2\}$ such that C_i is contained in the ball of radius $10P(d)Q(d)\sigma$ centered at z_i. So all components of $D - Z_t^+$ except at most one are contained in balls of radius $R(d)\sigma$, for some fixed polynomial R. Furthermore, the number of components of $D - Z_t^+$ is bounded by a value directly related to the number of balls making up Z_t^+, so that, increasing R if necessary, the number of components of $D - Z_t^+$ is also bounded by $R(d)$.

Assuming that δ is much larger than $R(d)^{3/2}\sigma$, the area $\pi\delta^2$ of D is much larger than $\pi R(d)^3\sigma^2$, so that the small components of $D - Z_t^+$ cannot fill it, and there must be a single large component. Getting back to $D - Z_t$, which was the set in which we had to choose w_t, it contains $D - Z_t^+$ and differs from it by at most $Q(d)\sigma$, so that, letting $U(t)$ be the component of $D - Z_t$ containing the large component of $D - Z_t^+$, it is the only large component of $D - Z_t$. The component $U(t)$ is characterized by the property that it is the only component of diameter more than $2R(d)\sigma$ in $D - Z_t$.

So the existence of a single large component $U(t)$ in $D - Z_t$ is proved upon the assumption that δ is large enough, namely larger than $\sigma.\Phi(d)$ where Φ is a given fixed polynomial that can be expressed in terms of P, Q and R (so Φ depends only on the dimension n). Since d is bounded by a constant times $\log \sigma^{-1}$, it is not hard to see that there exists an integer p such that, for all $0 < \delta < \frac{1}{2}$, the relation $\sigma = \delta. \log(\delta^{-1})^{-p}$ implies that $\delta > \sigma.\Phi(d)$. This is the value of p which we choose in the statement of the proposition, thus ensuring that the above statements always hold.

Since $\bigcup_t \{t\} \times Z_t$ is a closed subset of $T \times D$, the open set $U(t)$ depends semi-continuously on t : let $U^-(t,\epsilon)$ be the set of all points of $U(t)$ at distance more than ϵ from $Z_t \cup \partial D$. We claim that, given any t and any small $\epsilon > 0$, for all τ close enough to t, $U(\tau)$ contains $U^-(t,\epsilon)$. To see this, we first show for all τ close to t, $U^-(t,\epsilon) \cap Z_\tau = \emptyset$. Assuming that such is not the case, one can get a sequence of points of Z_τ for $\tau \to t$ that belong to $U^-(t,\epsilon)$. From this sequence one can extract a convergent subsequence, whose limit belongs to $\bar{U}^-(t,\epsilon)$ and thus lies outside of Z_t, in contradiction with the fact that $\bigcup_t \{t\} \times Z_t$ is closed. So $U^-(t,\epsilon) \subset D - Z_\tau$ for all τ close enough to t. Making ϵ smaller if necessary, one may assume that $U^-(t,\epsilon)$ is connected, so that for τ close to t, $U^-(t,\epsilon)$ is necessarily contained in the large component of $D - Z_\tau$, namely $U(\tau)$.

It follows that $U = \bigcup_t \{t\} \times U(t)$ is an open connected subset of $T \times D$, and is thus path-connected. So we get a path $s \mapsto (t(s), w(s))$ joining $(0, w(0))$ to $(1, w(1))$ inside U, for any given $w(0)$ and $w(1)$ in $U(0)$ and $U(1)$. We then only have to make sure that $s \mapsto t(s)$ is strictly increasing in order to define $w_{t(s)} = w(s)$.

Getting the t component to increase strictly is in fact quite easy. Indeed, we first get it to be weakly increasing, by considering values $s_1 < s_2$ of the parameter such that $t(s_1) = t(s_2) = t$ and simply replacing the portion of the path between s_1 and s_2 by a path joining $w(s_1)$ to $w(s_2)$ in the connected set $U(t)$. Then, we slightly shift the path, using the fact that U is open, to get the t component to increase slightly over the parts where it was constant. Thus we can define $w_{t(s)} = w(s)$ and end the proof of Proposition 3.

3. The globalization process

3.1. **Statement of the result.** We will now prove a semi-global result using Proposition 3. The globalization process we describe here is based on that used by Donaldson in [1], but a significantly higher amount of work is required because we have to deal with bundles of rank larger than one. The important fact we use is that transversality to 0 is a *local* and *open* property.

Theorem 3. *Let U be any open subset of X, and let E be a complex vector bundle of rank $r \geq 0$ over U. Let $(J_t)_{t \in T}$ be a family of almost-complex structures on X compatible with ω. Fix a constant $\epsilon > 0$. Let $W_{t,k}$ be a family of symplectic submanifolds in U, obtained as the zero sets of asymptotically J_t-holomorphic sections $w_{t,k}$ of the vector bundles $E \otimes L^k$ which are η-transverse to 0 over U for some $\eta > 0$ and depend continuously on $t \in T$ (if the rank is $r = 0$, then we simply define $W_{t,k} = U$). Finally, let $(\sigma_{t,k})$ be a family of asymptotically J_t-holomorphic sections of L^k which depend continuously on t. Define $U_k^=$ to be the set of all points of U at distance more than $4k^{-1/3}$ from the boundary of U.*

Then for some $\tilde{\eta} > 0$ and for all large k, there exist asymptotically J_t-holomorphic sections $\tilde{\sigma}_{t,k}$ of L^k over U, depending continuously on t, and such that

(a) for all $t \in T$, $\tilde{\sigma}_{t,k}$ is equal to $\sigma_{t,k}$ near the boundary of U,

(b) $|\tilde{\sigma}_{t,k} - \sigma_{t,k}| < \epsilon$ and $|\nabla\tilde{\sigma}_{t,k} - \nabla\sigma_{t,k}| < k^{1/2}\epsilon$ for all t,

(c) the sections $(w_{t,k} + \tilde{\sigma}_{t,k})$ of $(E \oplus \mathbb{C}) \otimes L^k$ are $\tilde{\eta}$-transverse to 0 over U_k^- for all t.

Basically, this result states that the construction of Theorem 2 can be carried out, in the line bundle case, in such a way that the resulting sections are transverse to a given family of symplectic submanifolds.

As remarked in the introduction, the choice of the metric in the statement of the theorem is not obvious. We choose to use always the same metric g on X, rather than trying to work directly with the metrics g_t induced by ω and J_t.

3.2. **Local coordinates and sections.** The proof of Theorem 3 is based on the existence of highly localized asymptotically holomorphic sections of L^k near every point $x \in X$. First, we notice that near any point $x \in X$, we can define local *complex* Darboux coordinates (z_i), that is to say a symplectomorphism from a neighborhood of x in (X, ω) to a neighborhood of 0 in $\mathbb{C}^n$ with the standard symplectic form. Furthermore it is well-known that, by composing the coordinate map with a ($\mathbb{R}$-linear) symplectic transformation of $\mathbb{C}^n$, one can ensure that its differential at x induces a *complex linear* map from (T_xX, J_t) to $\mathbb{C}^n$ with its standard complex structure.

Since the almost-complex structure J_t is not integrable, the coordinate map cannot be made pseudo-holomorphic on a whole neighborhood of x. However, since the manifold X and the parameter space T are compact, the Nijenhuis tensor, which is the obstruction to the integrability of the complex structure J_t on X, is bounded by a fixed constant, and so are its derivatives. It follows that for a suitable choice of the Darboux coordinates, the coordinate map can be made nearly pseudo-holomorphic around x, in the sense that the antiholomorphic part of its differential vanishes at x and grows no faster than a constant times the distance to x. Furthermore, it is easy to check that the coordinate map can be chosen to depend continuously on the parameter t. So, we have the following lemma :

Lemma 2. *Near any point $x \in X$, there exist for all $t \in T$ complex Darboux coordinates depending continuously on t, such that the inverse $\psi_t : (\mathbb{C}^n, 0) \to (X, x)$ of the coordinate map is nearly pseudo-holomorphic with respect to the almost-complex structure J_t on X and the canonical complex structure on $\mathbb{C}^n$. Namely, the map ψ_t, which trivially satisfies $|\nabla\psi_t| = O(1)$ and $|\nabla\nabla\psi_t| = O(1)$ on a ball of fixed radius around 0, fails to be pseudo-holomorphic by an amount that vanishes at 0 and thus grows no faster than the distance to the origin, i.e. $|\bar{\partial}\psi_t(z)| = O(|z|)$, and $|\nabla\bar{\partial}\psi_t| = O(1)$.*

Fix a certain value of the parameter $t \in T$, and consider the hermitian connections with curvature $-ik\omega$ that we have put on L^k in the introduction. Near any point $x \in X$, using the local complex Darboux coordinates (z_i) we have just constructed, a suitable choice of a local trivialization of L^k leads to the following connection 1-form :

$$A_k = \frac{k}{4} \sum_{j=1}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j)$$

(it can be readily checked that $dA_k = -ik\omega$).

On the standard $\mathbb{C}^n$ with connection A_k, the function $s(z) = \exp(-k|z|^2/4)$ satisfies the equation $\bar\partial_{A_k} s = 0$ and the bound $|\nabla_{A_k} s| = O(k^{1/2})$. Multiplying this section by a cut-off function at distance $k^{-1/3}$ from the origin whose derivative is small enough, we get a section $\tilde s$ with small compact support. Since the coordinate map near x has small antiholomorphic part where $\tilde s$ is large, the local sections $\tilde s \circ \psi_t^{-1}$ of L^k defined near x by pullback of $\tilde s$ through the coordinate map can be easily checked to be asymptotically holomorphic with respect to J_t and A_k. Thus, for all large k and for any point $x \in X$, extending $\tilde s \circ \psi_t^{-1}$ by 0 away from x, we obtain asymptotically holomorphic sections $s_{t,k,x}$ of L^k.

Since T is compact, the metrics g_t induced on X by ω and J_t differ from the chosen reference metric g by a bounded factor. Therefore, it is clear from the way we constructed the sections $s_{t,k,x}$ that the following statement holds :

Lemma 3. *There exist constants $\lambda > 0$ and $c_s > 0$ such that, given any $x \in X$, for all $t \in T$ and large k, there exist sections $s_{t,k,x}$ of L^k over X with the following properties : the sections $s_{t,k,x}$ are asymptotically J_t-holomorphic ; they depend continuously on t ; the bound $|s_{t,k,x}| \ge c_s$ holds over the ball of radius $10k^{-1/2}$ around x ; and finally, $|s_{t,k,x}| \le \exp(-\lambda k.\mathrm{dist}_g(x,.)^2)$ everywhere on X.*

3.3. **General setup and strategy of proof.** In a first step, we wish to obtain sections $\tilde\sigma_{t,k}$ of L^k over U satisfying all the requirements of Theorem 3, except that we replace *(c)* by the weaker condition that the restriction of $\tilde\sigma_{t,k}$ to $W_{t,k}$ must be $\hat\eta$-transverse to 0 over $W_{t,k} \cap U_k^-$ for some $\hat\eta > 0$, where U_k^- is the set of all points of U at distance more than $2k^{-1/3}$ from the boundary of U. It will be shown later that the transversality to 0 of the restriction to $W_{t,k} \cap U_k^-$ of $\tilde\sigma_{t,k}$, together with the bounds on the second derivatives, implies the transversality to 0 of $(w_{t,k} + \tilde\sigma_{t,k})$ over $U_k^=$.

To start with, we notice that there exists a constant $c > 0$ such that $W_{t,k}$ is trivial at small scale, namely in the ball of radius $10c.k^{-1/2}$ around any point. Indeed, if $r = 0$ we just take $c = 1$, and otherwise we use the fact that $w_{t,k}$ is η-transverse to 0, which implies that at any $x \in W_{t,k}$, $|\nabla w_{t,k}(x)| > \eta.k^{1/2}$. Since $|\nabla\nabla w_{t,k}| < C_2.k$ for some constant C_2, defining $c = \frac{1}{100}\eta.C_2^{-1}$, the derivative $\nabla w_{t,k}$ varies by a factor of at most $\frac{1}{10}$ in the ball B of radius $10c.k^{-1/2}$ around x. It follows that $B \cap W_{t,k}$ is diffeomorphic to a ball.

In all the following, we work with a given fixed value of k, while keeping in mind that all the constants appearing in the estimates have to be independent of k.

For fixed k, we consider a finite set of points x_i of $U_k^- \subset U$ such that the balls of radius $c.k^{-1/2}$ centered around x_i cover U_k^-. A suitable choice of the points ensures that their number is $O(k^n)$. For fixed $D > 0$, this set can be subdivided into N subsets S_j such that the distance between two points in the same subset is at least $D.k^{-1/2}$. Furthermore, $N = O(D^{2n})$ can be chosen independent of k. The precise value of D (and consequently of N) will be determined later in the proof.

The idea is to start with the sections $\sigma_{t,k}$ of L^k and proceed in steps. Let $\mathcal{N}_j$ be the union of all balls of radius $c.k^{-1/2}$ around the points of S_i for all $i < j$. During the j-th step, we start from asymptotically J_t-holomorphic sections $\sigma_{t,k,j}$ which satisfy conditions *(a)* and *(b)*, and such that the restriction of $\sigma_{t,k,j}$ to $W_{t,k}$ is η_j-transverse to 0 over $W_{t,k} \cap \mathcal{N}_j$, for some constant η_j independent of k. For the first step, this requirement is void, but we choose $\eta_0 = \frac{\epsilon}{2}$ in order to obtain a total perturbation smaller than ϵ at the end of the process. We wish to construct $\sigma_{t,k,j+1}$ from $\sigma_{t,k,j}$ by subtracting small multiples $c_{t,k,x}s_{t,k,x}$ of the sections $s_{t,k,x}$ for $x \in S_j$, in such a way that the restrictions of the resulting sections are η_{j+1}-transverse to 0, for some small η_{j+1}, over the intersection of $W_{t,k}$ with all balls of radius $c.k^{-1/2}$ around points in S_j. Furthermore, if the coefficients of the linear combination are chosen much smaller than η_j, transversality to 0 still holds over $W_{t,k} \cap \mathcal{N}_j$. Also, since the coefficients $c_{t,k,x}$ are bounded, the resulting sections, which are sums of asymptotically holomorphic sections, remain asymptotically holomorphic. So we need to find, for all $x \in S_j$, small coefficients $c_{t,k,x}$ so that $\sigma_{t,k,j} - c_{t,k,x}s_{t,k,x}$ has the desired properties near x.

3.4. **Obtaining transversality near a point of S_j.** In what follows, x is a given point in S_j, and B_x is the ball of radius $c.k^{-1/2}$ around x. Let Ω be the closure of the open subset of T containing all t such that $B_x \cap W_{t,k}$ is not empty (when $r = 0$, one gets $\Omega = T$). When Ω is empty, it is sufficient to define $c_{t,k,x} = 0$ for all t. Otherwise, $\Omega = \{0\}$ when $T = \{0\}$, and when $T = [0, 1]$ clearly Ω is a union of disjoint closed intervals. In any case, we choose a component I of Ω, i.e. either a closed interval or a point.

We can then define for all $t \in I$ a point x_t belonging to $\overline{B_x} \cap W_{t,k}$, in such a way that x_t depends continuously on t, since $W_{t,k}$ depends continuously on t and always intersects B_x in a nice way (when $r = 0$ one can simply choose $x_t = x$). Let $\hat{B}_t$ be the ball in $W_{t,k}$ of radius $3c.k^{-1/2}$ (for the metric induced by g) centered at x_t. Because of the bounds on the second derivatives of $w_{t,k}$, we know that $\hat{B}_t$ contains $B_x \cap W_{t,k}$ for all $t \in I$. We now want to define a nearly holomorphic diffeomorphism from a neighborhood of 0 in $\mathbb{C}^{n-r}$ to $\hat{B}_t$.

Let $\hat{B}$ be the ball of radius $4ck^{-1/2}$ around 0 in $\mathbb{C}^{n-r}$, and let $\hat{B}^-$ be the smaller ball of radius $3ck^{-1/2}$ around 0. We claim the following :

Lemma 4. *For all $t \in I$, there exist diffeomorphisms θ_t from $\hat{B}$ to a neighborhood of x_t in $W_{t,k}$, depending continuously on t, such that $\theta_t(0) = x_t$ and $\theta_t(\hat{B}^-) \supset \hat{B}_t$, and satisfying the following estimates over $\hat{B}$:*

$$|\bar\partial\theta_t| = O(k^{-1/2}), \quad |\nabla\theta_t| = O(1), \quad |\nabla\bar\partial\theta_t| = O(1), \quad |\nabla\nabla\theta_t| = O(k^{1/2}).$$

Proof. Recall that, by Lemma 2, there exist local complex Darboux coordinates on X near x depending continuously on t with the property that the inverse map $\psi_t : (\mathbb{C}^n, 0) \to (X, x)$ satisfies the following bounds at all points at distance $O(k^{-1/2})$ from x :

$$|\bar\partial\psi_t| = O(k^{-1/2}), \quad |\nabla\psi_t| = O(1), \quad |\nabla\bar\partial\psi_t| = O(1), \quad |\nabla\nabla\psi_t| = O(1).$$

Let $\mathcal{T}_t$ be the kernel of the complex linear map $\partial w_{t,k}(x_t)$ in $T_{x_t}X$: it is within $O(k^{-1/2})$ of the tangent space to $W_{t,k}$ at x_t, but $\mathcal{T}_t$ is preserved by J_t. Composing ψ_t with a translation and a rotation in $\mathbb{C}^n$, one gets maps $\tilde\psi_t$ satisfying the same requirements as ψ_t, but with $\tilde\psi_t(0) = x_t$ and such that the differential of $\tilde\psi_t$ at 0 maps the span of the $n - r$ first coordinates to $\mathcal{T}_t$.

Furthermore, X and T are compact, so the metrics g_t induced by ω and J_t differ from the reference metric g by at most a fixed constant. It follows that, composing $\tilde\psi_t$ with a fixed dilation of $\mathbb{C}^n$ if necessary, one may also require that the image by

$\tilde{\psi}_t$ of the ball of radius $3ck^{-1/2}$ around 0 contains the ball of radius $4ck^{-1/2}$ around x for the reference metric g. The only price to pay is that $\tilde{\psi}_t$ is no longer a local symplectomorphism ; all other properties still hold.

Since by definition of c the submanifolds $W_{t,k}$ are trivial over the considered balls, it follows from the implicit function theorem that $W_{t,k}$ can be parametrized around x_t in the chosen coordinates as the set of points of the form $\tilde{\psi}_t(z, \tau_t(z))$ for $z \in \mathbb{C}^{n-r}$, where $\tau_t : \mathbb{C}^{n-r} \to \mathbb{C}^r$ satisfies $\tau_t(0) = 0$ and $\nabla\tau_t(0) = O(k^{-1/2})$. The derivatives of τ_t can be easily computed, since it is characterized by the equation

$$w_{t,k}(\tilde{\psi}_t(z, \tau_t(z))) = 0.$$

Notice that it follows from the transversality to 0 of $w_{t,k}$ that $|\nabla w_{t,k} \circ d\tilde{\psi}_t(v))|$ is larger than a constant times $k^{1/2}|v|$ for all $v \in 0 \times \mathbb{C}^r$. Combining this estimate with the bounds on the derivatives of $w_{t,k}$ given by asymptotic holomorphicity and the above bounds on those of $\tilde{\psi}_t$, one gets the following estimates for τ_t over the ball $\hat{B}$:

$$|\bar{\partial}\tau_t| = O(k^{-1/2}), \quad |\nabla\tau_t| = O(1), \quad |\nabla\bar{\partial}\tau_t| = O(1), \quad |\nabla\nabla\tau_t| = O(k^{1/2}).$$

It is then clear that $\theta_t(z) = \tilde{\psi}_t(z, \tau_t(z))$ satisfies all the required properties. □

Now that a local identification between $W_{t,k}$ and $\mathbb{C}^{n-r}$ is available, we define the restricted sections $\hat{s}_{t,k,x}(z) = s_{t,k,x}(\theta_t(z))$ and $\hat{\sigma}_{t,k,j}(z) = \sigma_{t,k,j}(\theta_t(z))$. Since $s_{t,k,x}$ and $\sigma_{t,k,j}$ are both asymptotically holomorphic, the estimates on θ_t imply that $\hat{s}_{t,k,x}$ and $\hat{\sigma}_{t,k,j}$, as sections of the pull-back of L^k over the ball $\hat{B}$, are also asymptotically holomorphic. Furthermore, they clearly depend continuously on $t \in I$, and $\hat{s}_{t,k,x}$ remains larger than a fixed constant $c_s > 0$ over $\hat{B}$. We can then define the complex-valued functions $f_{t,k,x} = \hat{\sigma}_{t,k,j}/\hat{s}_{t,k,x}$ over $\hat{B}$, which are clearly asymptotically holomorphic too.

After dilation of $\hat{B}$ by a factor of $3c.k^{1/2}$, all hypotheses of Proposition 3 are satisfied with δ as small as desired, provided that k is large enough. Indeed, the asymptotic holomorphicity of $f_{t,k,x}$ implies that, for large k, the antiholomorphic part of the function over the dilated ball is smaller than $\sigma = \delta.(\log\delta^{-1})^{-p}$. So the local result implies that there exist complex numbers $c_{t,k,x}$ of norm less than δ and depending continuously on $t \in I$, such that the functions $f_{t,k,x} - c_{t,k,x}$ are σ-transverse to 0 over the ball $\hat{B}^-$ of radius $3c.k^{-1/2}$ around 0 in $\mathbb{C}^{n-r}$. We now notice that the sections $\hat{g}_{t,k,x} = \hat{\sigma}_{t,k,j} - c_{t,k,x}\hat{s}_{t,k,x}$, which clearly depend continuously on t and are asymptotically holomorphic, are σ'-transverse to 0 over $\hat{B}^-$, for some σ' differing from σ by at most a constant factor. Indeed,

$$\nabla\hat{g}_{t,k,x} = \nabla(\hat{s}_{t,k,x}(f_{t,k,x} - c_{t,k,x})) = \hat{s}_{t,k,x}\nabla f_{t,k,x} - (f_{t,k,x} - c_{t,k,x})\nabla\hat{s}_{t,k,x}.$$

Wherever $\hat{g}_{t,k,x}$ is very small, so is $f_{t,k,x} - c_{t,k,x}$, and $\nabla f_{t,k,x}$ is thus large. Since $\hat{s}_{t,k,x}$ remains larger than some $c_s > 0$ and $\nabla\hat{s}_{t,k,x}$ is bounded by a constant times $k^{1/2}$, it follows that $\nabla\hat{g}_{t,k,x}$ is large wherever $\hat{g}_{t,k,x}$ is very small. Putting the right constants in the right places, one easily checks that $\hat{g}_{t,k,x}$ is σ'-transverse to 0 with σ/σ' bounded by a fixed constant.

We now notice that the restrictions to $W_{t,k}$ of the sections $g_{t,k,x} = \sigma_{t,k,j} - c_{t,k,x}s_{t,k,x}$ of L^k over U, which clearly are asymptotically J_t-holomorphic and depend continuously and t, are also σ''-transverse to 0 over $\hat{B}_t$ for some σ'' differing from σ' by at most a constant factor. Indeed, $\hat{B}_t$ is contained in the set of all points of the form $\theta_t(z)$ for $z \in \hat{B}^-$, and

$$g_{t,k,x}(\theta_t(z)) = \hat{\sigma}_{t,k,j}(z) - c_{t,k,x}\hat{s}_{t,k,x}(z) = \hat{g}_{t,k,x}(z),$$

so wherever $g_{t,k,x}$ is smaller than σ', the derivative of $\hat{g}_{t,k,x}$ is larger than $\sigma'.k^{1/2}$, and since $\nabla\theta_t$ is bounded by a fixed constant, $\nabla g_{t,k,x}$ is large too.

Next we extend the definition of $c_{t,k,x}$ to all $t \in T$, in the case of $T = [0,1]$, since we have defined it only over the components of Ω. However when $t \notin \Omega$, $W_{t,k}$ does not meet the ball B_x, so that there is no transversality requirement. Thus the only constraints are that $c_{t,k,x}$ must depend continuously on t and remain smaller than δ for all t. These conditions are easy to satisfy, so we have proved the following :

Lemma 5. *For all large k there exist complex numbers $c_{t,k,x}$ smaller than δ and depending continuously on $t \in T$ such that the restriction to $W_{t,k}$ of $\sigma_{t,k,j} - c_{t,k,x}s_{t,k,x}$ is σ''-transverse to 0 over $W_{t,k} \cap B_x$. Furthermore, for some constant p' depending only on the dimension, σ'' is at least $\delta.(\log \delta^{-1})^{-p'}$.*

3.5. **Constructing $\sigma_{t,k,j+1}$ from $\sigma_{t,k,j}$.** We can now define the sections $\sigma_{t,k,j+1}$ of L^k over U by

$$\sigma_{t,k,j+1} = \sigma_{t,k,j} - \sum_{x \in S_j} c_{t,k,x}s_{t,k,x}.$$

Clearly the sections $\sigma_{t,k,j+1}$ are asymptotically holomorphic and depend continuously on $t \in T$. Furthermore, any two points in S_j are distant of at least $D.k^{-1/2}$ with $D > 0$, so the total size of the perturbation is bounded by a fixed multiple of δ. So, choosing δ smaller than η_j over a constant factor (recall that η_j is the transversality estimate of the previous step of the iterative process), we can ensure that $|\sigma_{t,k,j+1} - \sigma_{t,k,j}| < \frac{\eta_j}{2}$ and $|\nabla\sigma_{t,k,j+1} - \nabla\sigma_{t,k,j}| < \frac{\eta_j}{2}k^{1/2}$. As a direct consequence, the restriction to $W_{t,k}$ of $\sigma_{t,k,j+1}$ is $\frac{\eta_j}{2}$-transverse to 0 wherever the restriction of $\sigma_{t,k,j}$ is η_j-transverse to 0, including over $W_{t,k} \cap \mathcal{N}_j$ (recall that $\mathcal{N}_j = \bigcup_{i<j} \bigcup_{x \in S_i} B_x$).

Letting $\eta_{j+1} = \frac{1}{2}\sigma''$, it is known that for all $x \in S_j$ the restriction of $\sigma_{t,k,j} - c_{t,k,x}s_{t,k,x}$ to $W_{t,k}$ is $2\eta_{j+1}$-transverse to 0 over $B_x \cap W_{t,k}$. So, in order to prove that the restriction to $W_{t,k}$ of $\sigma_{t,k,j+1}$ is η_{j+1}-transverse to 0 over $W_{t,k} \cap \mathcal{N}_{j+1}$, it is sufficient to check that given $x \in S_j$, over B_x, the sum of the perturbations corresponding to all points $y \in S_j$ distinct from x is smaller than η_{j+1}, and the sum of their derivatives is smaller than $\eta_{j+1}k^{1/2}$. In other words, since several contributions were added at the same time (one at each point of S_j), we have to make sure that they cannot interfere.

This is where the parameter D (minimum distance between two points in S_j) is important : indeed, over B_x, by Lemma 3, each of the contributions of the other points in S_j is at most of the order of $\delta.\exp(-\lambda D^2)$, and the sum of these terms is $O(\eta_j.\exp(-\lambda D^2))$. Similarly, the derivative of that sum is $O(\eta_j.\exp(-\lambda D^2).k^{1/2})$. So the requirement that the sum of the contributions of all points of S_j distinct from x be smaller than η_{j+1} corresponds to an inequality of the form $K_0 \exp(-\lambda D^2) < \eta_{j+1}/\eta_j$, where K_0 is a fixed constant depending only on the geometry of X. Recalling that η_{j+1} is no smaller than $\eta_j.\log(\eta_j^{-1})^{-P}$ for some fixed integer P, the required inequality is

$$\exp(\lambda D^2) > K_0.\log(\eta_j^{-1})^P.$$

This inequality, which *does not depend on* k, must be satisfied by every η_j, for each of the N steps of the process.

To check that the condition on D can be enforced at all steps, we must recall that the number of steps in the process is $N = O(D^{2n})$, and study the sequence (η_j) given by a fixed $\eta_0 > 0$ and the inductive definition described above. It can be shown (see Lemma 24 of [1]) that the sequence (η_j) satisfies for all j a bound of the type $\log(\eta_j^{-1}) = O(j.\log(j))$. It follows that $\log(\eta_N^{-1})^P = O(D^{2nP}.log(D^{2n})^P)$, which is clearly subexponential : a choice of sufficiently large D thus ensures that the required inequality holds at all steps. So the inductive process described above is valid, and leads to sections $\tilde{\sigma}_{t,k} = \sigma_{t,k,N}$ which are asymptotically J_t-holomorphic, depend continuously on t, and whose restrictions to $W_{t,k}$ are $\hat{\eta}$-transverse to 0 over U_k^- for $\hat{\eta} = \eta_N$. Furthermore, $\tilde{\sigma}_{t,k}$ is equal to $\sigma_{t,k}$ near the boundary of U because we only added a linear combination of sections $s_{t,k,x}$ for $x \in U_k^-$, and $s_{t,k,x}$ vanishes by construction outside of the ball of radius $k^{-1/3}$ around x. Moreover, $\tilde{\sigma}_{t,k}$ differs from $\sigma_{t,k}$ by at most $\sum_j \eta_j$, which is less than $2\eta_0 = \epsilon$. So to complete the proof of Theorem 3 we only have to show that the transversality result on $\tilde{\sigma}_{t,k|W_{t,k}}$ implies the transversality to 0 of $(w_{t,k} + \tilde{\sigma}_{t,k})$ over $U_k^=$.

3.6. **Transversality to 0 over $U_k^=$.** At a point $x \in W_{t,k} \cap U_k^-$ where $|\tilde{\sigma}_{t,k}| < \hat{\eta}$, we know that $\nabla w_{t,k}$ is surjective and vanishes in all directions tangential to $W_{t,k}$, while $\nabla\tilde{\sigma}_{t,k}$ has a tangential component larger than $\hat{\eta}.k^{1/2}$. It follows that $\nabla(w_{t,k} + \tilde{\sigma}_{t,k})$ is surjective. We now construct a right inverse $R : (E_x \oplus \mathbb{C}) \otimes L_x^k \to T_xX$ whose norm is $O(k^{-1/2})$.

Considering a unit length element u of L_x^k, there exists a vector $\hat{u} \in T_xW_{t,k}$ of norm at most $(\hat{\eta}.k^{1/2})^{-1}$ such that $\nabla\tilde{\sigma}_{t,k}(\hat{u}) = u$. Clearly $\nabla w_{t,k}(\hat{u}) = 0$ because $\hat{u} \in T_xW_{t,k}$, so we define $R(u) = \hat{u}$. Now consider an orthonormal frame (v_i) in $E_x \otimes L_x^k$. It follows from the η-transversality to 0 of $w_{t,k}$ that $\nabla_x w_{t,k}$ has a right inverse of norm smaller than $(\eta.k^{1/2})^{-1}$, so we obtain vectors $\hat{v}_i$ in T_xX such that $\nabla w_{t,k}(\hat{v}_i) = v_i$ and $|\hat{v}_i| < (\eta.k^{1/2})^{-1}$. There exist coefficients λ_i such that $\nabla\tilde{\sigma}_{t,k}(\hat{v}_i) = \lambda_i.u$, with $|\lambda_i| < C.k^{1/2}.|\hat{v}_i| < C.\eta^{-1}$, for some constant C such that $|\nabla\tilde{\sigma}_{t,k}| < C.k^{1/2}$ everywhere. So we define $R(v_i) = \hat{v}_i - \lambda_i\hat{u}$, which completes the determination of R.

The norm of R is, by construction, smaller than $K.k^{-1/2}$ for some K depending only on the constants above (C, η and $\hat{\eta}$). We thus know that $\nabla(w_{t,k} + \tilde{\sigma}_{t,k})$ has a right inverse smaller than $K.k^{-1/2}$ at any point of $W_{t,k} \cap U_k^-$ where $|\tilde{\sigma}_{t,k}| < \hat{\eta}$. Furthermore we know, from the definition of asymptotic holomorphicity, that $|\nabla\nabla(w_{t,k} + \tilde{\sigma}_{t,k})| < K'.k$ for some constant K'.

Consider a point x of $U_k^=$ where $|w_{t,k}|$ and $|\tilde{\sigma}_{t,k}|$ are both smaller than some α which is simultaneously smaller than $\frac{\hat{\eta}}{2}$, $\frac{\eta\hat{\eta}}{2C}$ and $\frac{\eta}{2KK'}$. From the η-transversality to 0 of $w_{t,k}$, we know that $\nabla w_{t,k}$ is surjective at x and has a right inverse smaller than $(\eta.k^{1/2})^{-1}$. Since the connection ∇ is unitary, applying the right inverse to $w_{t,k}$ itself, we can follow the downward gradient flow of $|w_{t,k}|$, and we are certain to reach a point y of $W_{t,k}$ at a distance d from the starting point x no larger than $\alpha.(\eta.k^{1/2})^{-1}$, which is simultaneously smaller than $\frac{1}{2KK'}.k^{-1/2}$ and $\frac{\hat{\eta}}{2C}.k^{-1/2}$. Furthermore if k is large enough, $d < 2k^{-1/3}$ so that $y \in U_k^-$.

Since $|\nabla\tilde{\sigma}_{t,k}| < C.k^{1/2}$ everywhere, $|\tilde{\sigma}_{t,k}(y)| - |\tilde{\sigma}_{t,k}(x)| < C.k^{1/2}.d < \frac{\hat{\eta}}{2}$, so that $|\tilde{\sigma}_{t,k}(y)| < \hat{\eta}$, and the previous results apply at y. Also, since the second derivatives are bounded by $K'.k$ everywhere, $\nabla_x(w_{t,k}+\tilde{\sigma}_{t,k})$ differs from $\nabla_y(w_{t,k}+\tilde{\sigma}_{t,k})$ by at most $K'k.d$, which is smaller than $\frac{1}{2K}.k^{1/2}$, so that it is still surjective and admits a right inverse of norm $O(k^{-1/2})$. From this we infer immediately that $(w_{t,k}+\tilde{\sigma}_{t,k})$ is transverse to 0 over all of $U_k^{=}$, and the proof of Theorem 3 is complete.

4. The main result

4.1. **Proof of Theorem 2.** Theorem 2 follows from Theorem 3 by a simple induction argument. Indeed, to obtain asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0 over X for any vector bundle E, we start from the fact that E is locally trivial, so that there exists a finite covering of X by N open subsets U_j such that E is a trivial bundle on a small neighborhood of each U_j. We start initially from the sections $s_{t,k,0}=s_{t,k}$ of $E \otimes L^k$, and proceed iteratively, assuming at the beginning of the j-th step that we have constructed, for all large k, asymptotically holomorphic sections $s_{t,k,j}$ of $E \otimes L^k$ which are η_j-transverse to 0 on $\bigcup_{i<j} U_i$ for some $\eta_j > 0$ and differ from $s_{t,k}$ by at most $j\epsilon/N$.

Over a small neighborhood of U_j, we trivialize $E \simeq \underline{\mathbb{C}}^r$ and decompose the sections $s_{t,k,j}$ into their r components for this trivialization. Recall that, in order to define the connections on $E \otimes L^k$ for which asymptotic holomorphicity and transversality to 0 are expected, we have used a hermitian connection ∇^E on E. Because X is compact the connection 1-form of ∇^E in the chosen trivializations can be safely assumed to be bounded by a fixed constant. It follows that, up to a change in the constants, asymptotic holomorphicity and transversality to 0 over U_j with respect to the connections on $E \otimes L^k$ induced by ∇^E and ∇^L are equivalent to asymptotic holomorphicity and transversality to 0 with respect to the connections induced by ∇^L and the trivial connection on E in the chosen trivialization. So we actually do not have to worry about ∇^E.

Now let α be a constant smaller than both ϵ/rN and $\eta_j/2r$. First, using Theorem 3, we perturb the first component of $s_{t,k,j}$ over a neighborhood of U_j by at most α to make it transverse to 0 over a slightly smaller neighborhood. Next, using again Theorem 3, we perturb the second component by at most α so that the sum of the two first components is transverse to 0, and so on, perturbing the i-th component by at most α to make the sum of the i first components transverse to 0. The result of this process is a family of asymptotically J_t-holomorphic sections $s_{t,k,j+1}$ of $E \otimes L^k$ which are transverse to 0 over U_j. Furthermore, since the total perturbation is smaller than $r\alpha \le \eta_j/2$, transversality to 0 still holds over U_i for $i < j$, so that the hypotheses of the next step are satisfied. The construction thus leads to sections $\sigma_{t,k}=s_{t,k,N}$ which are transverse to 0 over all of X. Since at each of the N steps the total perturbation is less than ϵ/N, the sections $\sigma_{t,k}$ differ from $s_{t,k}$ by less than ϵ, and Theorem 2 is proved.

4.2. **Symplectic isotopies.** We now give the remaining part of the proof of Corollary 2, namely the following statement :

Proposition 4. *let $(W_t)_{t\in[0,1]}$ be a family of symplectic submanifolds in X. Then there exist symplectomorphisms $\Phi_t : X \to X$ depending continuously on t, such that $\Phi_0 = \mathrm{Id}$ and $\Phi_t(W_0) = W_t$.*

The following strategy of proof, based on Moser's ideas, was suggested to me by M. Gromov. The reader unfamiliar with these techniques may use [4] (pp. 91-101) as a reference.

It follows immediately from Moser's stability theorem that there exists a continuous family of symplectomorphisms $\phi_t : (W_0, \omega_{|W_0}) \to (W_t, \omega_{|W_t})$. Since the symplectic normal bundles to W_t are all isomorphic, Weinstein's symplectic neighborhood theorem allows one to extend these maps to symplectomorphisms $\psi_t : U_0 \to U_t$ such that $\psi_t(W_0) = W_t$, where U_t is a small tubular neighborhood of W_t for all t.

Let ρ_t be any family of *diffeomorphisms* of X extending ψ_t. Let $\omega_t = \rho_t^*\omega$ and $\Omega_t = -d\omega_t/dt$. We want to find vector fields ξ_t on X such that the 1-forms $\alpha_t = \iota_{\xi_t}\omega_t$ satisfy $d\alpha_t = \Omega_t$ and such that ξ_t is tangent to W_0 at any point of W_0. If this is possible, then define diffeomorphisms Ψ_t as the flow of the vector fields ξ_t, and notice that

$$\frac{d}{dt}(\Psi_t^*\rho_t^*\omega) = \Psi_t^*\left(\frac{d}{dt}(\rho_t^*\omega) + L_{\xi_t}(\rho_t^*\omega)\right) = \Psi_t^*(-\Omega_t + d\iota_{\xi_t}\omega_t) = 0.$$

So the diffeomorphisms $\rho_t \circ \Psi_t$ are actually symplectomorphisms of X. Furthermore Ψ_t preserves W_0 by construction, so $\rho_t \circ \Psi_t$ maps W_0 to W_t, thus giving the desired result.

So we are left with the problem of finding ξ_t, or equivalently α_t, such that $d\alpha_t = \Omega_t$ and $\xi_{t|W_0}$ is tangent to W_0. Note that, since ρ_t extends the symplectomorphisms ψ_t, one has $\omega_t = \omega$ and $\Omega_t = 0$ over U_0. It follows that the condition on $\xi_{t|W_0}$ is equivalent to the requirement that at any point $x \in W_0$, the ω-symplectic orthogonal N_xW_0 to T_xW_0 lies in the kernel of the 1-form α_t.

Since the closed 2-forms ω_t are all cohomologous, one has $[\Omega_t] = 0$ in $H^2(X, \mathbb{R})$, so there exist 1-forms β_t on X such that $d\beta_t = \Omega_t$. Remark that, although $\Omega_t = 0$ over U_0, one cannot ensure that $\beta_{t|U_0} = 0$ unless the class $[\Omega_t]$ also vanishes in the relative cohomology group $H^2(X, U_0; \mathbb{R})$. So we need to work a little more to find the proper 1-forms α_t.

Over U_0 one has $d\beta_t = \Omega_t = 0$, so β_t defines a class in $H^1(U_0, \mathbb{R})$. By further restriction, the forms $\beta_{t|W_0}$ are also closed 1-forms on W_0. Let π be a projection map $U_0 \to W_0$ such that at any point $x \in W_0$ the tangent space to $\pi^{-1}(x)$ is the symplectic normal space N_xW_0, and let $\gamma_t = \pi^*(\beta_{t|W_0})$. First we notice that, by construction, the 1-form γ_t is closed over U_0, and at any point $x \in W_0$ the space N_xW_0 lies in the kernel of γ_t. Furthermore the composition of π^* and the restriction map induces the identity map over $H^1(U_0, \mathbb{R})$, so $[\gamma_t] = [\beta_{t|U_0}]$ in $H^1(U_0, \mathbb{R})$. Therefore there exist functions f_t over U_0 such that $\gamma_t = \beta_t + df_t$ at any point of U_0.

Let g_t be any smooth functions over X extending f_t, and let $\alpha_t = \beta_t + dg_t$. The 1-forms α_t satisfy $d\alpha_t = d\beta_t = \Omega_t$, and since $\alpha_{t|U_0} = \gamma_t$ the space N_xW_0 also lies in the kernel of α_t at any $x \in W_0$. So Proposition 4 is proved.

5. Properties of the constructed submanifolds

5.1. **Proof of Proposition 2.** This proof is based on that of a similar result obtained by Donaldson [1] for the submanifolds obtained from Theorem 1 ($r = 1$). The result comes from a Morse theory argument, as described in [1]. Indeed, consider the real valued function $f = \log |s|^2$ over $X - W$ (where $W = s^{-1}(0)$). We only have to show that, if k is large enough, all its critical points are of index at least $n-r+1$. For this purpose, let x be a critical point of f, and let us compute the derivative $\bar\partial\partial f$ at x.

First we notice that x is also a critical point of $|s|^2$, so that s itself is not in the image of $\nabla_x s$. Recalling that s is η-transverse to 0 for some $\eta > 0$, it follows that $\nabla_x s$ is not surjective and thus $|s(x)| \geq \eta$.

Recalling that the scalar product is linear in the first variable and antilinear in the second variable, we compute the derivative

$$\partial \log |s|^2 = \frac{1}{|s|^2}(\langle \partial s, s\rangle + \langle s, \bar\partial s\rangle),$$

which equals zero at x. A first consequence is that, at x, $|\langle \partial s, s\rangle| = |\langle \bar\partial s, s\rangle| < C|s|$, where C is a constant bounding $\bar\partial s$ independently of k.

A second derivation, omitting the quantities that vanish at a critical point, yields that, at x,

$$\bar\partial\partial \log |s|^2 = \frac{1}{|s|^2}(\langle \bar\partial\partial s, s\rangle - \langle \partial s, \partial s\rangle + \langle \bar\partial s, \bar\partial s\rangle + \langle s, \partial\bar\partial s\rangle).$$

Recall that $\bar\partial\partial + \partial\bar\partial$ is equal to the part of type (1,1) of the curvature of the bundle $E \otimes L^k$. This is equal to $-ik\omega \otimes \mathrm{Id} + R$, where R is the part of type (1,1) of the curvature of E, so that at x,

$$\bar\partial\partial \log |s|^2 = -ik\omega + \frac{1}{|s|^2}(\langle R.s, s\rangle - \langle \partial\bar\partial s, s\rangle + \langle s, \partial\bar\partial s\rangle - \langle \partial s, \partial s\rangle + \langle \bar\partial s, \bar\partial s\rangle).$$

To go further, we have to restrict our choice of vectors to a subspace of the tangent space T_xX at x. Call Θ the space of all vectors v in T_xX such that $\partial s(v)$ belongs to the complex line generated by s in $(E \otimes L^k)_x$. The subspace Θ of T_xX is clearly stable by the almost-complex structure, and its complex dimension is at least $n-r+1$. For any vector $v \in \Theta$, $|\langle \partial s(v), s\rangle| = |\partial s(v)|.|s|$ is smaller than $|v|.|\langle \partial s, s\rangle| < C|v|.|s|$ where C is the same constant as above, so that ∂s is $O(1)$ over Θ.

Since $\bar\partial s = O(1)$ and $\partial\bar\partial s = O(k^{1/2})$ because of asymptotic holomorphicity, it is now known that the restriction to Θ of $\bar\partial\partial \log |s|^2$ is equal to $-ik\omega + O(k^{1/2})$. It follows that, for all large k, given any unit length vector $u \in \Theta$, the quantity $-2i\bar\partial\partial f(u, Ju)$, which equals $H_f(u) + H_f(Ju)$ where H_f is the Hessian of f at x, is negative. If the index of the critical point at x were less than $n-r+1$, there would exist a subspace $P \subset T_xX$ of real dimension at least $n+r$ over which H_f is non-negative, and the subspace $P \cap JP$ of real dimension at least $2r$ would necessarily intersect non-trivially with Θ whose real dimension is at least $2n-2r+2$, contradicting the previous remark. The index of the critical point x of f is thus at least $n-r+1$.

A standard Morse theory argument then implies that the inclusion $W \to X$ induces an isomorphism on all homotopy (and homology) groups up to π_{n-r-1} (resp. H_{n-r-1}), and a surjection on π_{n-r} (resp. H_{n-r}), which completes the proof of Proposition 2.

5.2. **Homology and Chern numbers of the submanifolds.** Proposition 2 allows one to compute the middle-dimensional Betti number $b_{n-r} = \dim H_{n-r}(W_k, \mathbb{R})$ of the constructed submanifolds. Indeed the tangent bundle TW_k and the normal bundle NW_k (isomorphic to the restriction to W_k of $E \otimes L^k$) are both symplectic vector bundles over W_k. So it is well-known (see e.g. [4], p. 67) that they admit underlying structures of complex vector bundles, uniquely determined up to homotopy (in our case there exist J-stable subspaces in TX very close to TW_k and NW_k, so after a small deformation one can think of these complex structures as induced by J). Furthermore one has $TW_k \oplus NW_k \simeq TX_{|W_k}$. It follows that, calling i the inclusion map $W_k \to X$, the Chern classes of the bundle TW_k can be computed from the relation

$$i^*c(TX) = i^*c(E \otimes L^k).c(TW_k).$$

Since $c_{n-r}(TW_k).[W_k]$ is equal to the Euler-Poincaré characteristic of W_k, and since the spaces $H_i(W_k, \mathbb{R})$ have the same dimension as $H_i(X, \mathbb{R})$ for $i < n - r$, the dimension of $H_{n-r}(W_k, \mathbb{R})$ follows immediately.

For further computations, we need an estimate on this dimension :

Proposition 5. *For any sequence of symplectic submanifolds $W_k \subset X$ of real codimension $2r$ obtained as the zero sets of asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0, the Chern classes of W_k are given by*

$$c_l(TW_k) = (-1)^l \binom{r+l-1}{l} (k\hat{\omega})^l + O(k^{l-1}),$$

where $\hat{\omega}$ denotes the class of $\frac{\omega}{2\pi}$ in the cohomology of W_k.

This can be proved by induction on l, starting from $c_0(TW_k) = 1$, since the above equality implies that

$$c_l(TW_k) = i^*c_l(TX) - \sum_{j=0}^{l-1} i^*c_{l-j}(E \otimes L^k).c_j(TW_k).$$

It can be checked that $i^*c_{l-j}(E \otimes L^k) = \binom{r}{l-j}(k\hat{\omega})^{l-j} + O(k^{l-j-1})$, so that the result follows from a combinatorial calculation showing that $\sum_{j=0}^{l}(-1)^j \binom{r}{l-j}\binom{r+j-1}{j} = 0$. □

Since $[W_k]$ is Poincaré dual in X to $c_r(E \otimes L^k)$, Proposition 5 yields that $\chi(W_k) = c_{n-r}(TW_k).[W_k] = (-1)^{n-r}\binom{n-1}{n-r}(k\hat{\omega})^{n-r}.(k\hat{\omega})^r + O(k^{n-1})$. Finally, Proposition 2 implies that $\chi(W_k) = (-1)^{n-r} \dim H_{n-r}(W_k, \mathbb{R}) + O(1)$, so that

$$\dim H_{n-r}(W_k, \mathbb{R}) = \binom{n-1}{n-r}[\tfrac{\omega}{2\pi}]^n.k^n + O(k^{n-1}).$$

5.3. **Geometry of the submanifolds.** Aside from the above topological information on the submanifolds, one can also try to characterize the *geometry* of W_k inside X. We prove the following result, expressing the fact that the middle-dimensional homology of W_k has many generators that are very "localized" around any given point of X :

Proposition 6. *There exists a constant $C > 0$ depending only on the geometry of the manifold X with the following property : let B be any ball of small enough radius $\rho > 0$ in X. For any sequence of symplectic submanifolds $W_k \subset X$ of real codimension $2r$ obtained as the zero sets of asymptotically holomorphic sections of $E \otimes L^k$ which are transverse to 0, let $N_k(B)$ be the number of independent generators of $H_{n-r}(W_k, \mathbb{R})$ which can be realized by cycles that are entirely included in $W_k \cap B$. Then, if k is large enough, one has*

$$N_k(B) > C.\rho^{2n}. \dim H_{n-r}(W_k, \mathbb{R}).$$

As a consequence, we can state that when k becomes large the submanifolds W_k tend to "fill out" all of X, since they must intersect non-trivially with any given ball.

The proof of Proposition 6 relies on the study of what happens when we perform a symplectic blow-up on the manifold X inside the ball B. Recall that the blown-up manifold $\tilde{X}$ is endowed with a symplectic form $\tilde{\omega}$ which is equal to ω outside of B, and can be described inside B using the following model on $\mathbb{C}^n$ around 0 : define on $\mathbb{C}^n \times (\mathbb{C}^n - \{0\})$ the 2-form

$$\phi = i\partial\bar{\partial}(p_1^*\beta . p_2^* \log \|\cdot\|^2),$$

where p_1 is the projection map to $\mathbb{C}^n$, β is a cut-off function around the blow-up point, and p_2 is the projection on the factor $\mathbb{C}^n - \{0\}$. The 2-form ϕ projects to $\mathbb{C}^n \times \mathbb{CP}^{n-1}$, and after restriction to the graph of the blown-up manifold (i.e. the set of all (x, y) such that x belongs to the complex line in $\mathbb{C}^n$ defined by y) one obtains a closed 2-form whose restriction to the exceptional divisor is positive. Calling θ the 2-form on $\tilde{X}$ supported in B defined by this procedure, it can be checked that, if $\epsilon > 0$ is small enough and π is the projection map $\tilde{X} \to X$, the 2-form $\tilde{\omega} = \pi^*\omega + \epsilon\theta$ is symplectic on $\tilde{X}$.

If we call $e \in H^2(\tilde{X}, \mathbb{Z})$ the Poincaré dual of the exceptional divisor, since its normal bundle is the inverse of the standard bundle over $\mathbb{CP}^{n-1}$, we have $(-e)^{n-1}.e = 1$, so that $e^n = (-1)^{n-1}$. Furthermore, the cohomology class of $\tilde{\omega}$ is given by $[\frac{\tilde{\omega}}{2\pi}] = \pi^*[\frac{\omega}{2\pi}] - \epsilon.e$. Now we consider the sections s_k of $E \otimes L^k$ over X which define W_k, and assuming ϵ^{-1} to be an integer we write $k = K + \tilde{k}$ with $0 \le \tilde{k} < \epsilon^{-1}$ and $\epsilon K \in \mathbb{N}$. Notice that $\tilde{\omega} = \pi^*\omega$ outside B and that we can safely choose a metric on $\tilde{X}$ with the same property. Considering that the line bundle $\tilde{L}^K$ on $\tilde{X}$ whose first Chern class is $K[\frac{\tilde{\omega}}{2\pi}]$ is isomorphic to $\pi^* L^K$ over $\tilde{X} - B$, the sections $\pi^* s_k$ of $\pi^*(E \otimes L^k) = \pi^*(E \otimes L^{\tilde{k}}) \otimes \pi^* L^K$ obtained by pull-back of s_k satisfy all desired conditions outside B, namely asymptotic holomorphicity and transversality to 0. If we multiply $\pi^* s_k$ by a cut-off function equal to 1 over $\tilde{X} - B$ and vanishing over the support of θ, we now obtain asymptotically holomorphic sections of $\pi^*(E \otimes L^{\tilde{k}}) \otimes \tilde{L}^K$ over $\tilde{X}$ which are transverse to 0 over $\tilde{X} - B$. So, if K is large enough, we can use the construction described in Theorems 2 and 3 to perturb these sections *over B only* to make them transverse to 0 over all of $\tilde{X}$. Since there are only finitely many values of $\tilde{k}$, the bounds on K required for each $\tilde{k}$ translate as a single bound on k. Considering the zero sets of the resulting sections, we thus obtain symplectic submanifolds $\tilde{W}_k \subset \tilde{X}$ to which we can again apply Propositions 2 and 5. The interesting remark is that, using the above estimate for $\dim H_{n-r}(\tilde{W}_k, \mathbb{R})$, since $[\frac{\tilde{\omega}}{2\pi}]^n = [\frac{\omega}{2\pi}]^n - \epsilon^n$ (symplectic blowups decrease the symplectic volume), we get for all large k

$$\dim H_{n-r}(\tilde{W}_k, \mathbb{R}) = \dim H_{n-r}(W_k, \mathbb{R}) - \epsilon^n \binom{n-1}{n-r} k^n + O(k^{n-1}).$$

This means that we have decreased the dimension of $H_{n-r}(W_k, \mathbb{R})$ by changing the picture only inside the ball B. To continue we need an estimate on the dependence of ϵ on the radius ρ of the ball. The main constraint on ϵ is that $\epsilon\theta$ should be much smaller than $\pi^*\omega$ so that the perturbation does not affect the positivity of $\pi^*\omega$. The norm of θ is directly related to that of the second derivative $\partial\bar{\partial}\beta$ of the cut-off function β. Since the only constraint on β is that it should be 0 outside B

and 1 near the blow-up point, an appropriate choice of β leads to a bound of the type $|\partial\bar{\partial}\beta| = O(\rho^{-2})$. It follows that ϵ can be chosen equal at least to a constant times ρ^2. So we obtain that, for a suitable value of C and for all large enough k,

$$\dim H_{n-r}(\tilde{W}_k, \mathbb{R}) < (1 - 2C\rho^{2n}) \dim H_{n-r}(W_k, \mathbb{R}).$$

Proposition 6 now follows immediately from the following general lemma by decomposing W_k into $(W_k - B) \cup (W_k \cap B)$ and perturbing slightly ρ if necessary so that the boundary of B is transverse to W_k :

Lemma 6. *Let W be a $2d$-dimensional compact manifold which decomposes into two pieces $W = A \cup B$ glued along their common boundary S, which is a smooth codimension 1 submanifold in W. Assume that there exists a manifold $\tilde{W}$ which is identical to W outside of B, and such that $\dim H_d(\tilde{W}, \mathbb{R}) \le \dim H_d(W, \mathbb{R}) - N$. Then there exists a $\frac{N}{2}$-dimensional subspace in $H_d(W, \mathbb{R})$ consisting of classes which can be represented by cycles contained in B.*

To prove this lemma, let $H = H_d(W, \mathbb{R})$ and consider its subspaces F consisting of all classes which can be represented by a cycle contained in A and G consisting of all classes representable in B. We have to show that $\dim G \ge \frac{N}{2}$. Let $G^\perp$ be the subspace of H orthogonal to G with respect to the intersection pairing, namely the set of classes which intersect trivially with all classes in G. We claim that $G^\perp \subset F + G$.

Indeed, let α be a cycle realizing a class in $G^\perp$. Subdividing α along its intersection with the common boundary S of A and B, we have $\alpha = \alpha_1 + \alpha_2$ where α_1 and α_2 are chains respectively in A and B, such that $\partial\alpha_1 = -\partial\alpha_2 = \beta$ is a $(d-1)$-cycle contained in S. However β intersects trivially with any d-cycle in S since α intersects trivially with all cycles that have a representative in B. So the homology class represented by β in $H_{d-1}(S, \mathbb{R})$ is trivial, and we have $\beta = \partial\gamma$ for some d-chain γ in S. Writing $\alpha = (\alpha_1 - \gamma) + (\alpha_2 + \gamma)$ and shifting slightly the two copies of γ on either side of S, we get that $[\alpha] \in F + G$.

It follows that, if F_G is a supplementary of $F \cap G$ in F, $\dim F_G + \dim G = \dim(F + G)$ is larger than $\dim G^\perp \ge \dim H - \dim G$, so that $\dim G \ge \frac{1}{2}(\dim H - \dim F_G)$. Thus it only remains to show that $\dim F_G \le \dim H_d(\tilde{W}, \mathbb{R})$ to complete the proof of the lemma. To do this, we remark that the morphism $h : H_d(W; \mathbb{R}) \to H_d(W, B; \mathbb{R})$ in the relative homology sequence is injective on F_G, since its kernel is precisely G. However, if we define $\tilde{F}$ and $\tilde{G}$ inside $H_d(\tilde{W}, \mathbb{R})$ similarly to F and G, the subspace $\tilde{F}_{\tilde{G}}$ similarly injects into $H_d(\tilde{W}, \tilde{B}; \mathbb{R})$. Furthermore, the images of the two injections are both equal to the image of the morphism $H_d(A; \mathbb{R}) \to H_d(A, S; \mathbb{R})$ under the identification $H_d(\tilde{W}, \tilde{B}; \mathbb{R}) \simeq H_d(A, S; \mathbb{R}) \simeq H_d(W, B; \mathbb{R})$, so that $\dim H_d(\tilde{W}, \mathbb{R}) \ge \dim \tilde{F}_{\tilde{G}} = \dim F_G$ and the proof is complete.

6. Conclusion

This paper has extended the field of applicability of the construction outlined by Donaldson [1] to more general vector bundles. It is in fact probable that similar methods can be used in other situations involving sequences of vector bundles whose curvatures become very positive.

The statement that, in spite of the high flexibility of the construction, the submanifolds obtained as zero sets of asymptotically holomorphic sections of $E \otimes L^k$

which are transverse to 0 are all isotopic for a given large enough k, has important consequences. Indeed, as suggested by Donaldson, it may allow the definition of relatively easily computable invariants of higher-dimensional symplectic manifolds from the topology of their submanifolds, for example from the Seiberg-Witten invariants of 4-dimensional submanifolds [5][6]. Furthermore, it facilitates the characterization of the topology of the constructed submanifolds in many cases, thus leading the way to many possibly new examples of symplectic manifolds.

References

[1] S.K. Donaldson, *Symplectic Submanifolds and Almost-complex Geometry*, J. Diff. Geom. **44** (1996), 666–705.
[2] R.E. Gompf, *A New Construction of Symplectic Manifolds*, Ann. of Math. **142** (1995), 527–595.
[3] M. Gromov, *Partial Differential Relations*, Springer, 1986.
[4] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 1995.
[5] C.H. Taubes, *The Seiberg-Witten and Gromov Invariants*, Math. Res. Lett. **2** (1995), 221–238.
[6] E. Witten, *Monopoles and 4-manifolds*, Math. Res. Lett. **1** (1994), 769–796.
[7] Y. Yomdin, *The Geometry of Critical and Near-critical Values of Differentiable Mappings*, Math. Annalen **264** (1983), 495–515.

Centre de Mathématiques, Ecole Polytechnique, 91128 Palaiseau, France
E-mail address: auroux@math.polytechnique.fr

SYMPLECTIC 4-MANIFOLDS AS BRANCHED COVERINGS OF $\mathbb{CP}^2$

DENIS AUROUX

Abstract. We show that every compact symplectic 4-manifold X can be topologically realized as a covering of $\mathbb{CP}^2$ branched along a smooth symplectic curve in X which projects as an immersed curve with cusps in $\mathbb{CP}^2$. Furthermore, the covering map can be chosen to be approximately pseudo-holomorphic with respect to any given almost-complex structure on X.

1. Introduction

It has recently been shown by Donaldson [3] that the existence of approximately holomorphic sections of very positive line bundles over compact symplectic manifolds allows the construction not only of symplectic submanifolds ([2], see also [1],[5]) but also of symplectic Lefschetz pencil structures. The aim of this paper is to show how similar techniques can be applied in the case of 4-manifolds to obtain maps to $\mathbb{CP}^2$, thus proving that every compact symplectic 4-manifold is topologically a (singular) branched covering of $\mathbb{CP}^2$.

Let (X, ω) be a compact symplectic 4-manifold such that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ is integral. This integrality condition does not restrict the diffeomorphism type of X in any way, since starting from an arbitrary symplectic structure one can always perturb it so that $\frac{1}{2\pi}[\omega]$ becomes rational, and then multiply ω by a constant factor to obtain integrality. A compatible almost-complex structure J on X and the corresponding Riemannian metric g are also fixed.

Let L be the complex line bundle on X whose first Chern class is $c_1(L) = \frac{1}{2\pi}[\omega]$. Fix a Hermitian structure on L, and let ∇^L be a Hermitian connection on L whose curvature 2-form is equal to $-i\omega$ (it is clear that such a connection always exists). The key observation is that, for large values of an integer parameter k, the line bundles L^k admit many approximately holomorphic sections, thus making it possible to obtain sections which have nice transversality properties.

For example, one such section can be used to define an approximately holomorphic symplectic submanifold in X [2]. Similarly, constructing two sections satisfying certain transversality requirements yields a Lefschetz pencil structure [3]. In our case, the aim is to construct, for large enough k, *three* sections s^0_k, s^1_k and s^2_k of L^k satisfying certain transversality properties, in such a way that the three sections do not vanish simultaneously and that the map from X to $\mathbb{CP}^2$ defined by $x \mapsto [s^0_k(x) : s^1_k(x) : s^2_k(x)]$ is a branched covering.

Let us now describe more precisely the notion of approximately holomorphic singular branched covering. Fix a constant $\epsilon > 0$, and let U be a neighborhood of a point x in an almost-complex 4-manifold. We say that a local complex coordinate map $\phi : U \to \mathbb{C}^2$ is *ϵ-approximately holomorphic* if, at every point, $|\phi_* J - \mathbb{J}_0| \leq \epsilon$, where $\mathbb{J}_0$ is the canonical complex structure on $\mathbb{C}^2$. Another equivalent way to state the same property is the bound $|\bar{\partial}\phi(u)| \leq \frac{1}{2}\epsilon|d\phi(u)|$ for every tangent vector u (this definition does not depend on the choice of a metric on the almost-complex 4-manifold ; $\mathbb{C}^2$ is endowed with its usual Euclidean metric).

Definition 1. *A map $f : X \to \mathbb{CP}^2$ is* locally ϵ-holomorphically modelled *at x on a map $g : \mathbb{C}^2 \to \mathbb{C}^2$ if there exist neighborhoods U of x in X and V of $f(x)$ in $\mathbb{CP}^2$, and ϵ-approximately holomorphic C^1 coordinate maps $\phi : U \to \mathbb{C}^2$ and $\psi : V \to \mathbb{C}^2$ such that $f = \psi^{-1} \circ g \circ \phi$ over U.*

Definition 2. *A map $f : X \to \mathbb{CP}^2$ is an* ϵ-holomorphic singular covering *branched along a submanifold $R \subset X$ if its differential is surjective everywhere except at the points of R, where* $\mathrm{rank}(df) = 2$, *and if at any point $x \in X$ it is locally ϵ-holomorphically modelled on one of the three following maps :*
(i) *local diffeomorphism : $(z_1, z_2) \mapsto (z_1, z_2)$;*
(ii) *branched covering : $(z_1, z_2) \mapsto (z_1^2, z_2)$;*
(iii) *cusp covering : $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$.*

In particular it is clear that the cusp model occurs only in a neighborhood of a finite set of points $\mathcal{C} \subset R$, and that the branched covering model occurs only in a neighborhood of R (away from $\mathcal{C}$), while f is a local diffeomorphism everywhere outside of a neighborhood of R. Moreover, the set of branch points R and its projection $f(R)$ can be described as follows in the local models : for the branched covering model, $R = \{(z_1, z_2),\ z_1 = 0\}$ and $f(R) = \{(x, y),\ x = 0\}$; for the cusp covering model, $R = \{(z_1, z_2),\ 3z_1^2 - z_2 = 0\}$ and $f(R) = \{(x, y),\ 27x^2 - 4y^3 = 0\}$.

It follows that, if $\epsilon < 1$, R is a *smooth 2-dimensional submanifold in X, approximately J-holomorphic, and therefore symplectic*, and that $f(R)$ is an *immersed symplectic curve in $\mathbb{CP}^2$ except for a finite number of cusps.*

We now state our main result :

Theorem 1. *For any $\epsilon > 0$ there exists an ϵ-holomorphic singular covering map $f : X \to \mathbb{CP}^2$.*

The techniques involved in the proof of this result are similar to those introduced by Donaldson in [2] : the first ingredient is a local transversality result stating roughly that, given approximately holomorphic sections of certain bundles, it is possible to ensure that they satisfy certain transversality estimates over a small ball in X by adding to them small and localized perturbations. The other ingredient is a globalization principle, which, if the small perturbations providing local transversality are sufficiently well localized, ensures that a transversality estimate can be obtained over all of X by combining the local perturbations.

We now define more precisely the notions of approximately holomorphic sections and of transversality with estimates. We will be considering sequences of sections of complex vector bundles E_k over X, for all large values of the integer k, where each of the bundles E_k carries naturally a Hermitian metric and a Hermitian connection. These connections together with the almost complex structure J on X yield ∂ and $\bar{\partial}$ operators on E_k. Moreover, we choose to rescale the metric on X, and use $g_k = k\, g$: for example, the diameter of X is multiplied by $k^{1/2}$, and all derivatives of order p are divided by $k^{p/2}$. The reason for this rescaling is that the vector bundles E_k we will consider are derived from L^k, on which the natural Hermitian connection induced by ∇^L has curvature $-ik\omega$.

Definition 3. *Let $(s_k)_{k\gg 0}$ be a sequence of sections of complex vector bundles E_k over X. The sections s_k are said to be* asymptotically holomorphic *if there exist constants $(C_p)_{p\in\mathbb{N}}$ such that, for all k and at every point of X, $|s_k| \le C_0$, $|\nabla^p s_k| \le C_p$ and $|\nabla^{p-1}\bar\partial s_k| \le C_p k^{-1/2}$ for all $p \ge 1$, where the norms of the derivatives are evaluated with respect to the metrics $g_k = k\,g$.*

Definition 4. *Let s_k be a section of a complex vector bundle E_k, and let $\eta > 0$ be a constant. The section s_k is said to be* η-transverse to 0 *if, at any point $x \in X$ where $|s_k(x)| < \eta$, the covariant derivative $\nabla s_k(x) : T_xX \to (E_k)_x$ is surjective and has a right inverse of norm less than η^{-1} w.r.t. the metric g_k.*

We will often say that a sequence $(s_k)_{k\gg 0}$ of sections of E_k is *transverse to* 0 (without precising the constant) if there exists a constant $\eta > 0$ independent of k such that η-transversality to 0 holds for all large k.

In this definition of transversality, two cases are of specific interest. First, when E_k is a line bundle, and if one assumes the sections to be asymptotically holomorphic, transversality to 0 can be equivalently expressed by the property

$$\forall x \in X,\ |s_k(x)| < \eta \Rightarrow |\nabla s_k(x)|_{g_k} > \eta.$$

Next, when E_k has rank greater than 2 (or more generally than the complex dimension of X), the property actually means that $|s_k(x)| \ge \eta$ for all $x \in X$.

An important point to keep in mind is that transversality to 0 is an *open* property : if s is η-transverse to 0, then any section σ such that $|s-\sigma|_{C^1} < \epsilon$ is $(\eta-\epsilon)$-transverse to 0.

The interest of such a notion of transversality with estimates is made clear by the following observation :

Lemma 1. *Let γ_k be asymptotically holomorphic sections of vector bundles E_k over X, and assume that the sections γ_k are transverse to 0. Then, for large enough k, the zero set of γ_k is a smooth symplectic submanifold in X.*

This lemma follows from the observation that, where γ_k vanishes, $|\bar\partial\gamma_k| = O(k^{-1/2})$ by the asymptotic holomorphicity property while $\partial\gamma_k$ is bounded from below by the transversality property, thus ensuring that for large enough k the zero set is smooth and symplectic, and even asymptotically J-holomorphic.

We can now write our second result, which is a one-parameter version of Theorem 1 :

Theorem 2. *Let $(J_t)_{t\in[0,1]}$ be a family of almost-complex structures on X compatible with ω. Fix a constant $\epsilon > 0$, and let $(s_{t,k})_{t\in[0,1],k\gg 0}$ be asymptotically J_t-holomorphic sections of $\mathbb{C}^3 \otimes L^k$, such that the sections $s_{t,k}$ and their derivatives depend continuously on t.*

Then, for all large enough values of k, there exist asymptotically J_t-holomorphic sections $\sigma_{t,k}$ of $\mathbb{C}^3 \otimes L^k$, nowhere vanishing, depending continuously on t, and such that, for all $t \in [0,1]$, $|\sigma_{t,k} - s_{t,k}|_{C^3,g_k} \le \epsilon$ and the map $X \to \mathbb{CP}^2$ defined by $\sigma_{t,k}$ is an approximately holomorphic singular covering with respect to J_t.

Note that, although we allow the almost-complex structure on X to depend on t, we always use the same metric $g_k = k\,g$ independently of t. Therefore, there is no special relation between g_k and J_t. However, since the parameter space $[0,1]$ is compact, we know that the metric defined by ω and J_t differs from g by at most a constant factor, and therefore up to a change in the constants this has no real influence on the transversality and holomorphicity properties.

We now describe more precisely the properties of the approximately holomorphic singular coverings constructed in Theorems 1 and 2, in order to state a uniqueness result for such coverings.

Definition 5. *Let s_k be nowhere vanishing asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Define the corresponding projective maps $f_k = \mathbb{P}s_k$ from X to $\mathbb{CP}^2$ by $f_k(x) = [s_k^0(x) : s_k^1(x) : s_k^2(x)]$. Define the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$, which is a section of $\Lambda^{2,0}T^*X \otimes f_k^*\Lambda^{2,0}T\mathbb{CP}^2 = K_X \otimes L^{3k}$. Finally, define $R(s_k)$ to be the set of points of X where $\mathrm{Jac}(f_k)$ vanishes, i.e. where ∂f_k is not surjective.*

Given a constant $\gamma > 0$, we say that s_k satisfies the transversality property $\mathcal{P}_3(\gamma)$ if $|s_k| \geq \gamma$ and $|\partial f_k|_{g_k} \geq \gamma$ at every point of X, and if $\mathrm{Jac}(f_k)$ is γ-transverse to 0.

If s_k satisfies $\mathcal{P}_3(\gamma)$ for some $\gamma > 0$ and if k is large enough, then it follows from Lemma 1 that $R(s_k)$ is a smooth symplectic submanifold in X. By analogy with the expected properties of the set of branch points, it is therefore natural to require such a property for the sections which define our covering maps.

Furthermore, recall that one expects the projection to $\mathbb{CP}^2$ of the set of branch points to be an immersed curve except at only finitely many non-degenerate cusps. Forget temporarily the antiholomorphic derivative $\bar\partial f_k$, and consider only the holomorphic part. Then the cusps correspond to the points of $R(s_k)$ where the kernel of ∂f_k and the tangent space to $R(s_k)$ coincide (in other words, the points where the tangent space to $R(s_k)$ becomes "vertical"). Since $R(s_k)$ is the set of points where $\mathrm{Jac}(f_k) = 0$, the cusp points are those where the quantity $\partial f_k \wedge \partial\mathrm{Jac}(f_k)$ vanishes ; in this notation ∂f_k and $\partial \mathrm{Jac}(f_k)$ are both seen as $(1,0)$-forms with values in vector bundles ($f_k^*T\mathbb{CP}^2$ and $K_X \otimes L^{3k}$, respectively), and their exterior product is a $(2,0)$-form with values in the tensor product $(f_k^*T\mathbb{CP}^2) \otimes (K_X \otimes L^{3k})$.

Note that, along $R(s_k)$, ∂f_k has complex rank 1 and so is actually a nowhere vanishing $(1,0)$-form with values in the rank 1 subbundle $\mathcal{L}(s_k) = \mathrm{Im}\ \partial f_k \subset f_k^*T\mathbb{CP}^2$. In a neighborhood of $R(s_k)$, this is no longer true, but one can project ∂f_k onto a rank 1 subbundle in $f_k^*T\mathbb{CP}^2$ (which will still be called $\mathcal{L}(s_k)$), thus obtaining a nonvanishing $(1,0)$-form $\pi(\partial f_k)$ with values in the line bundle $\mathcal{L}(s_k)$. The quantity $\pi(\partial f_k) \wedge \partial \mathrm{Jac}(f_k)$ (where the wedge notation denotes as above the exterior product of two $(1,0)$-forms with values in line bundles), which is a section of a line bundle over $R(s_k)$, can under the above-described transversality assumptions be thought of as a measurement of the angle between the kernel of ∂f_k and the tangent space to $R(s_k)$. Its vanishing over $R(s_k)$ is therefore characteristic of cusp points, and so it is natural to require that its restriction to $R(s_k)$ be transverse to 0, as it implies that the cusp points are isolated and in some sense non-degenerate.

It is worth noting that, up to a change of constants in the estimates, this transversality property is actually independent of the choice of the subbundle of $f_k^*T\mathbb{CP}^2$ on which one projects ∂f_k, as long as $\pi(\partial f_k)$ remains bounded from below.

For convenience, we introduce the following notations :

Definition 6. *Let s_k be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ and $f_k = \mathbb{P}s_k$. Assume that s_k satisfies $\mathcal{P}_3(\gamma)$ for some $\gamma > 0$. Consider the rank one subbundle $\mathcal{L}(s_k) = (\mathrm{Im}\ \partial f_k)_{|R(s_k)}$ of $f_k^*T\mathbb{CP}^2$ over $R(s_k)$, and let $\pi : f_k^*T\mathbb{CP}^2 \to \mathcal{L}(s_k)$*

be the orthogonal projection. Finally define, over $R(s_k)$, the quantity $\mathcal{T}(s_k) = \pi(\partial f_k) \wedge \partial \mathrm{Jac}(f_k)$.

We say that asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ are γ-generic *if they satisfy $\mathcal{P}_3(\gamma)$ and if the quantity $\mathcal{T}(s_k)$ is γ-transverse to 0 over $R(s_k)$. We then define the set of cusp points $\mathcal{C}(s_k)$ as the set of points of $R(s_k)$ where $\mathcal{T}(s_k) = 0$.*

In a holomorphic setting, such a genericity property would be sufficient to ensure that the map $f_k = \mathbb{P}s_k$ is a singular branched covering. However, in our case, extra difficulties arise because we only have approximately holomorphic sections. This means that at a point of $R(s_k)$, although ∂f_k has rank 1, we have no control over the rank of $\bar\partial f_k$, and the local picture may be very different from what one expects. Therefore, we need to control the antiholomorphic part of the derivative along the set of branch points by adding the following requirement :

Definition 7. *Let s_k be γ-generic asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. We say that s_k is* $\bar\partial$-tame *if there exist constants $(C_p)_{p\in\mathbb{N}}$ and $c > 0$, depending only on the geometry of X and the bounds on s_k and its derivatives, and an ω-compatible almost complex structure $\tilde J_k$, such that the following properties hold :*

(1) *$\forall p \in \mathbb{N}$, $|\nabla^p(\tilde J_k - J)|_{g_k} \le C_p k^{-1/2}$;*

(2) *the almost-complex structure $\tilde J_k$ is integrable over the set of points whose g_k-distance to $\mathcal{C}_{\tilde J_k}(s_k)$ is less than c (the subscript indicates that one uses $\partial_{\tilde J_k}$ rather than ∂_J to define $\mathcal{C}(s_k)$) ;*

(3) *the map $f_k = \mathbb{P}s_k$ is $\tilde J_k$-holomorphic at every point of X whose g_k-distance to $\mathcal{C}_{\tilde J_k}(s_k)$ is less than c ;*

(4) *at every point of $R_{\tilde J_k}(s_k)$, the antiholomorphic derivative $\bar\partial_{\tilde J_k}(\mathbb{P}s_k)$ vanishes over the kernel of $\partial_{\tilde J_k}(\mathbb{P}s_k)$.*

Note that since $\tilde J_k$ is within $O(k^{-1/2})$ of J, the notions of asymptotic J-holomorphicity and asymptotic $\tilde J_k$-holomorphicity actually coincide, because the ∂ and $\bar\partial$ operators differ only by $O(k^{-1/2})$. Furthermore, if k is large enough, then γ-genericity for J implies γ'-genericity for $\tilde J_k$ as well for some γ' slightly smaller than γ ; and, because of the transversality properties, the sets $R_{\tilde J_k}(s_k)$ and $\mathcal{C}_{\tilde J_k}(s_k)$ lie within $O(k^{-1/2})$ of $R_J(s_k)$ and $\mathcal{C}_J(s_k)$.

In the case of families of sections depending continuously on a parameter $t \in [0, 1]$, it is natural to also require that the almost complex structures $\tilde J_{t,k}$ close to J_t for every t depend continuously on t. We claim the following :

Theorem 3. *Let s_k be asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Assume that the sections s_k are γ-generic and $\bar\partial$-tame. Then, for all large enough values of k, the maps $f_k = \mathbb{P}s_k$ are ϵ_k-holomorphic singular branched coverings, for some constants $\epsilon_k = O(k^{-1/2})$.*

Therefore, in order to prove Theorems 1 and 2 it is sufficient to construct γ-generic and $\bar\partial$-tame sections (resp. one-parameter families of sections) of $\mathbb{C}^3 \otimes L^k$. Even better, we have the following uniqueness result for these particular singular branched coverings :

Theorem 4. *Let $s_{0,k}$ and $s_{1,k}$ be sections of $\mathbb{C}^3 \otimes L^k$, asymptotically holomorphic with respect to ω-compatible almost-complex structures J_0 and J_1 respectively. Assume that $s_{0,k}$ and $s_{1,k}$ are γ-generic and $\bar\partial$-tame. Then there exist almost-complex* *structures $(J_t)_{t\in[0,1]}$ interpolating between J_0 and J_1, and a constant $\eta > 0$, with the following property : for all large enough k, there exist sections $(s_{t,k})_{t\in[0,1],k\gg 0}$ of $\mathbb{C}^3 \otimes L^k$ interpolating between $s_{0,k}$ and $s_{1,k}$, depending continuously on t, which are, for all $t \in [0,1]$, asymptotically J_t-holomorphic, η-generic and $\bar\partial$-tame with respect to J_t.*

In particular, for large k the approximately holomorphic singular branched coverings $\mathbb{P}s_{0,k}$ and $\mathbb{P}s_{1,k}$ are isotopic among approximately holomorphic singular branched coverings.

Therefore, there exists for all large k a canonical isotopy class of singular branched coverings $X \to \mathbb{CP}^2$, which could potentially be used to define symplectic invariants of X.

The remainder of this article is organized as follows : §2 describes the process of perturbing asymptotically holomorphic sections of bundles of rank greater than 2 to make sure that they remain away from zero. §3 deals with further perturbation in order to obtain γ-genericity. §4 describes a way of achieving $\bar\partial$-tameness, and therefore completes the proofs of Theorems 1, 2 and 4. Finally, Theorem 3 is proved in §5, and §6 deals with various related remarks.

Acknowledgments. The author wishes to thank Misha Gromov for valuable suggestions and comments, and Christophe Margerin for helpful discussions.

2. Nowhere vanishing sections

2.1. Non-vanishing of s_k.

Our strategy to prove Theorem 1 is to start with given asymptotically holomorphic sections s_k (for example $s_k = 0$) and perturb them in order to obtain the required properties ; the proof of Theorem 2 then relies on the same arguments, with the added difficulty that all statements must apply to 1-parameter families of sections.

The first step is to ensure that the three components s_k^0, s_k^1 and s_k^2 do not vanish simultaneously, and more precisely that, for some constant $\eta > 0$ independent of k, the sections s_k are η-transverse to 0, i.e. $|s_k| \geq \eta$ over all of X. Therefore, the first ingredient in the proof of Theorems 1 and 2 is the following result :

Proposition 1. *Let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3\otimes L^k$, and fix a constant $\epsilon > 0$. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3,g_k} \leq \epsilon$ and that $|\sigma_k| \geq \eta$ at every point of X. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proposition 1 is a direct consequence of the main theorem in [1], where it is proved that, given any complex vector bundle E, asymptotically holomorphic sections of $E \otimes L^k$ (or 1-parameter families of such sections) can be made transverse to 0 by small perturbations : Proposition 1 follows simply by considering the case where E is the trivial bundle of rank 3. However, for the sake of completeness and in order to introduce tools which will also be used in later parts of the proof, we give here a shorter argument dealing with the specific case at hand.

There are three ingredients in the proof of Proposition 1. The first one is the existence of many localized asymptotically holomorphic sections of the line bundle L^k for sufficiently large k.

Definition 8. *A section s of a vector bundle E_k has* Gaussian decay *in C^r norm away from a point $x \in X$ if there exists a polynomial P and a constant $\lambda > 0$ such that for all $y \in X$, $|s(y)|$, $|\nabla s(y)|_{g_k}$, ..., $|\nabla^r s(y)|_{g_k}$ are all bounded by $P(d(x,y)) \exp(-\lambda\, d(x,y)^2)$, where $d(.,.)$ is the distance induced by g_k.*

The decay properties of a family of sections are said to be uniform *if there exist P and λ such that the above bounds hold for all sections of the family, independently of k and of the point x at which decay occurs for a given section.*

Lemma 2 ([2],[1])**.** *Given any point $x \in X$, for all large enough k, there exist asymptotically holomorphic sections $s^{\text{ref}}_{k,x}$ of L^k over X satisfying the following bounds : $|s^{\text{ref}}_{k,x}| \geq c_s$ at every point of the ball of g_k-radius 1 centered at x, for some universal constant $c_s > 0$; and the sections $s^{\text{ref}}_{k,x}$ have uniform Gaussian decay away from x in C^3 norm.*

Moreover, given a one-parameter family of ω-compatible almost-complex structures $(J_t)_{t\in[0,1]}$, there exist one-parameter families of sections $s^{\text{ref}}_{t,k,x}$ which are asymptotically J_t-holomorphic for all t, depend continuously on t and satisfy the same bounds.

The first part of this statement is Proposition 11 of [2], while the extension to one-parameter families is carried out in Lemma 3 of [1]. Note that here we require decay with respect to the C^3 norm instead of C^0, but the bounds on all derivatives follow immediately from the construction of these sections : indeed, they are modelled on $f(z) = \exp(-|z|^2/4)$ in a local approximately holomorphic Darboux coordinate chart for $k\omega$ at x and in a suitable local trivialization of L^k where the connection 1-form is $\frac{1}{4}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$. Therefore, it is sufficient to notice that the model function has Gaussian decay and that all derivatives of the coordinate map are uniformly bounded because of the compactness of X.

More precisely, the result of existence of local approximately holomorphic Darboux coordinate charts needed for Lemma 2 (and throughout the proofs of the main theorems as well) is the following (see also [2]) :

Lemma 3. *Near any point $x \in X$, for any integer k, there exist local complex Darboux coordinates $(z^1_k, z^2_k) : (X,x) \to (\mathbb{C}^2, 0)$ for the symplectic structure $k\omega$ (i.e. such that the pullback of the standard symplectic structure of $\mathbb{C}^2$ is $k\omega$) such that, denoting by $\psi_k : (\mathbb{C}^2,0) \to (X,x)$ the inverse of the coordinate map, the following bounds hold uniformly in x and k : $|z^1_k(y)| + |z^2_k(y)| = O(dist_{g_k}(x,y))$ on a ball of fixed radius around x ; $|\nabla^r \psi_k|_{g_k} = O(1)$ for all $r \geq 1$ on a ball of fixed radius around 0 ; and, with respect to the almost-complex structure J on X and the canonical complex structure $\mathbb{J}_0$ on $\mathbb{C}^2$, $|\bar\partial \psi_k(z)|_{g_k} = O(k^{-1/2}|z|)$ and $|\nabla^r \bar\partial\psi|_{g_k} = O(k^{-1/2})$ for all $r \geq 1$ on a ball of fixed radius around 0.*

Moreover, given a continuous 1-parameter family of ω-compatible almost-complex structures $(J_t)_{t\in[0,1]}$ and a continuous family of points $(x_t)_{t\in[0,1]}$, one can find for all t coordinate maps near x_t satisfying the same estimates and depending continuously on t.

Proof. By Darboux's theorem, there exists a local symplectomorphism ϕ from a neighborhood of 0 in $\mathbb{C}^2$ with its standard symplectic structure to a neighborhood of x in (X,ω). It is well-known that the space of symplectic $\mathbb{R}$-linear endomorphisms of $\mathbb{C}^2$ which intertwine the complex structures $\mathbb{J}_0$ and $\phi^* J(x)$ is non-empty (and actually isomorphic to U(2)). So, choosing such a linear map Ψ and defining $\psi =$

$\phi \circ \Psi$, one gets a local symplectomorphism such that $\bar\partial\psi(0) = 0$. Moreover, because of the compactness of X, it is possible to carry out the construction in such a way that, with respect to the metric g, all derivatives of ψ are bounded over a neighborhood of x by uniform constants which do not depend on x. Therefore, over a neighborhood of x one can assume that $|\nabla(\psi^{-1})|_g = O(1)$, as well as $|\nabla^r\psi|_g = O(1)$ and $|\nabla^r\bar\partial\psi|_g = O(1)$ $\forall r \geq 1$.

Define $\psi_k(z) = \psi(k^{-1/2}z)$, and switch to the metric g_k : then $\bar\partial\psi_k(0) = 0$, and at every point near x, $|\nabla(\psi_k^{-1})|_{g_k} = |\nabla(\psi^{-1})|_g = O(1)$. Moreover, $|\nabla^r\psi_k|_{g_k} = O(k^{(1-r)/2}) = O(1)$ and $|\nabla^r\bar\partial\psi_k|_{g_k} = O(k^{-r/2}) = O(k^{-1/2})$ for all $r \geq 1$. Finally, since $|\nabla\bar\partial\psi_k|_{g_k} = O(k^{-1/2})$ and $\bar\partial\psi_k(0) = 0$ we have $|\bar\partial\psi_k(z)|_{g_k} = O(k^{-1/2}|z|)$, so that all expected estimates hold. Because of the compactness of X, the estimates are uniform in x, and because the maps ψ_k for different values of k differ only by a rescaling, the estimates are also uniform in k.

In the case of a one-parameter family of almost-complex structures, there is only one thing to check in order to carry out the same construction for every value of $t \in [0, 1]$ while ensuring continuity in t : given a one-parameter family of local Darboux maps ϕ_t near x_t (the existence of such maps depending continuously on t is trivial), one must check the existence of a continuous one-parameter family of $\mathbb{R}$-linear symplectic endomorphisms Ψ_t of $\mathbb{C}^2$ intertwining the complex structures $\mathbb{J}_0$ and $\phi_t^* J_t(x_t)$ on $\mathbb{C}^2$. To prove this, remark that for every t the set of these endomorphisms of $\mathbb{C}^2$ can be identified with the group U(2). Therefore, what we are looking for is a continuous section $(\Psi_t)_{t\in[0,1]}$ of a principal U(2)-bundle over $[0, 1]$. Since $[0, 1]$ is contractible, this bundle is necessarily trivial and therefore has a continuous section. This proves the existence of the required maps Ψ_t, so one can define $\psi_t = \phi_t \circ \Psi_t$, and set $\psi_{t,k}(z) = \psi_t(k^{-1/2}z)$ as above. The expected bounds follow naturally ; the estimates are uniform in t because of the compactness of $[0, 1]$. □

The second tool we need for Proposition 1 is the following local transversality result, which involves ideas similar to those in [2] and in §2 of [1] but applies to maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ with $m > n$ rather than $m = 1$:

Proposition 2. *Let f be a function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}^m$, with $m > n$. Let δ be a constant with $0 < \delta < \frac{1}{2}$, and let $\eta = \delta \log(\delta^{-1})^{-p}$ where p is a suitable fixed integer depending only on the dimension n. Assume that f satisfies the following bounds over B^+ :*

$$|f| \leq 1, \qquad |\bar\partial f| \leq \eta, \qquad |\nabla\bar\partial f| \leq \eta.$$

Then, there exists $w \in \mathbb{C}^m$, with $|w| \leq \delta$, such that $|f - w| \geq \eta$ over the interior ball B of radius 1.

Moreover, if one considers a one-parameter family of functions $(f_t)_{t\in[0,1]}$ satisfying the same bounds, then one can find for all t elements $w_t \in \mathbb{C}^m$ depending continuously on t such that $|w_t| \leq \delta$ and $|f_t - w_t| \geq \eta$ over B.

This statement is proved in §2.3. The last, and most crucial, ingredient of the proof of Proposition 1 is a globalization principle due to Donaldson [2] which we state here in a general form.

Definition 9. *A family of properties $\mathcal{P}(\epsilon, x)_{x\in X,\epsilon>0}$ of sections of bundles over X is* local and C^r-open *if, given a section s satisfying $\mathcal{P}(\epsilon, x)$, any section σ such that*

$|\sigma(x) - s(x)|$, $|\nabla\sigma(x) - \nabla s(x)|$, ... , $|\nabla^r\sigma(x) - \nabla^r s(x)|$ are smaller than η satisfies $\mathcal{P}(\epsilon - C\eta, x)$, where C is a constant (independent of x and ϵ).

For example, the property $|s(x)| \geq \epsilon$ is local and C^0-open ; ϵ-transversality to 0 of s at x is local and C^1-open.

Proposition 3 ([2]). *Let $\mathcal{P}(\epsilon, x)_{x\in X,\epsilon>0}$ be a local and C^r-open family of properties of sections of vector bundles E_k over X. Assume that there exist constants c, c' and p such that, given any $x \in X$, any small enough $\delta > 0$, and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections $\tau_{k,x}$ of E_k with the following properties :* (a) $|\tau_{k,x}|_{C^r,g_k} < \delta$, (b) *the sections $\frac{1}{\delta}\tau_{k,x}$ have uniform Gaussian decay away from x in C^r-norm, and* (c) *the sections $s_k + \tau_{k,x}$ satisfy the property $\mathcal{P}(\eta, y)$ for all $y \in B_{g_k}(x, c)$, with $\eta = c'\delta\log(\delta^{-1})^{-p}$.*

Then, given any $\alpha > 0$ and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections σ_k of E_k such that $|s_k - \sigma_k|_{C^r,g_k} < \alpha$ and the sections σ_k satisfy $\mathcal{P}(\epsilon, x)$ $\forall x \in X$ for some $\epsilon > 0$ independent of k.

Moreover the same result holds for one-parameter families of sections, provided the existence of sections $\tau_{t,k,x}$ satisfying properties (a), (b), (c) *and depending continuously on $t \in [0, 1]$.*

This result is a general formulation of the argument contained in §3 of [2] (see also [1], §3.3 and 3.5). For the sake of completeness, let us recall just a brief outline of the construction. To achieve property $\mathcal{P}$ over all of X, the idea is to proceed iteratively : in step j, one starts from sections $s_k^{(j)}$ satisfying $\mathcal{P}(\delta_j, x)$ for all x in a certain (possibly empty) subset $U_k^{(j)} \subset X$, and perturbs them by less than $\frac{1}{2C}\delta_j$ (where C is the same constant as in Definition 9) to get sections $s_k^{(j+1)}$ satisfying $\mathcal{P}(\delta_{j+1}, x)$ over certain balls of g_k-radius c, with $\delta_{j+1} = c'(\frac{\delta_j}{2C})\log((\frac{\delta_j}{2C})^{-1})^{-p}$. Because the property $\mathcal{P}$ is open, $s_k^{(j+1)}$ also satisfies $\mathcal{P}(\delta_{j+1}, x)$ over $U_k^{(j)}$, therefore allowing one to obtain $\mathcal{P}$ everywhere after a certain number N of steps.

The catch is that, since the value of δ_j decreases after each step and we want $\mathcal{P}(\epsilon, x)$ with ϵ independent of k, the number of steps needs to be bounded independently of k. However, the size of X for the metric g_k increases as k increases, and the number of balls of radius c needed to cover X therefore also increases. The key observation due to Donaldson [2] is that, because of the Gaussian decay of the perturbations, if one chooses a sufficiently large constant D, one can in a single step carry out perturbations centered at as many points as one wants, provided that any two of these points are distant of at least D with respect to g_k : the idea is that each of the perturbations becomes sufficiently small in the vicinity of the other perturbations in order to have no influence on property $\mathcal{P}$ there (up to a slight decrease of δ_{j+1}). Therefore the construction is possible with a bounded number of steps N and yields property $\mathcal{P}(\epsilon, x)$ for all $x \in X$ and for all large enough k, with $\epsilon = \delta_N$ independent of k. □

We now show how to derive Proposition 1 from Lemma 2 and Propositions 2 and 3, following the ideas contained in [2]. Proposition 1 follows directly from Proposition 3 by considering the property $\mathcal{P}$ defined as follows : say that a section s_k of $\mathbb{C}^3 \otimes L^k$ satisfies $\mathcal{P}(\epsilon, x)$ if $|s_k(x)| \geq \epsilon$. This property is local and open in C^0-sense, and therefore also in C^3-sense. So it is sufficient to check that the assumptions of Proposition 3 hold for $\mathcal{P}$.

Let $x \in X$, $0 < \delta < \frac{1}{2}$, and consider asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ (or 1-parameter families of sections $s_{t,k}$). Recall that Lemma 2 provides asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x}$ of L^k which have Gaussian decay away from x and remain larger than a constant c_s over $B_{g_k}(x,1)$. Therefore, dividing s_k by $s^{\mathrm{ref}}_{k,x}$ yields asymptotically holomorphic functions u_k on $B_{g_k}(x,1)$ with values in $\mathbb{C}^3$. Next, one uses a local approximately holomorphic coordinate chart as given by Lemma 3 to obtain, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions v_k defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}^3$, and satisfying the estimates $|v_k| = O(1)$, $|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$.

Let C_0 be a constant bounding $|s^{\mathrm{ref}}_{k,x}|_{C^3,g_k}$, and let $\alpha = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$. Provided that k is large enough, Proposition 2 yields constants $w_k \in \mathbb{C}^3$, with $|w_k| \le \frac{\delta}{C_0}$, such that $|v_k - w_k| \ge \alpha$ over the unit ball in $\mathbb{C}^2$. Equivalently, one has $|u_k - w_k| \ge \alpha$ over $B_{g_k}(x,c)$ for some constant c. Multiplying by $s^{\mathrm{ref}}_{k,x}$ again, one gets that $|s_k - w_k\, s^{\mathrm{ref}}_{k,x}| \ge c_s\alpha$ over $B_{g_k}(x,c)$.

The assumptions of Proposition 3 are therefore satisfied if one chooses $\eta = c_s\alpha$ (larger than $c'\delta\log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$) and $\tau_{k,x} = -w_k\, s^{\mathrm{ref}}_{k,x}$. Moreover, the same argument applies to one-parameter families of sections $s_{t,k}$ (one similarly constructs perturbations $\tau_{t,k,x} = -w_{t,k}\, s^{\mathrm{ref}}_{t,k,x}$). So Proposition 3 applies, which ends the proof of Proposition 1.

2.2. **Non-vanishing of ∂f_k.** We have constructed asymptotically holomorphic sections $s_k = (s^0_k, s^1_k, s^2_k)$ of $\mathbb{C}^3 \otimes L^k$ for all large enough k which remain away from zero. Therefore, the maps $f_k = \mathbb{P}s_k$ from X to $\mathbb{CP}^2$ are well defined, and they are asymptotically holomorphic, because the lower bound on $|s_k|$ implies that the derivatives of f_k are $O(1)$ and that $\bar\partial f_k$ and its derivatives are $O(k^{-1/2})$ (taking the metric g_k on X and the standard metric on $\mathbb{CP}^2$). Our next step is to ensure, by further perturbation of the sections s_k, that ∂f_k vanishes nowhere and remains far from zero :

Proposition 4. *Let δ and γ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ such that $|s_k| \ge \gamma$ at every point of X and for all k. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3,g_k} \le \delta$ and that the maps $f_k = \mathbb{P}\sigma_k$ satisfy the bound $|\partial f_k|_{g_k} \ge \eta$ at every point of X. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proposition 4 is proved in the same manner as Proposition 1 and uses the same three ingredients, namely Lemma 2 and Propositions 2 and 3. Proposition 4 follows directly from Proposition 3 by considering the following property : say that a section s of $\mathbb{C}^3 \otimes L^k$ of norm everywhere larger than $\frac{\gamma}{2}$ satisfies $\mathcal{P}(\eta, x)$ if the map $f = \mathbb{P}s$ satisfies $|\partial f(x)|_{g_k} \ge \eta$. This property is local and open in C^1-sense, and therefore also in C^3-sense, because the lower bound on $|s|$ makes f depend nicely on s (by the way, note that the bound $|s| \ge \frac{\gamma}{2}$ is always satisfied in our setting since one considers only sections differing from s_k by less than $\frac{\gamma}{4}$). So one only needs to check that the assumptions of Proposition 3 hold for this property $\mathcal{P}$.

Therefore, let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider nonvanishing asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ and the corresponding maps $f_k = \mathbb{P}s_k$. Without loss of generality, composing with a rotation in $\mathbb{C}^3$ (constant over X), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s_k^1(x) = s_k^2(x) = 0$ and therefore $|s_k^0(x)| \geq \frac{\gamma}{2}$. Because one has a uniform bound on $|\nabla s_k|$, there exists a constant $r > 0$ (independent of k) such that $|s_k^0| \geq \frac{\gamma}{3}$ over $B_{g_k}(x, r)$. Therefore, over this ball one can define a map to $\mathbb{C}^2$ by

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

It is quite easy to see that, at any point $y \in B_{g_k}(x, r)$, the ratio between $|\partial h_k(y)|$ and $|\partial f_k(y)|$ is bounded by a uniform constant. Therefore, what one actually needs to prove is that, for large enough k, a perturbation of s_k with Gaussian decay and smaller than δ can make $|\partial h_k|$ larger than $\eta = c'\delta\,(\log \delta^{-1})^{-p}$ over a ball $B_{g_k}(x, c)$, for some constants c, c' and p.

Recall that Lemma 2 provides asymptotically holomorphic sections $s_{k,x}^{\text{ref}}$ of L^k which have Gaussian decay away from x and remain larger than a constant c_s over $B_{g_k}(x, 1)$. Moreover, consider a local approximately holomorphic coordinate chart (as given by Lemma 3) on a neighborhood of x, and call z_k^1 and z_k^2 the two complex coordinate functions. Define the two 1-forms

$$\mu_k^1 = \partial\Big(\frac{z_k^1 s_{k,x}^{\text{ref}}}{s_k^0}\Big) \quad \text{and} \quad \mu_k^2 = \partial\Big(\frac{z_k^2 s_{k,x}^{\text{ref}}}{s_k^0}\Big),$$

and notice that at x they are both of norm larger than a fixed constant (which can be expressed as a function of c_s and the uniform C^0 bound on s_k), and mutually orthogonal. Moreover μ_k^1, μ_k^2 and their derivatives are uniformly bounded because of the bounds on $s_{k,x}^{\text{ref}}$, on s_k^0 and on the coordinate map ; these bounds are independent of k. Finally, μ_k^1 and μ_k^2 are asymptotically holomorphic because all the ingredients in their definition are asymptotically holomorphic and $|s_k^0|$ is bounded from below.

If follows that, for some constant r', one can express ∂h_k on the ball $B_{g_k}(x, r')$ as $(\partial h_k^1, \partial h_k^2) = (u_k^{11}\mu_k^1 + u_k^{12}\mu_k^2, u_k^{21}\mu_k^1 + u_k^{22}\mu_k^2)$, thus defining a function u_k on $B_{g_k}(x, r')$ with values in $\mathbb{C}^4$. The properties of μ_k^i described above imply that the ratio between $|\partial h_k|$ and $|u_k|$ is bounded between two constants which do not depend on k (because of the bounds on μ_k^1 and μ_k^2, and of their orthogonality at x), and that the map u_k is asymptotically holomorphic (because of the asymptotic holomorphicity of μ_k^i).

Next, one uses the local approximately holomorphic coordinate chart to obtain from u_k, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions v_k defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}^4$, and satisfying the estimates $|v_k| = O(1)$, $|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$. Let C_0 be a constant larger than $|z_k^i s_{k,x}^{\text{ref}}|_{C^3, g_k}$, and let $\alpha = \frac{\delta}{4C_0}.\log((\frac{\delta}{4C_0})^{-1})^{-p}$. Then, by Proposition 2, for all large enough k there exist constants $w_k = (w_k^{11}, w_k^{12}, w_k^{21}, w_k^{22}) \in \mathbb{C}^4$, with $|w_k| \leq \frac{\delta}{4C_0}$, such that $|v_k - w_k| \geq \alpha$ over the unit ball in $\mathbb{C}^2$.

Equivalently, one has $|u_k - w_k| \geq \alpha$ over $B_{g_k}(x, c)$ for some constant c. Multiplying by μ_k^i, one therefore gets that, over $B_{g_k}(x, c)$,

$$\left|\left(\partial\Big(h_k^1 - w_k^{11}\frac{z_k^1 s_{k,x}^{\text{ref}}}{s_k^0} - w_k^{12}\frac{z_k^2 s_{k,x}^{\text{ref}}}{s_k^0}\Big), \partial\Big(h_k^2 - w_k^{21}\frac{z_k^1 s_{k,x}^{\text{ref}}}{s_k^0} - w_k^{22}\frac{z_k^2 s_{k,x}^{\text{ref}}}{s_k^0}\Big)\right)\right| \geq \frac{\alpha}{C}$$

where C is a fixed constant determined by the bounds on μ_k^i. In other terms, letting

$$(\tau_{k,x}^0, \tau_{k,x}^1, \tau_{k,x}^2) = (0, -(w_k^{11} z_k^1 + w_k^{12} z_k^2) s_{k,x}^{\mathrm{ref}}, -(w_k^{21} z_k^1 + w_k^{22} z_k^2) s_{k,x}^{\mathrm{ref}}),$$

and defining $\tilde{h}_k$ similarly to h_k starting with $s_k + \tau_{k,x}$ instead of s_k, the above formula can be rewritten as $|\partial \tilde{h}_k| \ge \frac{\alpha}{C}$. Therefore, one has managed to make $|\partial \tilde{h}_k|$ larger than $\eta = \frac{\alpha}{C}$ over $B_{g_k}(x, c)$ by adding to s_k the perturbation $\tau_{k,x}$. Moreover, $|\tau_{k,x}| \le \sum |w_k^{ij}|.|z_k^i s_{k,x}^{\mathrm{ref}}| \le \delta$, and the sections $z_k^i s_{k,x}^{\mathrm{ref}}$ have uniform Gaussian decay away from x.

As remarked above, setting $\tilde{f}_k = \mathbb{P}(s_k + \tau_{k,x})$, the bound $|\partial \tilde{h}_k| \ge \eta$ implies that $|\partial \tilde{f}_k|$ is larger than some η' differing from η by at most a constant factor. The assumptions of Proposition 3 are therefore satisfied, since one has $\eta' \ge c' \delta \log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$. Moreover, the whole argument also applies to one-parameter families of sections $s_{t,k}$ as well (considering one-parameter families of coordinate charts, reference sections $s_{t,k,x}^{\mathrm{ref}}$, and constants $w_{t,k}$). So Proposition 3 applies. This ends the proof of Proposition 4.

2.3. **Proof of Proposition 2.** The proof of Proposition 2 goes along the same lines as that of the local transversality result introduced in [2] and extended to one-parameter families in [1] (see Proposition 6 below). To start with, notice that it is sufficient to prove the result in the case where $m = n + 1$. Indeed, given a map $f = (f^1, \dots, f^m) : B^+ \to \mathbb{C}^m$ with $m > n + 1$ satisfying the hypotheses of Proposition 2, one can define $f' = (f^1, \dots, f^{n+1}) : B^+ \to \mathbb{C}^{n+1}$, and notice that f' also satisfies the required bounds. Therefore, if it is possible to find $w' = (w^1, \dots, w^{n+1}) \in \mathbb{C}^{n+1}$ of norm at most δ such that $|f' - w'| \ge \eta$ over the unit ball B, then setting $w = (w^1, \dots, w^{n+1}, 0, \dots, 0) \in \mathbb{C}^m$ one gets $|w| = |w'| \le \delta$ and $|f - w| \ge |f' - w'| \ge \eta$ at all points of B, which is the desired result. The same argument applies to one-parameter families $(f_t)_{t \in [0,1]}$.

So we are now reduced to the case $m = n + 1$. Let us start with the case of a single map f, before moving on to the case of one-parameter families. The first step in the proof is to replace f by a complex polynomial g approximating f. For this, one approximates each of the $n+1$ components f^i by a polynomial g^i, in such a way that g differs from f by at most a fixed multiple of η over the unit ball B and that the degree d of g is less than a constant times $\log(\eta^{-1})$. The process is the same as the one described in [2] for asymptotically holomorphic maps to $\mathbb{C}$, so we skip the details. To obtain polynomial functions, one first constructs holomorphic functions $\tilde{f}^i$ differing from f^i by at most a fixed multiple of η, using the given bounds on $\bar{\partial} f^i$. The polynomials g^i are then obtained by truncating the Taylor series expansion of $\tilde{f}^i$ to a given degree. It can be shown that by this method one can obtain polynomial functions g^i of degree less than a constant times $\log(\eta^{-1})$ and differing from $\tilde{f}^i$ by at most a constant times η (see Lemmas 27 and 28 of [2]). The approximation process does not hold on the whole ball where f is defined ; this is why one needs f to be defined on B^+ to get a result over the slightly smaller ball B.

Therefore, we now have a polynomial map g of degree $d = O(\log(\eta^{-1}))$ such that $|f - g| \le c\,\eta$ for some constant c. In particular, if one finds $w \in \mathbb{C}^{n+1}$ with $|w| \le \delta$ such that $|g - w| \ge (c+1)\eta$ over the ball B, then it follows immediately that $|f - w| \ge \eta$ everywhere, which is the desired result. The key observation for finding such a w is that the image $g(B) \subset \mathbb{C}^{n+1}$ is contained in an algebraic hypersurface H

in $\mathbb{C}^{n+1}$ of degree at most $D=(n+1)d^n$. Indeed, if such were not the case, then for every nonzero polynomial P of degree at most D in $n+1$ variables, $P(g^1,\dots,g^{n+1})$ would be a non identically zero polynomial function of degree at most dD in n variables ; since the space of polynomials of degree at most D in $n+1$ variables is of dimension $\binom{D+n+1}{n+1}$ while the space of polynomials of degree at most dD in n variables is of dimension $\binom{dD+n}{n}$, the injectivity of the map $P\mapsto P(g^1,\dots,g^{n+1})$ from the first space to the second would imply that $\binom{D+n+1}{n+1}\le\binom{dD+n}{n}$. However since $D=(n+1)d^n$ one has

$$\frac{\binom{D+n+1}{n+1}}{\binom{dD+n}{n}}=\frac{(n+1)d^n+(n+1)}{n+1}\cdot\frac{D+n}{dD+n}\cdots\frac{D+1}{dD+1}\ge(d^n+1)\cdot\left(\frac{1}{d}\right)^n>1,$$

which gives a contradiction. So $g(B)\subset H$ for a certain hypersurface $H\subset\mathbb{C}^{n+1}$ of degree at most $D=(n+1)d^n$. Therefore the following classical result of algebraic geometry (see e.g. [4], pp. 11–15) can be used to provide control on the size of H inside any ball in $\mathbb{C}^{n+1}$:

Lemma 4. *Let $H\subset\mathbb{C}^{n+1}$ be a complex algebraic hypersurface of degree D. Then, given any $r>0$ and any $x\in\mathbb{C}^{n+1}$, the $2n$-dimensional volume of $H\cap B(x,r)$ is at most $DV_0\,r^{2n}$, where V_0 is the volume of the unit ball of dimension $2n$. Moreover, if $x\in H$, then one also has $vol_{2n}(H\cap B(x,r))\ge V_0\,r^{2n}$.*

In particular, we are interested in the intersection of H with the ball $\hat{B}$ of radius δ centered at the origin. Lemma 4 implies that the volume of this intersection is bounded by $(n+1)V_0\,d^n\delta^{2n}$. Cover $\hat{B}$ by a finite number of balls $B(x_i,\eta)$, in such a way that no point is contained in more than a fixed constant number (depending only on n) of the balls $B(x_i,2\eta)$. Then, for every i such that $B(x_i,\eta)\cap H$ is non-empty, $B(x_i,2\eta)$ contains a ball of radius η centered at a point of H, so by Lemma 4 the volume of $B(x_i,2\eta)\cap H$ is at least $V_0\,\eta^{2n}$. Summing the volumes of these intersections and comparing with the total volume of $H\cap\hat{B}$, one gets that the number of balls $B(x_i,\eta)$ which meet H is bounded by $N=Cd^n\delta^{2n}\eta^{-2n}$, where C is a constant depending only on n. Therefore, $H\cap\hat{B}$ is contained in the union of N balls of radius η.

Since our goal is to find $w\in\hat{B}$ at distance more than $(c+1)\eta$ of $g(B)\subset H$, the set Z of values which we want to avoid is contained in a set Z^+ which is the union of $N=Cd^n\delta^{2n}\eta^{-2n}$ balls of radius $(c+2)\eta$. The volume of Z^+ is bounded by $C'd^n\delta^{2n}\eta^2$ for some constant C' depending only on n. Therefore, there exists a constant C'' such that, if one assumes δ to be larger than $C''d^{n/2}\eta$, the volume of $\hat{B}$ is strictly larger than that of Z^+, and so $\hat{B}-Z^+$ is not empty. Calling w any element of $\hat{B}-Z^+$, one has $|w|\le\delta$, and $|g-w|\ge(c+1)\eta$ at every point of B, and therefore $|f-w|\ge\eta$ at every point of B, which is the desired result.

Since d is bounded by a constant times $\log(\eta^{-1})$, it is not hard to see that there exists an integer p such that, for all $0<\delta<\frac{1}{2}$, the relation $\eta=\delta\log(\delta^{-1})^{-p}$ implies that $\delta>C''d^{n/2}\eta$. This is the value of p which we choose in the statement of the proposition, thus ensuring that $\hat{B}-Z^+$ is not empty and therefore that there exists w with $|w|\le\delta$ such that $|f-w|\ge\eta$ at every point of B.

We now consider the case of a one-parameter family of functions $(f_t)_{t\in[0,1]}$. The first part of the above argument also applies to this case, so there exist polynomial maps g_t of degree $d = O(\log(\eta^{-1}))$, depending continuously on t, such that $|f_t - g_t| \leq c\,\eta$ for some constant c and for all t. In particular, if one finds $w_t \in \mathbb{C}^{n+1}$ with $|w_t| \leq \delta$ and depending continuously on t such that $|g_t - w_t| \geq (c+1)\eta$ over the ball B, then it follows immediately that $|f_t - w_t| \geq \eta$ everywhere, which is the desired result.

As before, $g_t(B)$ is contained in a hypersurface of degree at most $(n+1)d^n$ in $\mathbb{C}^{n+1}$, and the same argument as above implies that the set Z_t of values which we want to avoid for w_t (i.e. all the points of $\hat{B}$ at distance less than $(c+1)\eta$ from $g_t(B)$) is contained in a set Z_t^+ which is the union of $N = Cd^n\delta^{2n}\eta^{-2n}$ balls of radius $(c+2)\eta$. The rest of the proof is now a higher-dimensional analogue of the argument used in [1] : the crucial point is to show that, if δ is large enough, $\hat{B} - Z_t^+$ splits into several small connected components and only *one* large component, because the boundary $Y_t = \partial Z_t^+$ is much smaller than a $(2n+1)$-ball of radius δ and therefore cannot split $\hat{B}$ into components of comparable sizes.

Each component of $\hat{B} - Z_t^+$ is delimited by a subset of the sphere $\partial\hat{B}$ and by a union of components of Y_t. Each component $Y_{t,i}$ of Y_t is a real hypersurface in $\hat{B}$ (with corners at the points where the boundaries of the various balls of Z_t^+ intersect) whose boundary is contained in $\partial\hat{B}$, and therefore splits $\hat{B}$ into two components C_i' and C_i''. So each component of $\hat{B} - Z_t^+$ is an intersection of components C_i' or C_i'' where i ranges over a certain subset of the set of components of Y_t. Let us now state the following isoperimetric inequality :

Lemma 5. *Let Y be a connected (singular) submanifold of real codimension 1 in the unit ball of dimension $2n+2$, with (possibly empty) boundary contained in the boundary of the ball. Let A be the $(2n+1)$-dimensional area of Y. Then the volume V of the smallest of the two components delimited by Y in the ball satisfies the bound $V \leq K\,A^{(2n+2)/(2n+1)}$, where K is a fixed constant depending only on the dimension.*

Proof. The stereographic projection maps the unit ball quasi-isometrically onto a half-sphere. Therefore, up to a change in the constant, it is sufficient to prove the result on the half-sphere. By doubling Y along its intersection with the boundary of the half-sphere, which doubles both the volume delimited by Y and its area, one reduces to the case of a closed connected (singular) real hypersurface in the sphere S^{2n+2} (if Y does not meet the boundary, then it is not necessary to consider the double). Next, one notices that the singular hypersurfaces we consider can be smoothed in such a way that the area of Y and the volume it delimits are changed by less than any fixed constant ; therefore, Lemma 5 follows from the classical spherical isoperimetric inequality (see e.g. [6]). □

It follows that, letting A_i be the $(2n+1)$-dimensional area of $Y_{t,i}$, the smallest of the two components delimited by $Y_{t,i}$, e.g. C_i', has volume $V_i \leq K\,A_i^{(2n+2)/(2n+1)}$. Therefore, the volume of the set $\bigcup_i C_i'$ is bounded by

$$K \textstyle\sum_i A_i^{(2n+2)/(2n+1)} \leq K\,(\sum_i A_i)^{(2n+2)/(2n+1)}.$$

However, $\sum_i A_i$ is the total area of the boundary Y_t of Z_t^+, so it is less than the total area of the boundaries of the balls composing Z_t^+, which is at most a fixed constant times $Cd^n\delta^{2n}\eta^{-2n}((c+2)\eta)^{2n+1}$, i.e. at most a fixed constant times $d^n\delta^{2n}\eta$.

Therefore, one has

$$\mathrm{vol}(\bigcup_i C'_i) \le K' \Big(d^n \frac{\eta}{\delta}\Big)^{\frac{2n+2}{2n+1}} \delta^{2n+2}$$

for some constant K' depending only on n. So there exists a constant K'' depending only on n such that, if $\delta > K'' d^n \eta$, then $\mathrm{vol}(\bigcup_i C'_i) \le \frac{1}{10}\mathrm{vol}(\hat{B})$, and therefore $\mathrm{vol}(\bigcap_i C''_i) \ge \frac{8}{10}\mathrm{vol}(\hat{B})$.

Since d is bounded by a constant times $\log(\eta^{-1})$, it is not hard to see that there exists an integer p such that, for all $0 < \delta < \frac{1}{2}$, the relation $\eta = \delta \log(\delta^{-1})^{-p}$ implies that $\delta > K'' d^n \eta$. This is the value of p which we choose in the statement of the proposition, thus ensuring that the above volume bounds on $\bigcup_i C'_i$ and $\bigcap_i C''_i$ hold.

Now, recall that every component of $\hat{B} - Z_t^+$ is an intersection of sets C'_i and C''_i for certain values of i. Therefore, every component of $\hat{B} - Z_t^+$ either is contained in $\bigcup_i C'_i$ or contains $\bigcap_i C''_i$. However, because $\bigcup_i C'_i$ is much smaller than the ball $\hat{B}$, one cannot have $\hat{B} - Z_t^+ \subset \bigcup_i C'_i$. Therefore, there exists a component in $\hat{B} - Z_t^+$ containing $\bigcup_i C''_i$. Since its volume is at least $\frac{8}{10}\mathrm{vol}(\hat{B})$, this large component is necessarily unique.

Let $U(t)$ be the connected component of $\hat{B} - Z_t$ which contains the large component of $\hat{B} - Z_t^+$: it is the only large component of $\hat{B} - Z_t$. We now follow the same argument as in [1]. Since $g_t(B)$ depends continuously on t, so does its $(c+1)\eta$-neighborhood Z_t, and the set $\bigcup_t \{t\} \times Z_t$ is therefore a closed subset of $[0,1] \times \hat{B}$. Let $U^-(t,\epsilon)$ be the set of all points of $U(t)$ at distance more than ϵ from $Z_t \cup \partial\hat{B}$. Then, given any t and any small $\epsilon > 0$, for all τ close to t, $U(\tau)$ contains $U^-(t,\epsilon)$. To see this, we first notice that, for all τ close to t, $U^-(t,\epsilon) \cap Z_\tau = \emptyset$. Indeed, if such were not the case, one could take a sequence of points of $Z_\tau \cap U^-(t,\epsilon)$ for $\tau \to t$, and extract a convergent subsequence whose limit belongs to $\overline{U}^-(t,\epsilon)$ and therefore lies outside of Z_t, in contradiction with the fact that $\bigcup_t \{t\} \times Z_t$ is closed. So $U^-(t,\epsilon) \subset \hat{B} - Z_\tau$ for all τ close enough to t. Making ϵ smaller if necessary, one may assume that $U^-(t,\epsilon)$ is connected, so that for all τ close to t, $U^-(t,\epsilon)$ is necessarily contained in the large component of $\hat{B} - Z_\tau$, namely $U(\tau)$.

It follows that $U = \bigcup_t \{t\} \times U(t)$ is an open connected subset of $[0,1] \times \hat{B}$, and is therefore path-connected. So we get a path $s \mapsto (t(s), w(s))$ joining $(0, w(0))$ to $(1, w(1))$ inside U, for any given $w(0)$ and $w(1)$ in $U(0)$ and $U(1)$. We then only have to make sure that $s \mapsto t(s)$ is strictly increasing in order to define $w_{t(s)} = w(s)$.

Getting the t component to increase strictly is not hard. Indeed, one first gets it to be weakly increasing, by considering values $s_1 < s_2$ of the parameter such that $t(s_1) = t(s_2) = t$ and replacing the portion of the path between s_1 and s_2 by a path joining $w(s_1)$ to $w(s_2)$ in the connected set $U(t)$. Then, we slightly shift the path, using the fact that U is open, to get the t component to increase slightly over the parts where it was constant. Thus we can define $w_{t(s)} = w(s)$ and end the proof of Proposition 2.

3. Transversality of derivatives

3.1. **Transversality to 0 of** $\mathrm{Jac}(f_k)$**.** At this point in the proofs of Theorems 1 and 2, we have constructed for all large k asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ (or families of sections), bounded away from 0, and such that the holomorphic derivative of the map $f_k = \mathbb{P}s_k$ is bounded away from 0. The next property we wish to ensure by perturbing the sections s_k is the transversality to 0 of the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$. The main result of this section is :

Proposition 5. *Let δ and γ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k \gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ such that $|s_k| \geq \gamma$ and $|\partial(\mathbb{P}s_k)|_{g_k} \geq \gamma$ at every point of X. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3, g_k} \leq \delta$ and $\mathrm{Jac}(\mathbb{P}\sigma_k)$ is η-transverse to 0. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

The proof of Proposition 5 uses once more the same techniques and globalization argument as Propositions 1 and 4. The local transversality result one uses in conjunction with Proposition 3 is now the following statement for complex valued functions :

Proposition 6 ([2],[1])**.** *Let f be a function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}$. Let δ be a constant such that $0 < \delta < \frac{1}{2}$, and let $\eta = \delta \log(\delta^{-1})^{-p}$ where p is a suitable fixed integer depending only on the dimension n. Assume that f satisfies the following bounds over B^+ :*

$$|f| \leq 1, \qquad |\bar\partial f| \leq \eta, \qquad |\nabla \bar\partial f| \leq \eta.$$

Then there exists $w \in \mathbb{C}$, with $|w| \leq \delta$, such that $f - w$ is η-transverse to 0 over the interior ball B of radius 1, i.e. $f - w$ has derivative larger than η at any point of B where $|f - w| < \eta$.

Moreover, the same statement remains true for a one-parameter family of functions $(f_t)_{t \in [0,1]}$ satisfying the same bounds, i.e. for all t one can find elements $w_t \in \mathbb{C}$ depending continuously on t such that $|w_t| \leq \delta$ and $f_t - w_t$ is η-transverse to 0 over B.

The first part of this statement is exactly Theorem 20 of [2], and the version for one-parameter families is Proposition 3 of [1]. □

Proposition 5 is proved by applying Proposition 3 to the following property : say that a section s of $\mathbb{C}^3 \otimes L^k$ everywhere larger than $\frac{\gamma}{2}$ and such that $|\partial \mathbb{P}s| \geq \frac{\gamma}{2}$ everywhere satisfies $\mathcal{P}(\eta, x)$ if $\mathrm{Jac}(\mathbb{P}s)$ is η-transverse to 0 at x, i.e. either $|\mathrm{Jac}(\mathbb{P}s)(x)| \geq \eta$ or $|\nabla \mathrm{Jac}(\mathbb{P}s)(x)| > \eta$. This property is local and C^2-open, and therefore also C^3-open, because the lower bound on s makes $\mathrm{Jac}(\mathbb{P}s)$ depend nicely on s. Note that, since one considers only sections differing from s_k by less than δ in C^3 norm, decreasing δ if necessary, one can safely assume that the two hypotheses $|s| \geq \frac{\gamma}{2}$ and $|\partial(\mathbb{P}s)| \geq \frac{\gamma}{2}$ are satisfied everywhere by all the sections appearing in the construction of σ_k. So one only needs to check that the assumptions of Proposition 3 hold for the property $\mathcal{P}$ defined above.

Therefore, let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ and the corresponding maps $f_k = \mathbb{P}s_k$, such that $|s_k| \geq \frac{\gamma}{2}$ and $|\partial f_k| \geq \frac{\gamma}{2}$ everywhere. The setup is similar to that of §2.2. Without loss of generality, composing with a rotation in $\mathbb{C}^3$ (constant over X), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s_k^1(x) = s_k^2(x) = 0$ and therefore $|s_k^0(x)| \geq \frac{\gamma}{2}$. Because of the uniform bound on $|\nabla s_k|$, there exists $r > 0$

(independent of k) such that $|s_k^0| \geq \frac{\gamma}{3}$, $|s_k^1| < \frac{\gamma}{3}$ and $|s_k^2| < \frac{\gamma}{3}$ over the ball $B_{g_k}(x,r)$. Therefore, over this ball one can define the map

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

Note that f_k is the composition of h_k with the map $\iota : (z_1, z_2) \mapsto [1 : z_1 : z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$, which is a quasi-isometry over the unit ball in $\mathbb{C}^2$. Therefore, at any point $y \in B_{g_k}(x,r)$, the bound $|\partial f_k(y)| \geq \frac{\gamma}{2}$ implies that $|\partial h_k(y)| \geq \gamma'$ for some constant $\gamma' > 0$. Moreover, the $(2,0)$-Jacobians $\mathrm{Jac}(f_k) = \det(\partial f_k)$ and $\mathrm{Jac}(h_k) = \det(\partial h_k)$ are related to each other : $\mathrm{Jac}(f_k)(y) = \phi(y)\,\mathrm{Jac}(h_k)(y)$, where $\phi(y)$ is the Jacobian of ι at $h_k(y)$. In particular, $|\phi|$ is bounded between two universal constants over $B_{g_k}(x,r)$, and $\nabla\phi$ is also bounded.

Since $\nabla \mathrm{Jac}(h_k) = \phi^{-1}\nabla\mathrm{Jac}(f_k) - \phi^{-2}\mathrm{Jac}(f_k)\nabla\phi$, it follows from the bounds on ϕ that, if $\mathrm{Jac}(f_k)$ fails to be α-transverse to 0 at y for some α, i.e. if $|\mathrm{Jac}(f_k)(y)| < \alpha$ and $|\nabla\mathrm{Jac}(f_k)(y)| \leq \alpha$, then $|\mathrm{Jac}(h_k)(y)| < C\alpha$ and $|\nabla\mathrm{Jac}(h_k)(y)| \leq C\alpha$ for some constant C independent of k and α. This means that, if $\mathrm{Jac}(h_k)$ is $C\alpha$-transverse to 0 at y, then $\mathrm{Jac}(f_k)$ is α-transverse to 0 at y. Therefore, what one actually needs to prove is that, for large enough k, a perturbation of s_k with Gaussian decay and smaller than δ allows one to obtain the η-transversality to 0 of $\mathrm{Jac}(h_k)$ over a ball $B_{g_k}(x,c)$, with $\eta = c'\delta\,(\log\delta^{-1})^{-p}$, for some constants c, c' and p ; the $\frac{\eta}{C}$-transversality to 0 of $\mathrm{Jac}(f_k)$ then follows by the above remark.

Since $|\partial h_k(x)| \geq \gamma'$, one can assume, after composing with a rotation in $\mathbb{C}^2$ (constant over X) acting on the two components (s_k^1, s_k^2) or equivalently on (h_k^1, h_k^2), that $|\partial h_k^2(x)| \geq \frac{\gamma'}{2}$. As in §2.2, consider the asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of L^k with Gaussian decay away from x given by Lemma 2, and the complex coordinate functions z_k^1 and z_k^2 of a local approximately holomorphic Darboux coordinate chart on a neighborhood of x. Recall that the two asymptotically holomorphic 1-forms

$$\mu_k^1 = \partial\Big(\frac{z_k^1 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big) \quad \text{and} \quad \mu_k^2 = \partial\Big(\frac{z_k^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big)$$

are, at x, both of norm larger than a fixed constant and mutually orthogonal, and that μ_k^1, μ_k^2 and their derivatives are uniformly bounded independently of k.

Because $\mu_k^1(x)$ and $\mu_k^2(x)$ define an orthogonal frame in $\Lambda^{1,0}T_x^*X$, there exist complex numbers a_k and b_k such that $\partial h_k^2(x) = a_k\mu_k^1(x) + b_k\mu_k^2(x)$. Let $\lambda_{k,x} = (\bar b_k z_k^1 - \bar a_k z_k^2)s_{k,x}^{\mathrm{ref}}$. The properties of $\lambda_{k,x}$ of importance to us are the following : the sections $\lambda_{k,x}$ are asymptotically holomorphic because the coordinates z_k^i are asymptotically holomorphic ; they are uniformly bounded in C^3 norm by a constant C_0, because of the bounds on $s_{k,x}^{\mathrm{ref}}$, on the coordinate chart and on $\partial h_k^2(x)$; they have uniform Gaussian decay away from x ; and, letting

$$\Theta_{k,x} = \partial\Big(\frac{\lambda_{k,x}}{s_k^0}\Big) \wedge \partial h_k^2,$$

one has $|\Theta_{k,x}(x)| = |(\bar b_k\mu_k^1(x) - \bar a_k\mu_k^2(x)) \wedge (a_k\mu_k^1(x) + b_k\mu_k^2(x))| \geq \gamma''$ for some constant $\gamma'' > 0$, because of the lower bounds on $|\mu_k^i(x)|$ and $|\partial h_k^2(x)|$.

Because $\nabla\Theta_{k,x}$ is uniformly bounded and $|\Theta_{k,x}(x)| \geq \gamma''$, there exists a constant $r' > 0$ independent of k such that $|\Theta_{k,x}|$ remains larger than $\frac{\gamma''}{2}$ over the ball $B_{g_k}(x,r')$. Define on $B_{g_k}(x,r')$ the function $u_k = \Theta_{k,x}^{-1}\mathrm{Jac}(h_k)$ with values in $\mathbb{C}$:

because $\Theta_{k,x}$ is bounded from above and below and has bounded derivative, the transversality to 0 of u_k is equivalent to that of $\mathrm{Jac}(h_k)$. Moreover, for any $w_k \in \mathbb{C}$, adding $w_k\lambda_{k,x}$ to s_k^1 is equivalent to adding $w_k\Theta_{k,x}$ to $\mathrm{Jac}(h_k) = \partial h_k^1 \wedge \partial h_k^2$, i.e. adding w_k to u_k. Therefore, to prove Proposition 5 we only need to find $w_k \in \mathbb{C}$ with $|w_k| \le \frac{\delta}{C_0}$ such that the functions $u_k - w_k$ are transverse to 0.

Using the local approximately holomorphic coordinate chart, one can obtain from u_k, after composing with a fixed dilation of $\mathbb{C}^2$ if necessary, functions v_k defined on the ball $B^+ \subset \mathbb{C}^2$, with values in $\mathbb{C}$, and satisfying the estimates $|v_k| = O(1)$, $|\bar\partial v_k| = O(k^{-1/2})$ and $|\nabla\bar\partial v_k| = O(k^{-1/2})$. One can then apply Proposition 6, provided that k is large enough, to obtain constants $w_k \in \mathbb{C}$, with $|w_k| \le \frac{\delta}{C_0}$, such that $v_k - w_k$ is α-transverse to 0 over the unit ball in $\mathbb{C}^2$, where $\alpha = \frac{\delta}{C_0}\log((\frac{\delta}{C_0})^{-1})^{-p}$. Therefore, $u_k - w_k$ is $\frac{\alpha}{C'}$-transverse to 0 over $B_{g_k}(x,c)$ for some constants c and C'. Multiplying by $\Theta_{k,x}$, one finally gets that, over $B_{g_k}(x,c)$, $\mathrm{Jac}(h_k) - w_k\Theta_{k,x}$ is η-transverse to 0, where $\eta = \frac{\alpha}{C''}$ for some constant C''.

In other terms, let $(\tau^0_{k,x}, \tau^1_{k,x}, \tau^2_{k,x}) = (0, -w_k\lambda_{k,x}, 0)$, and define $\tilde h_k$ similarly to h_k starting with $s_k + \tau_{k,x}$ instead of s_k : then the above discussion shows that $\mathrm{Jac}(\tilde h_k)$ is η-transverse to 0 over $B_{g_k}(x,c)$. Moreover, $|\tau_{k,x}|_{C^3} = |w_k|\,|\lambda_{k,x}|_{C^3} \le \delta$, and the sections $\tau_{k,x}$ have uniform Gaussian decay away from x. As remarked above, the η-transversality to 0 of $\mathrm{Jac}(\tilde h_k)$ implies that $\mathrm{Jac}(\mathbb{P}(s_k + \tau_{k,x}))$ is η'-transverse to 0 for some η' differing from η by at most a constant factor. The assumptions of Proposition 3 are therefore satisfied, since $\eta' \ge c'\delta\log(\delta^{-1})^{-p}$ for a suitable constant $c' > 0$.

Moreover, the whole argument also applies to one-parameter families of sections $s_{t,k}$ as well. The only nontrivial point to check, in order to apply the above construction for each $t \in [0,1]$ in such a way that everything depends continuously on t, is the existence of a continuous family of rotations of $\mathbb{C}^2$ acting on (h_k^1, h_k^2) allowing one to assume that $|\partial h^2_{t,k}(x)| > \frac{\gamma'}{2}$ for all t. For this, observe that, for every t, such rotations in $\mathrm{SU}(2)$ are in one-to-one correspondence with pairs $(\alpha,\beta) \in \mathbb{C}^2$ such that $|\alpha|^2 + |\beta|^2 = 1$ and $|\alpha\,\partial h^1_{t,k}(x) + \beta\,\partial h^2_{t,k}(x)| > \frac{\gamma'}{2}$. The set Γ_t of such pairs (α,β) is non-empty because $|\partial h_{t,k}(x)| \ge \gamma'$; let us now prove that it is connected.

First, notice that Γ_t is invariant under the diagonal S^1 action on $\mathbb{C}^2$. Therefore, it is sufficient to prove that the set of $(\alpha:\beta) \in \mathbb{CP}^1$ such that

$$\phi(\alpha:\beta) := \frac{|\alpha\,\partial h^1_{t,k}(x) + \beta\,\partial h^2_{t,k}(x)|^2}{|\alpha|^2 + |\beta|^2} > \frac{(\gamma')^2}{4}$$

is connected. For this, consider a critical point of ϕ over $\mathbb{CP}^1$. Composing with a rotation in $\mathbb{CP}^1$, one may assume that this critical point is $(1:0)$. Then it follows from the property $\frac{\partial}{\partial\beta}\phi(1:\beta)_{|\beta=0} = 0$ that $\partial h^1_{t,k}(x)$ and $\partial h^2_{t,k}(x)$ must necessarily be orthogonal to each other. Therefore, one has

$$\phi(1:\beta) = \frac{|\partial h^1_{t,k}(x)|^2 + |\beta|^2|\partial h^2_{t,k}(x)|^2}{1 + |\beta|^2},$$

and it follows that either ϕ is constant over $\mathbb{CP}^1$ (if $|\partial h^1_{t,k}(x)| = |\partial h^2_{t,k}(x)|$), or the critical point is nondegenerate of index 0 (if $|\partial h^1_{t,k}(x)| < |\partial h^2_{t,k}(x)|$), or it is nondegenerate of index 2 (if $|\partial h^1_{t,k}(x)| > |\partial h^2_{t,k}(x)|$). As a consequence, since ϕ has no critical point of index 1, all nonempty sets of the form $\{(\alpha:\beta)\in\mathbb{CP}^1,\ \phi(\alpha,\beta)>\text{constant}\}$ are connected.

Lifting back from $\mathbb{CP}^1$ to the unit sphere in $\mathbb{C}^2$, it follows that Γ_t is connected. Therefore, for each t the open set $\Gamma_t\subset \mathrm{SU}(2)$ of admissible rotations of $\mathbb{C}^2$ is connected. Since $h_{t,k}$ depends continuously on t, the sets Γ_t also depend continuously on t (with respect to nearly every conceivable topology), and therefore $\bigcup_t\{t\}\times\Gamma_t$ is connected. The same argument as in the end of §2.3 then implies the existence of a continuous section of $\bigcup_t\{t\}\times\Gamma_t$ over $[0,1]$, i.e. the existence of a continuous one-parameter family of rotations of $\mathbb{C}^2$ which allows one to ensure that $|\partial h^2_{t,k}(x)|>\frac{\gamma'}{2}$ for all t. Therefore, the argument described in this section also applies to the case of one-parameter families, and the assumptions of Proposition 3 are satisfied by the property $\mathcal{P}$ even in the case of one-parameter families of sections. Proposition 5 follows immediately.

3.2. **Nondegeneracy of cusps.** At this point in the proof, we have obtained sections satisfying the transversality property $\mathcal{P}_3(\gamma)$. The only missing property in order to obtain η-genericity for some $\eta>0$ is the transversality to 0 of $\mathcal{T}(s_k)$ over $R(s_k)$. The main result of this section is therefore the following :

Proposition 7. *Let δ and γ be two constants such that $0<\delta<\frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3\otimes L^k$ satisfying $\mathcal{P}_3(\gamma)$ for all k. Then there exists a constant $\eta>0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3\otimes L^k$ such that $|\sigma_k-s_k|_{C^3,g_k}\le\delta$ and that the restrictions to $R(\sigma_k)$ of the sections $\mathcal{T}(\sigma_k)$ are η-transverse to 0 over $R(\sigma_k)$. Moreover, the same statement holds for families of sections indexed by a parameter $t\in[0,1]$.*

Note that, decreasing δ if necessary in the statement of Proposition 7, it is safe to assume that all sections lying within δ of s_k in C^3 norm, and in particular the sections σ_k, satisfy $\mathcal{P}_3(\frac{\gamma}{2})$.

For technical reasons that will be clear below, we need to extend the definition of the quantity $\mathcal{T}(s_k)$ to a neighborhood of $R(s_k)$. As suggested in the introduction, this can be done by extending to a neighborhood of $R(s_k)$ the rank 1 subbundle $\mathcal{L}(s_k)$ of $f_k^*T\mathbb{CP}^2$ over which the quantity ∂f_k is projected. Recall from the introduction that $\mathcal{L}(s_k)$ has been defined over $R(s_k)$ to be the line bundle $\operatorname{Im}\partial f_k$, and denote again by $\mathcal{L}(s_k)$ its extension over a neighborhood of $R(s_k)$ as a subbundle of $f_k^*T\mathbb{CP}^2$, constructed by radial parallel transport along directions normal to $R(s_k)$. Finally define, over the same neighborhood of $R(s_k)$ and as in the introduction, $\mathcal{T}(s_k)=\pi(\partial f_k)\wedge\partial\mathrm{Jac}(f_k)$, where $\pi: f_k^*T\mathbb{CP}^2\to\mathcal{L}(s_k)$ is the orthogonal projection.

There are several ways of obtaining transversality to 0 of certain sections restricted to asymptotically holomorphic symplectic submanifolds : for example, one such technique is described in the main argument of [1]. However in our case, the perturbations we will add to s_k in order to get the transversality to 0 of $\mathcal{T}(s_k)$ have the side effect of moving the submanifolds $R(s_k)$ along which the transversality conditions have to hold, which makes things slightly more complicated. Therefore, we choose to use the equivalence between two different transversality properties :

Lemma 6. *Let σ_k and σ'_k be asymptotically holomorphic sections of vector bundles E_k and E'_k respectively over X. Assume that σ'_k is γ-transverse to 0 over X for some $\gamma > 0$, and let Σ'_k be its (smooth) zero set. Fix a constant $r > 0$ and a point $x \in X$. Then :*

(1) *There exists a constant $c > 0$, depending only on r, γ and the bounds on the sections, such that, if the restriction of σ_k to Σ'_k is η-transverse to 0 over $B_{g_k}(x,r) \cap \Sigma'_k$ for some $\eta < \gamma$, then $\sigma_k \oplus \sigma'_k$ is $c\,\eta$-transverse to 0 at x as a section of $E_k \oplus E'_k$.*

(2) *If $\sigma_k \oplus \sigma'_k$ is η-transverse to 0 at x and x belongs to Σ'_k, then the restriction of σ_k to Σ'_k is η-transverse to 0 at x.*

Proof. We start with (1), whose proof follows the ideas of §3.6 of [1] with improved estimates. Let C_1 be a constant bounding $|\nabla\sigma_k|$ everywhere, and let C_2 be a constant bounding $|\nabla\nabla\sigma_k|$ and $|\nabla\nabla\sigma'_k|$ everywhere. Fix two constants $0 < c < c' < \frac{1}{2}$, such that the following inequalities hold : $c < r$, $c < \frac{1}{2}\gamma\, C_1^{-1}$, $c' < (2+\gamma^{-1}C_1)^{-1}$, and $(2C_2\gamma^{-1}+1)c < c'$. Clearly, these constants depend only on r, γ, C_1 and C_2.

Assume that $|\sigma_k(x)|$ and $|\sigma'_k(x)|$ are both smaller than $c\,\eta$. Because of the γ-transversality to 0 of σ'_k and because $|\sigma'_k(x)| < c\,\eta < \gamma$, the covariant derivative of σ'_k is surjective at x, and admits a right inverse $(E'_k)_x \to T_xX$ of norm less than γ^{-1}. Since the connection is unitary, applying this right inverse to σ'_k itself one can follow the downward gradient flow of $|\sigma'_k|$, and since one remains in the region where $|\sigma'_k| < \gamma$ this gradient flow converges to a point y where σ'_k vanishes, at a distance d from the starting point x no larger than $\gamma^{-1}c\,\eta$. In particular, $d < c < r$, so $y \in B_{g_k}(x,r) \cap \Sigma'_k$, and therefore the restriction of σ_k to Σ'_k is η-transverse to 0 at y.

Since $c < \frac{1}{2}\gamma\, C_1^{-1}$, the norm of $\sigma_k(y)$ differs from that of $\sigma_k(x)$ by at most $C_1 d < \frac{\eta}{2}$, and so $|\sigma_k(y)| < \eta$. Since $y \in B_{g_k}(x,r)\cap\Sigma'_k$, we therefore know that $\nabla\sigma'_k$ is surjective at y and vanishes in all directions tangential to Σ'_k, while $\nabla\sigma_k$ restricted to $T_y\Sigma'_k$ is surjective and larger than η. It follows that $\nabla(\sigma_k \oplus \sigma'_k)$ is surjective at y. Let $\rho : (E_k)_y \to T_y\Sigma'_k$ and $\rho' : (E'_k)_y \to T_yX$ be the right inverses of $\nabla_y\sigma_{k|\Sigma'_k}$ and $\nabla_y\sigma'_k$ given by the transversality properties of $\sigma_{k|\Sigma'_k}$ and σ'_k. We now construct a right inverse $\hat\rho : (E_k \oplus E'_k)_y \to T_yX$ of $\nabla_y(\sigma_k \oplus \sigma'_k)$ with bounded norm.

Considering any element $u \in (E_k)_y$, the vector $\hat u = \rho(u) \in T_y\Sigma'_k$ has norm at most $\eta^{-1}|u|$ and satisfies $\nabla\sigma_k(\hat u) = u$. Clearly $\nabla\sigma'_k(\hat u) = 0$ because $\hat u$ is tangent to Σ'_k, so we define $\hat\rho(u) = \hat u$. Now consider an element v of $(E'_k)_y$, and let $\hat v = \rho'(v)$: we have $|\hat v| \le \gamma^{-1}|v|$ and $\nabla\sigma'_k(\hat v) = v$. Let $\hat w = \rho(\nabla\sigma_k(\hat v))$: then $\nabla\sigma_k(\hat w) = \nabla\sigma_k(\hat v)$ and $\nabla\sigma'_k(\hat w) = 0$, while $|\hat w| \le \eta^{-1}C_1|\hat v| \le \eta^{-1}\gamma^{-1}C_1|v|$. Therefore $\nabla(\sigma_k\oplus\sigma'_k)(\hat v - \hat w) = v$, and we define $\hat\rho(v) = \hat v - \hat w$.

Therefore $\nabla(\sigma_k \oplus \sigma'_k)$ admits at y a right inverse $\hat\rho$ of norm bounded by $\eta^{-1} + \gamma^{-1} + \eta^{-1}\gamma^{-1}C_1 \le (2+\gamma^{-1}C_1)\eta^{-1} < (c'\eta)^{-1}$. Finally, note that $\nabla_x(\sigma_k \oplus \sigma'_k)$ differs from $\nabla_y(\sigma_k \oplus \sigma'_k)$ by at most $2C_2 d < 2C_2\gamma^{-1}c\,\eta < (c'-c)\eta$. Therefore, $\nabla_x(\sigma_k \oplus \sigma'_k)$ is also surjective, and is larger than $(c'\eta) - ((c'-c)\eta) = c\,\eta$. In other terms, we have shown that $\sigma_k \oplus \sigma'_k$ is $c\,\eta$-transverse to 0 at x, which is what we sought to prove.

The proof of (2) is much easier : we know that $x \in \Sigma'_k$, i.e. $\sigma'_k(x) = 0$, and let us assume that $|\sigma_k(x)| < \eta$. Then $|\sigma_k(x) \oplus \sigma'_k(x)| = |\sigma_k(x)| < \eta$, and the η-transversality to 0 of $\sigma_k \oplus \sigma'_k$ at x implies that $\nabla_x(\sigma_k \oplus \sigma'_k)$ has a right inverse $\hat\rho$ of norm less than η^{-1}. Choose any $u \in (E_k)_x$, and let $\rho(u) = \hat\rho(u \oplus 0)$. One has $\nabla\sigma'_k(\rho(u)) = 0$, therefore $\rho(u)$ lies in $T_x\Sigma'_k$, and $\nabla\sigma_k(\rho(u)) = u$ by construction. So $(\nabla\sigma_k)_{|T_x\Sigma'_k}$ is surjective and admits ρ as a right inverse. Moreover, $|\rho(u)| =$

$|\hat\rho(u \oplus 0)| \le \eta^{-1}|u|$, so the norm of ρ is less than η^{-1}, which shows that $\sigma_{k|\Sigma'_k}$ is η-transverse to 0 at x. □

It follows from assertion (2) of Lemma 6 that, in order to obtain the transversality to 0 of $\mathcal{T}(\sigma_k)_{|R(\sigma_k)}$, it is sufficient to make $\mathcal{T}(\sigma_k) \oplus \mathrm{Jac}(\mathbb{P}\sigma_k)$ transverse to 0 over a neighborhood of $R(\sigma_k)$. Therefore, we can use once more the globalization principle of Proposition 3 to prove Proposition 7. Indeed, consider a section s of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$, a point $x \in X$ and a constant $\eta > 0$, and say that s satisfies the property $\mathcal{P}(\eta, x)$ if either x is at distance more than η of $R(s)$, or x lies close to $R(s)$ and $\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s)$ is η-transverse to 0 at x (i.e. one of the two quantities $|(\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s))(x)|$ and $|\nabla(\mathcal{T}(s) \oplus \mathrm{Jac}(\mathbb{P}s))(x)|$ is larger than η). Since $\mathrm{Jac}(\mathbb{P}s) \oplus \mathcal{T}(s)$ is, under the assumption $\mathcal{P}_3(\frac{\gamma}{2})$, a smooth function of s and its first two derivatives, and since $R(s)$ depends nicely on s, it is easy to show that the property $\mathcal{P}$ is local and C^3-open. So one only needs to check that $\mathcal{P}$ satisfies the assumptions of Proposition 3. Our next remark is :

Lemma 7. *There exists a constant $r'_0 > 0$ (independent of k) with the following property : choose $x \in X$ and $r' < r'_0$, and let s_k be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$. Assume that $\overline{B}_{g_k}(x, r')$ intersects $R(s_k)$. Then there exists an approximately holomorphic map $\theta_{k,x}$ from the disc D^+ of radius $\frac{11}{10}$ in $\mathbb{C}$ to $R(s_k)$ such that :* (i) *the image by $\theta_{k,x}$ of the unit disc D contains $B_{g_k}(x, r') \cap R(s_k)$;* (ii) *$|\nabla\theta_{k,x}|_{C^1,g_k} = O(1)$ and $|\bar\partial\theta_{k,x}|_{C^1,g_k} = O(k^{-1/2})$;* (iii) *$\theta_{k,x}(D^+)$ is contained in a ball of radius $O(r')$ centered at x.*

Moreover the same statement holds for one-parameter families of sections : given sections $(s_{t,k})_{t\in[0,1]}$ depending continuously on t, satisfying $\mathcal{P}_3(\frac{\gamma}{2})$ and such that $B_{g_k}(x, r')$ intersects $R(s_{t,k})$ for all t, there exist approximately J_t-holomorphic maps $\theta_{t,k,x}$ depending continuously on t and with the same properties as above.

Proof. We work directly with the case of one-parameter families (the result for isolated sections follows trivially) and let $j_{t,k} = \mathrm{Jac}(\mathbb{P}s_{t,k})$. First note that $R(s_{t,k})$ is the zero set of $j_{t,k}$, which is $\frac{\gamma}{2}$-transverse to 0 and has uniformly bounded second derivative. So, given any point $y \in R(s_{t,k})$, $|\nabla j_{t,k}(y)| > \frac{\gamma}{2}$, and therefore there exists $c > 0$, depending only on γ and the bound on $\nabla\nabla j_{t,k}$, such that $\nabla j_{t,k}$ varies by a factor of at most $\frac{1}{10}$ in the ball of radius c centered at y. It follows that $\overline{B}_{g_k}(y, c) \cap R(s_{t,k})$ is diffeomorphic to a ball (in other words, $R(s_{t,k})$ is "trivial at small scale").

Assume first that $3r' < c$. For all t, choose a point $y_{t,k}$ (not necessarily depending continuously on t) in $\overline{B}_{g_k}(x, r') \cap R(s_{t,k}) \neq \emptyset$. The intersection $B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k})$ is diffeomorphic to a ball and therefore connected, and contains $\overline{B}_{g_k}(x, r') \cap R(s_{t,k})$ which is nonempty and depends continuously on t. Therefore, the set $\bigcup_t \{t\} \times B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k})$ is connected, which implies the existence of points $x_{t,k} \in B_{g_k}(y_{t,k}, 3r') \cap R(s_{t,k}) \subset B_{g_k}(x, 4r') \cap R(s_{t,k})$ which depend continuously on t.

Consider local approximately J_t-holomorphic coordinate charts over a neighborhood of $x_{t,k}$, depending continuously on t, as given by Lemma 3, and call $\psi_{t,k} : (\mathbb{C}^2, 0) \to (X, x_{t,k})$ the inverse of the coordinate map. Because of asymptotic holomorphicity, the tangent space to $R(s_{t,k})$ at $x_{t,k}$ lies within $O(k^{-1/2})$ of the complex subspace $\tilde{T}_{x_{t,k}} R(s_{t,k}) = \mathrm{Ker}\ \partial j_{t,k}(x_{t,k})$ of $T_{x_{t,k}}X$. Composing $\psi_{t,k}$ with a rotation in $\mathbb{C}^2$, one can get maps $\psi'_{t,k}$ satisfying the same bounds as $\psi_{t,k}$ and such that the differential of $\psi'_{t,k}$ at 0 maps $\mathbb{C} \times \{0\}$ to $\tilde{T}_{x_{t,k}} R(s_{t,k})$.

The estimates of Lemma 3 imply that there exists a constant $\lambda = O(r')$ such that $\psi'_{t,k}(B_{\mathbb{C}^2}(0,\lambda)) \supset B_{g_k}(x,r')$. Define $\tilde{\psi}_{t,k}(z) = \psi'_{t,k}(\lambda z)$: if r' is sufficiently small, this map is well-defined over the ball $B_{\mathbb{C}^2}(0,2)$. Over $B_{\mathbb{C}^2}(0,2)$ the estimates of Lemma 3 imply that $|\bar\partial\tilde{\psi}_{t,k}|_{C^1,g_k} = O(\lambda k^{-1/2})$ and $|\nabla\tilde{\psi}_{t,k}|_{C^1,g_k} = O(\lambda)$. Moreover, because $\lambda = O(r')$ the image by $\tilde{\psi}_{t,k}$ of $B_{\mathbb{C}^2}(0,2)$ is contained in a ball of radius $O(r')$ around x.

Assuming r' to be sufficiently small, one can also require that the image of $B_{\mathbb{C}^2}(0,2)$ by $\tilde{\psi}_{t,k}$ has diameter less than c. The submanifolds $R(s_{t,k})$ are then trivial over the considered balls, so it follows from the implicit function theorem that $R(s_{t,k}) \cap \tilde{\psi}_{t,k}(D^+ \times D^+)$ can be parametrized in the chosen coordinates as the set of points of the form $\tilde{\psi}_{t,k}(z,\tau_{t,k}(z))$ for $z \in D^+$, where $\tau_{t,k} : D^+ \to D^+$ satisfies $\tau_{t,k}(0) = 0$ and $\nabla\tau_{t,k}(0) = O(k^{-1/2})$.

The derivatives of $\tau_{t,k}$ can be easily computed, since they are characterized by the equation $j_{t,k}(\tilde{\psi}_{t,k}(z,\tau_{t,k}(z))) = 0$. Notice that, if r' is small enough, it follows from the transversality to 0 of $j_{t,k}$ that $|\nabla j_{t,k} \circ d\tilde{\psi}_{t,k}(v)|$ is larger than a constant times $\lambda|v|$ for all $v \in \{0\} \times \mathbb{C}$ and at any point of $D^+ \times D^+$. Combining this estimate with the bounds on the derivatives of $j_{t,k}$ given by asymptotic holomorphicity and the above bounds on the derivatives of $\tilde{\psi}_{t,k}$, one gets that $|\nabla\tau_{t,k}|_{C^1} = O(1)$ and $|\bar\partial\tau_{t,k}|_{C^1} = O(k^{-1/2})$ over D^+.

One then defines $\theta_{t,k}(z) = \tilde{\psi}_{t,k}(z,\tau_{t,k}(z))$ over D^+, which satisfies all the required properties : the image $\theta_{t,k}(D^+)$ is contained in $R(s_{t,k})$ and in a ball of radius $O(r')$ centered at x ; $\theta_{t,k}(D)$ contains the intersection of $R(s_{t,k})$ with $\tilde{\psi}_{t,k}(D \times D^+) \supset \psi'_{t,k}(B_{\mathbb{C}^2}(0,\lambda)) \supset B_{g_k}(x,r')$; and the required bounds on derivatives follow directly from those on derivatives of $\tau_{t,k}$ and $\tilde{\psi}_{t,k}$. Therefore, Lemma 7 is proved under the assumption that r' is small enough. We set r'_0 in the statement of the lemma to be the bound on r' which ensures that all the assumptions we have made on r' are satisfied. □

We now prove that the assumptions of Proposition 3 hold for property $\mathcal{P}$ in the case of single sections s_k (the case of one-parameter families is discussed later). Let $x \in X$, $0 < \delta < \frac{\gamma}{4}$, and consider asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$ satisfying $\mathcal{P}_3(\frac{\gamma}{2})$ and the corresponding maps $f_k = \mathbb{P}s_k$. We have to show that, for large enough k, a perturbation of s_k with Gaussian decay and smaller than δ in C^3 norm can make property $\mathcal{P}$ hold over a ball centered at x. Because of assertion (1) of Lemma 6, it is actually sufficient to show that there exist constants c, c' and p independent of k and δ such that, if x lies within distance c of $R(s_k)$, then s_k can be perturbed to make the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$ η-transverse to 0 over the intersection of $R(s_k)$ with a ball $B_{g_k}(x,c)$, where $\eta = c'\delta\,(\log\delta^{-1})^{-p}$. Such a result is then sufficient to imply the transversality to 0 of $\mathcal{T}(s_k) \oplus \mathrm{Jac}(f_k)$ over the ball $B_{g_k}(x,\frac{c}{2})$, with a transversality constant decreased by a bounded factor.

As in previous sections, composing with a rotation in $\mathbb{C}^3$ (constant over X), one can assume that $s_k(x)$ is directed along the first component in $\mathbb{C}^3$, i.e. that $s^1_k(x) = s^2_k(x) = 0$ and therefore $|s^0_k(x)| \geq \frac{\gamma}{2}$. Because of the uniform bound on $|\nabla s_k|$, there exists $r > 0$ (independent of k) such that $|s^0_k| \geq \frac{\gamma}{3}$, $|s^1_k| < \frac{\gamma}{3}$ and

$|s_k^2| < \frac{\gamma}{3}$ over the ball $B_{g_k}(x,r)$. Therefore, over this ball one can define the map

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

The map f_k is the composition of h_k with the map $\iota : (z_1, z_2) \mapsto [1 : z_1 : z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$, which is a quasi-isometry over the unit ball in $\mathbb{C}^2$. Therefore, at any point $y \in B_{g_k}(x,r)$, the bound $|\partial f_k(y)| \geq \frac{\gamma}{2}$ implies that $|\partial h_k(y)| \geq \gamma'$ for some constant $\gamma' > 0$. Moreover, one has $\mathrm{Jac}(f_k) = \phi\,\mathrm{Jac}(h_k)$, where $\phi(y)$ is the Jacobian of ι at $h_k(y)$. In particular, $\mathrm{Jac}(h_k)$ vanishes at exactly the same points of $B_{g_k}(x,r)$ as $\mathrm{Jac}(f_k)$. Since $|\phi|$ is bounded between two universal constants over $B_{g_k}(x,r)$ and $\nabla\phi$ is bounded too, it follows from the $\frac{\gamma}{2}$-transversality to 0 of $\mathrm{Jac}(f_k)$ that, decreasing γ' if necessary, $\mathrm{Jac}(h_k)$ is γ'-transverse to 0 over $B_{g_k}(x,r)$.

Since $|\partial h_k(x)| \geq \gamma'$, after composing with a rotation in $\mathbb{C}^2$ (constant over X) acting on the two components (s_k^1, s_k^2) one can assume that $|\partial h_k^2(x)| \geq \frac{\gamma'}{2}$. Since $\nabla\nabla h_k$ is uniformly bounded, decreasing r if necessary one can ensure that $|\partial h_k^2|$ remains larger than $\frac{\gamma'}{4}$ at every point of $B_{g_k}(x,r)$.

Let us now show that, over $\hat{R}_x(s_k) = B_{g_k}(x,r) \cap R(s_k)$, the transversality to 0 of $\mathcal{T}(s_k)$ follows from that of $\hat{\mathcal{T}}(s_k) = \partial h_k^2 \wedge \partial\mathrm{Jac}(h_k)$.

It follows from the identity $\mathrm{Jac}(f_k) = \phi\,\mathrm{Jac}(h_k)$ and the vanishing of $\mathrm{Jac}(h_k)$ over $\hat{R}_x(s_k)$ that $\partial\mathrm{Jac}(f_k) = \phi\,\partial\mathrm{Jac}(h_k)$ over $\hat{R}_x(s_k)$. Moreover the two $(1,0)$-forms ∂f_k and ∂h_k have complex rank one at any point of $\hat{R}_x(s_k)$ and are related by $\partial f_k = d\iota(\partial h_k)$, so they have the same kernel (in some sense they are "colinear"). Because $|\partial h_k^2|$ is bounded from below over $B_{g_k}(x,r)$, the ratio between $|\partial h_k|$ and $|\partial h_k^2|$ is bounded. Because the line bundle $\mathcal{L}(s_k)$ on which one projects ∂f_k coincides with $\mathrm{Im}\,\partial f_k$ over $R(s_k)$, we have $|\pi(\partial f_k)| = |\partial f_k|$ over $R(s_k)$. Since ι is a quasi-isometry over the unit ball, it follows that the ratio between $|\pi(\partial f_k)|$ and $|\partial h_k^2|$ is bounded from above and below over $\hat{R}_x(s_k)$. Moreover, the two 1-forms $\pi(\partial f_k)$ and ∂h_k^2 have same kernel, so one can write $\pi(\partial f_k) = \psi\,\partial h_k^2$ over $\hat{R}_x(s_k)$, with ψ bounded from above and below. Because of the uniform bounds on derivatives of s_k and therefore f_k and h_k, it is easy to check that the derivatives of ψ are bounded.

So $\mathcal{T}(s_k) = \phi\psi\,\hat{\mathcal{T}}(s_k)$ over $\hat{R}_x(s_k)$. Therefore, assume that $\hat{\mathcal{T}}(s_k)_{|R(s_k)}$ is η-transverse to 0 at a given point $y \in \hat{R}_x(s_k)$, and let $C > 1$ be a constant such that $\frac{1}{C} < |\phi\psi| < C$ and $|\nabla(\phi\psi)| < C$ over $\hat{R}_x(s_k)$. If $|\mathcal{T}(s_k)(y)| < \frac{\eta}{2C^3}$, then $|\hat{\mathcal{T}}(s_k)(y)| < \frac{\eta}{2C^2} < \eta$, and therefore $|\partial(\hat{\mathcal{T}}(s_k))(y)| > \eta$, so at y one has $|\partial(\mathcal{T}(s_k))| \geq |\phi\psi\,\partial(\hat{\mathcal{T}}(s_k))| - |\hat{\mathcal{T}}(s_k)\partial(\phi\psi)| > \frac{1}{C}\eta - \frac{\eta}{2C^2}C = \frac{\eta}{2C} > \frac{\eta}{2C^3}$. In other terms, the restriction to $R(s_k)$ of $\mathcal{T}(s_k)$ is $\frac{\eta}{2C^3}$-transverse to 0 at y.

Therefore, we only need to show that there exists a constant $c > 0$ such that, if $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, then by perturbing s_k it is possible to ensure that $\hat{\mathcal{T}}(s_k)_{|R(s_k)}$ is transverse to 0 over $B_{g_k}(x,c) \cap R(s_k)$.

By Lemma 7, given any sufficiently small constant $c > 0$ and assuming that $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$, there exists an approximately holomorphic map $\theta_k : D^+ \to R(s_k)$ such that $\theta_k(D)$ contains $B_{g_k}(x,c) \cap R(s_k)$ and satisfying bounds $|\nabla\theta_k|_{C^1,g_k} = O(1)$ and $|\bar\partial\theta_k|_{C^1,g_k} = O(k^{-1/2})$. We call $\bar{c} = O(c)$ the size of the ball such that $\theta_k(D^+) \subset B_{g_k}(x,\bar{c})$, and assume that c is small enough to have $\bar{c} < r$.

From now on, we assume that $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$.

Let $s_{k,x}^{\mathrm{ref}}$ be the asymptotically holomorphic sections of L^k with Gaussian decay away from x given by Lemma 2, and let z_k^1 and z_k^2 be the complex coordinate functions of a local approximately holomorphic Darboux coordinate chart on a neighborhood of x. There exist two complex numbers a and b such that $\partial h_k^2(x) = a\,\partial z_k^1(x) + b\,\partial z_k^2(x)$. Composing the coordinate chart (z_k^1, z_k^2) with the rotation

$$\frac{1}{|a|^2+|b^2|}\begin{pmatrix}\bar{b} & -\bar{a}\\ a & b\end{pmatrix},$$

we can actually write $\partial h_k^2(x) = \lambda\,\partial z_k^2(x)$, with $|\lambda|$ bounded from below independently of k and x. We now define $Q_{k,x} = \big(0, (z_k^1)^2 s_{k,x}^{\mathrm{ref}}, 0\big)$ and study the behavior of $\hat{\mathcal{T}}(s_k + wQ_{k,x})$ for small $w \in \mathbb{C}$.

First we look at how adding $wQ_{k,x}$ to s_k affects the submanifold $R(s_k)$: for small enough w, $R(s_k + wQ_{k,x})$ is a small deformation of $R(s_k)$ and can therefore be seen as a section of $TX_{|R(s_k)}$. Because the derivative of $\mathrm{Jac}(h_k)$ is uniformly bounded and $B_{g_k}(x,c) \cap R(s_k)$ is not empty, if c is small enough then $|\mathrm{Jac}(h_k)|$ remains less than γ' over $B_{g_k}(x,\bar{c})$. Recall that $\mathrm{Jac}(h_k)$ is γ'-transverse to 0 over $B_{g_k}(x,r)$: therefore, at every point $y \in B_{g_k}(x,\bar{c})$, $\nabla\mathrm{Jac}(h_k)$ admits a right inverse $\rho : \Lambda^{2,0}T_y^*X \to T_yX$ of norm less than $\frac{1}{\gamma'}$. Adding $wQ_{k,x}$ to s_k increases $\mathrm{Jac}(h_k)$ by $w\Delta_{k,x}$, where

$$\Delta_{k,x} = \partial\Big(\frac{(z_k^1)^2 s_{k,x}^{\mathrm{ref}}}{s_k^0}\Big) \wedge \partial h_k^2.$$

Therefore, $R(s_k + wQ_{k,x})$ is obtained by shifting $R(s_k)$ by an amount equal to $-\rho(w\Delta_{k,x}) + O(|w\Delta_{k,x}|^2)$. It follows immediately that the value of $\hat{\mathcal{T}}(s_k + wQ_{k,x})$ at a point of $R(s_k + wQ_{k,x})$ differs from the value of $\hat{\mathcal{T}}(s_k)$ at the corresponding point of $R(s_k)$ by an amount

$$\Theta_{k,x}(w) = w\,\partial h_k^2 \wedge \partial\Delta_{k,x} - \nabla(\hat{\mathcal{T}}(s_k)).\rho(w\Delta_{k,x}) + O(w^2).$$

Our aim is therefore to show that, if c is small enough, for a suitable value of w the quantity $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w)$ is transverse to 0 over $R(s_k) \cap B_{g_k}(x,c)$.

Notice that the quantities $\hat{\mathcal{T}}(s_k)$ and $\mathrm{Jac}(h_k)$ are asymptotically holomorphic, so that $\nabla(\hat{\mathcal{T}}(s_k))$ and ρ are approximately complex linear. Therefore,

$$\nabla(\hat{\mathcal{T}}(s_k)).\rho(w\Delta_{k,x}) = w\nabla(\hat{\mathcal{T}}(s_k)).\rho(\Delta_{k,x}) + O(k^{-1/2}).$$

It follows that $\Theta_{k,x}(w) = w\Theta_{k,x}^0 + O(w^2) + O(k^{-1/2})$, where

$$\Theta_{k,x}^0 = \partial h_k^2 \wedge \partial\Delta_{k,x} - \nabla(\hat{\mathcal{T}}(s_k)).\rho(\Delta_{k,x}).$$

We start by computing the value of $\Theta_{k,x}^0$ at x, using the fact that $\partial h_k^2(x) = \lambda\,\partial z_k^2(x)$ while $z_k^1(x) = 0$ and therefore $\Delta_{k,x}(x) = 0$. Because of the identity $\Delta_{k,x} = \frac{s_{k,x}^{\mathrm{ref}}}{s_k^0}\,2z_k^1\partial z_k^1 \wedge \partial h_k^2 + O(|z_k^1|^2)$, an easy calculation yields that

$$\partial\Delta_{k,x} = 2\frac{s_{k,x}^{\mathrm{ref}}}{s_k^0}(\partial z_k^1 \wedge \partial h_k^2)\,\partial z_k^1 + O(|z_k^1|)$$

and therefore

$$\Theta_{k,x}^0(x) = -2\lambda^2\,\frac{s_{k,x}^{\mathrm{ref}}(x)}{s_k^0(x)}\big(\partial z_k^1(x) \wedge \partial z_k^2(x)\big)^2.$$

The important point is that there exists a constant $\gamma'' > 0$ independent of k and x such that $|\Theta^0_{k,x}(x)| \ge \gamma''$.

Since the derivatives of $\Theta^0_{k,x}$ are uniformly bounded, $|\Theta^0_{k,x}|$ remains larger than $\frac{\gamma''}{2}$ at every point of $B_{g_k}(x, \bar{c})$ if c is small enough. It follows that, over $R(s_k) \cap B_{g_k}(x, c)$, the transversality to 0 of $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w)$ is equivalent to that of $(\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w))/\Theta^0_{k,x}$. The value of c we finally choose to use in Lemma 7 for the construction of θ_k is one small enough to ensure that all the above statements hold (but still independent of k, x and δ). Now define, over the disc $D^+ \subset \mathbb{C}$, the function

$$v_k(z) = \frac{\hat{\mathcal{T}}(s_k)(\theta_k(z))}{\Theta^0_{k,x}(\theta_k(z))}$$

with values in $\mathbb{C}$. Because $\Theta^0_{k,x}$ is bounded from below over $B_{g_k}(x, \bar{c})$ and because of the bounds on the derivatives of θ_k given by Lemma 7, the functions $v_k : D^+ \to \mathbb{C}$ satisfy the hypotheses of Proposition 6 for all large enough k. Therefore, if C_0 is a constant larger than $|Q_{k,x}|_{C^3, g_k}$, and if k is large enough, there exists $w_k \in \mathbb{C}$, with $|w_k| \le \frac{\delta}{C_0}$, such that $v_k + w_k$ is α-transverse to 0 over the unit disc D in $\mathbb{C}$, where $\alpha = \frac{\delta}{C_0} \log((\frac{\delta}{C_0})^{-1})^{-p}$.

Multiplying again by $\Theta^0_{k,x}$ and recalling that θ_k maps diffeomorphically D to a subset of $R(s_k)$ containing $R(s_k) \cap B_{g_k}(x, c)$, we get that the restriction to $R(s_k)$ of $\hat{\mathcal{T}}(s_k) + w_k \Theta^0_{k,x}$ is α'-transverse to 0 over $R(s_k) \cap B_{g_k}(x, c)$ for some α' differing from α by at most a constant factor. Recall that $\Theta_{k,x}(w_k) = w_k \Theta^0_{k,x} + O(|w_k|^2) + O(k^{-1/2})$, and note that $|w_k|^2$ is at most of the order of δ^2, while α' is of the order of $\delta \log(\delta^{-1})^{-p}$: so, if δ is small enough, one can assume that $|w_k|^2$ is much smaller than α'. If k is large enough, $k^{-1/2}$ is also much smaller than α', so that $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w_k)$ differs from $\hat{\mathcal{T}}(s_k) + w_k \Theta^0_{k,x}$ by less than $\frac{\alpha'}{2}$, and is therefore $\frac{\alpha'}{2}$-transverse to 0 over $R(s_k) \cap B_{g_k}(x, c)$.

Next, recall that $R(s_k + w_k Q_{k,x})$ is obtained by shifting $R(s_k)$ by an amount $-\rho(w_k \Delta_{k,x}) + O(|w_k \Delta_{k,x}|^2) = O(|w_k|)$ (because $|\Delta_{k,x}|$ is uniformly bounded, or more generally because the perturbation of s_k is $O(|w_k|)$ in C^3 norm). So, if δ is small enough, one can safely assume that the distance by which one shifts the points of $R(s_k)$ is less than $\frac{c}{2}$. Therefore, given any point in $R(s_k + w_k Q_{k,x}) \cap B_{g_k}(x, \frac{c}{2})$, the corresponding point in $R(s_k)$ belongs to $B_{g_k}(x, c)$.

We have seen above that the value of $\hat{\mathcal{T}}(s_k + w_k Q_{k,x})$ at a point of $R(s_k + w_k Q_{k,x})$ differs from the value of $\hat{\mathcal{T}}(s_k)$ at the corresponding point of $R(s_k)$ by $\Theta_{k,x}(w_k)$; therefore it follows from the transversality properties of $\hat{\mathcal{T}}(s_k) + \Theta_{k,x}(w_k)$ that the restriction to $R(s_k + w_k Q_{k,x})$ of $\hat{\mathcal{T}}(s_k + w_k Q_{k,x})$ is α''-transverse to 0 over $R(s_k + w_k Q_{k,x}) \cap B_{g_k}(x, \frac{c}{2})$ for some $\alpha'' > 0$ differing from α' by at most a constant factor.

By the remarks above, this transversality property implies transversality to 0 of the restriction of $\mathcal{T}(s_k + w_k Q_{k,x})$ over $R(s_k + w_k Q_{k,x}) \cap B_{g_k}(x, \frac{c}{2})$; therefore, by Lemma 6, $\mathcal{T}(s_k + w_k Q_{k,x}) \oplus \mathrm{Jac}(\mathbb{P}(s_k + w_k Q_{k,x}))$ is η-transverse to 0 over $B_{g_k}(x, \frac{c}{4})$, with a transversality constant η differing from α'' by at most a constant factor. So, if δ is small enough and k large enough, in the case where $B_{g_k}(x, c) \cap R(s_k) \neq \emptyset$, we have constructed w_k such that $s_k + w_k Q_{k,x}$ satisfies the required property $\mathcal{P}(\eta, y)$ at every point $y \in B_{g_k}(x, \frac{c}{4})$. By construction, $|w_k Q_{k,x}|_{C^3, g_k} \le \delta$, the asymptotically holomorphic sections $Q_{k,x}$ have uniform Gaussian decay away from x, and η is larger than $c'\delta\log(\delta^{-1})^{-p}$ for some constant $c' > 0$, so all required properties hold in this case.

Moreover, in the case where $B_{g_k}(x,c)$ does not intersect $R(s_k)$, the section s_k already satisfies the property $\mathcal{P}(\frac{3}{4}c, y)$ at every point y of $B_{g_k}(x, \frac{c}{4})$ and no perturbation is necessary. Therefore, the property $\mathcal{P}$ under consideration satisfies the hypotheses of Proposition 3 whether $B_{g_k}(x,c)$ intersects $R(s_k)$ or not. This ends the proof of Proposition 7 for isolated sections s_k.

In the case of one-parameter families of sections, the argument still works similarly : we are now given sections $s_{t,k}$ depending continuously on a parameter $t \in [0,1]$, and try to perform the same construction as above for each value of t, in such a way that everything depends continuously on t. As previously, we have to show that one can perturb $s_{t,k}$ in order to ensure that, for all t such that x lies in a neighborhood of $R(s_{t,k})$, $\mathcal{T}(s_{t,k})_{|R(s_{t,k})}$ is transverse to 0 over the intersection of $R(s_{t,k})$ with a ball centered at x.

As before, a continuous family of rotations of $\mathbb{C}^3$ can be used to ensure that $s^1_{t,k}(x)$ and $s^2_{t,k}(x)$ vanish for all t, allowing one to define $h_{t,k}$ for all t. Moreover the argument at the end of §3.1 proves the existence of a continuous one-parameter family of rotations of $\mathbb{C}^2$ acting on the two components $(s^1_{t,k}, s^2_{t,k})$ allowing one to assume that $|\partial h^2_{t,k}(x)| \geq \frac{\gamma'}{2}$ for all t. Therefore, as in the case of isolated sections, the problem is reduced to that of perturbing $s_{t,k}$ when x lies in a neighborhood of $R(s_{t,k})$ in order to obtain the transversality to 0 of $\hat{\mathcal{T}}(s_{t,k})_{|R(s_{t,k})}$ over the intersection of $R(s_{t,k})$ with a ball centered at x.

Because Lemma 7 and Proposition 6 also apply in the case of 1-parameter families of sections, the argument used above to obtain the expected transversality result for isolated sections also works here for all t such that x lies in the neighborhood of $R(s_{t,k})$. However, the ball $B_{g_k}(x,c)$ intersects $R(s_{t,k})$ only for certain values of $t \in [0,1]$, which makes it necessary to work more carefully.

Define $\Omega_k \subset [0,1]$ as the set of all t for which $B_{g_k}(x,c) \cap R(s_{t,k}) \neq \emptyset$. For all large enough k and for all $t \in \Omega_k$, Lemma 7 allows one to define maps $\theta_{t,k} : D^+ \to R(s_{t,k})$ depending continuously on t and with the same properties as in the case of isolated sections. Using local coordinates $z^i_{t,k}$ depending continuously on t given by Lemma 3 and sections $s^{\text{ref}}_{t,k,x}$ given by Lemma 2, the quantities $Q_{t,k,x}$, $\Delta_{t,k,x}$, $\Theta_{t,k,x}(w)$, $\Theta^0_{t,k,x}$ and $v_{t,k}$ can be defined for all $t \in \Omega_k$ by the same formulae as above and depend continuously on t.

Proposition 6 then gives, for all large k and for all $t \in \Omega_k$, complex numbers $w_{t,k}$ of norm at most $\frac{\delta}{C_0}$ and depending continuously on t, such that the functions $v_{t,k} + w_{t,k}$ are transverse to 0 over D. As in the case of isolated sections, this implies that $s_{t,k} + w_{t,k}Q_{t,k,x}$ satisfies the required transversality property over $B_{g_k}(x, \frac{c}{4})$.

Our problem is to define asymptotically holomorphic sections $\tau_{t,k,x}$ of $\mathbb{C}^3 \otimes L^k$ *for all values of* $t \in [0,1]$, of C^3-norm less than δ and with Gaussian decay away from x, in such a way that the sections $s_{t,k} + \tau_{t,k,x}$ depend continuously on $t \in [0,1]$ and satisfy the property $\mathcal{P}$ over $B_{g_k}(x, \frac{c}{4})$ for all t. For this, let $\beta : \mathbb{R}_+ \to [0,1]$ be a continuous cut-off function equal to 1 over $[0, \frac{3c}{4}]$ and to 0 over $[c, +\infty)$. Define, for all $t \in \Omega_k$,

$$\tau_{t,k,x} = \beta\big(dist_{g_k}(x, R(s_{t,k}))\big)\, w_{t,k} Q_{t,k,x},$$

and $\tau_{t,k,x} = 0$ for all $t \notin \Omega_k$. It is clear that, for all $t \in [0,1]$, the sections $\tau_{t,k,x}$ are asymptotically holomorphic, have Gaussian decay away from x, depend continuously on t and are smaller than δ in C^3 norm. Moreover, for all t such that $dist_{g_k}(x, R(s_{t,k})) \leq \frac{3c}{4}$, one has $\tau_{t,k,x} = w_{t,k} Q_{t,k,x}$, so the sections $s_{t,k} + \tau_{t,k,x}$ satisfy property $\mathcal{P}$ over $B_{g_k}(x, \frac{c}{4})$ for all such values of t.

For the remaining values of t, namely those such that x is at distance more than $\frac{3c}{4}$ from $R(s_{t,k})$, the argument is the following : since the perturbation $\tau_{t,k,x}$ is smaller than δ, every point of $R(s_{t,k} + \tau_{t,k,x})$ lies within distance $O(\delta)$ of $R(s_{t,k})$. Therefore, decreasing the maximum allowable value of δ in Proposition 3 if necessary, one can safely assume that this distance is less than $\frac{c}{4}$. It follows that x is at distance more than $\frac{c}{2}$ of $R(s_{t,k} + \tau_{t,k,x})$, and so that the property $\mathcal{P}(\frac{c}{4}, y)$ holds at every point $y \in B_{g_k}(x, \frac{c}{4})$.

Therefore, for all large enough k and for all $t \in [0,1]$, the perturbed sections $s_{t,k} + \tau_{t,k,x}$ satisfy property $\mathcal{P}$ over the ball $B_{g_k}(x, \frac{c}{4})$. It follows that the assumptions of Proposition 3 also hold for $\mathcal{P}$ in the case of one-parameter families, and so Proposition 7 is proved.

4. Dealing with the antiholomorphic part

4.1. **Holomorphicity in the neighborhood of cusp points.** At this point in the proof, we have constructed asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ satisfying all the required transversality properties. We now need to show that, by further perturbation, one can obtain $\bar\partial$-tameness. We first handle the case of cusp points :

Proposition 8. *Let $(s_k)_{k\gg 0}$ be γ-generic asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Then there exist constants $(C_p)_{p\in\mathbb{N}}$ and $c > 0$ such that, for all large k, there exist ω-compatible almost-complex structures $\tilde J_k$ on X and asymptotically J-holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ with the following properties : at any point whose g_k-distance to $\mathcal{C}_{\tilde J_k}(\sigma_k)$ is less than c, the almost-complex structure $\tilde J_k$ is integrable and the map $\mathbb{P}\sigma_k$ is $\tilde J_k$-holomorphic ; and for all $p \in \mathbb{N}$, $|\tilde J_k - J|_{C^p, g_k} \leq C_p k^{-1/2}$ and $|\sigma_k - s_k|_{C^p, g_k} \leq C_p k^{-1/2}$.*

Furthermore, the result also applies to one-parameter families of γ-generic asymptotically J_t-holomorphic sections $(s_{t,k})_{t\in[0,1],k\gg 0}$: for all large k there exist almost-complex structures $\tilde J_{t,k}$ and asymptotically J_t-holomorphic sections $\sigma_{t,k}$ depending continuously on t and such that the above properties hold for all values of t. Moreover, if $s_{0,k}$ and $s_{1,k}$ already satisfy the required properties, and if one assumes that, for some $\epsilon > 0$, J_t and $s_{t,k}$ are respectively equal to J_0 and $s_{0,k}$ for all $t \in [0, \epsilon]$ and to J_1 and $s_{1,k}$ for all $t \in [1-\epsilon, 1]$, then it is possible to ensure that $\sigma_{0,k} = s_{0,k}$ and $\sigma_{1,k} = s_{1,k}$.

The proof of this result relies on the following analysis lemma, which states that any approximately holomorphic complex-valued function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^2$ can be approximated over the interior ball B of unit radius by a holomorphic function :

Lemma 8. *There exist an operator $P : C^\infty(B^+, \mathbb{C}) \to C^\infty(B, \mathbb{C})$ and constants $(K_p)_{p\in\mathbb{N}}$ such that, given any function $f \in C^\infty(B^+, \mathbb{C})$, the function $\tilde f = P(f)$ is holomorphic over the unit ball B and satisfies $|f - \tilde f|_{C^p(B)} \leq K_p\, |\bar\partial f|_{C^p(B^+)}$ for every $p \in \mathbb{N}$.*

Proof. (see also [2]). This is a standard fact which can be proved e.g. using the Hörmander theory of weighted L^2 spaces. Using a suitable weighted L^2 norm on B^+ which compares uniformly with the standard norm on the interior ball B' of radius $1+\frac{1}{20}$ ($B \subset B' \subset B^+$), one obtains a bounded solution to the Cauchy-Riemann equation : for any $\bar\partial$-closed $(0,1)$-form ρ on B^+ there exists a function $T(\rho)$ such that $\bar\partial T(\rho)=\rho$ and $|T(\rho)|_{L^2(B')} \le C|\rho|_{L^2(B^+)}$ for some constant C.

Take $\rho = \bar\partial f$ and let $h = T(\rho)$: since $\bar\partial h = \rho = \bar\partial f$, the function $\tilde f = f - h$ is holomorphic (in other words, we set $P = \mathrm{Id} - T\bar\partial$). Moreover the L^2 norm of h and the C^p norm of $\bar\partial h = \bar\partial f$ over B' are bounded by multiples of $|\bar\partial f|_{C^p(B^+)}$; therefore, by standard elliptic theory, the same is true for the C^p norm of h over the interior ball B, which gives the desired result. □

We first prove Proposition 8 in the case when there is no parameter, where the argument is fairly easy. Because s_k is γ-generic, the set of points of $R(s_k)$ where $\mathcal{T}(s_k)$ vanishes, i.e. $\mathcal{C}_J(s_k)$, is finite. Moreover $\nabla\mathcal{T}(s_k)_{|R(s_k)}$ is larger than γ at all cusp points and $\nabla\nabla\mathcal{T}(s_k)$ is uniformly bounded, so there exists a constant $r>0$ such that the g_k-distance between any two points of $\mathcal{C}_J(s_k)$ is larger than $4r$.

Let x be a point of $\mathcal{C}_J(s_k)$, and consider a local approximately J-holomorphic Darboux map $\psi_k : (\mathbb{C}^2,0) \to (X,x)$ as given by Lemma 3. Because of the bounds on $\bar\partial\psi_k$, the ω-compatible almost-complex structure J'_k on the ball $B_{g_k}(x,2r)$ defined by pulling back the standard complex structure of $\mathbb{C}^2$ satisfies bounds of the type $|J'_k - J|_{C^p,g_k} = O(k^{-1/2})$ over $B_{g_k}(x,2r)$ for all $p\in\mathbb{N}$.

Recall that the set of ω-skew-symmetric endomorphisms of square -1 of the tangent bundle TX (i.e. ω-compatible almost-complex structures) is a subbundle of $\mathrm{End}(TX)$ whose fibers are contractible. Therefore, there exists a one-parameter family $(J_k^\tau)_{\tau\in[0,1]}$ of ω-compatible almost-complex structures over $B_{g_k}(x,2r)$ depending smoothly on τ and such that $J_k^0 = J$ and $J_k^1 = J'_k$. Also, let $\tau_x : B_{g_k}(x,2r) \to [0,1]$ be a smooth cut-off function with bounded derivatives such that $\tau_x = 1$ over $B_{g_k}(x,r)$ and $\tau_x = 0$ outside of $B_{g_k}(x,\frac{3}{2}r)$.

Then, define $\tilde J_k$ to be the almost-complex structure which equals J outside of the $2r$-neighborhood of $\mathcal{C}_J(s_k)$, and which at any point y of a ball $B_{g_k}(x,2r)$ centered at $x\in\mathcal{C}_J(s_k)$ coincides with $J_k^{\tau_x(y)}$: it is quite easy to check that $\tilde J_k$ is integrable over the r-neighborhood of $\mathcal{C}_J(s_k)$ where it coincides with J'_k, and satisfies bounds of the type $|\tilde J_k - J|_{C^p,g_k} = O(k^{-1/2})$ $\forall p\in\mathbb{N}$.

Let us now return to a neighborhood of $x\in\mathcal{C}_J(s_k)$, where we need to perturb s_k to make the corresponding projective map locally $\tilde J_k$-holomorphic. First notice that, by composing with a rotation of $\mathbb{C}^3$ (constant over X), one can safely assume that $s_k^1(x) = s_k^2(x) = 0$. Therefore, $|s_k^0(x)| \ge \gamma$, and decreasing r if necessary one can assume that $|s_k^0|$ remains larger than $\frac{\gamma}{2}$ at every point of $B_{g_k}(x,r)$. The $\tilde J_k$-holomorphicity of $\mathbb{P}s_k$ over a neighborhood of x is then equivalent to that of the map h_k with values in $\mathbb{C}^2$ defined by

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

Because of the properties of the map ψ_k given by Lemma 3, there exist constants $\lambda>0$ and $r'>0$, independent of k, such that $\psi_k(B_{\mathbb{C}^2}(0,\frac{11}{10}\lambda))$ is contained in

$B_{g_k}(x,r)$ while $\psi_k(B_{\mathbb{C}^2}(0,\frac{1}{2}\lambda))$ contains $B_{g_k}(x,r')$. We now define the two complex-valued functions $f_k^1(z) = h_k^1(\psi_k(\lambda z))$ and $f_k^2(z) = h_k^2(\psi_k(\lambda z))$ over the ball $B^+ \subset \mathbb{C}^2$. By definition of $\tilde{J}_k$, the map ψ_k intertwines the almost-complex structure $\tilde{J}_k$ over $B_{g_k}(x,r)$ and the standard complex structure of $\mathbb{C}^2$, so our goal is to make the functions f_k^1 and f_k^2 holomorphic in the usual sense over a ball in $\mathbb{C}^2$.

This is where we use Lemma 8. Remark that, because of the estimates on $\bar{\partial}_J\psi_k$ given by Lemma 3 and those on $\bar{\partial}_J h_k$ coming from asymptotic holomorphicity, we have $|\bar{\partial} f_k^i|_{C^p(B^+)} = O(k^{-1/2})$ for every $p \in \mathbb{N}$ and $i \in \{1,2\}$. Therefore, by Lemma 8 there exist two holomorphic functions $\tilde{f}_k^1$ and $\tilde{f}_k^2$, defined over the unit ball $B \subset \mathbb{C}^2$, such that $|f_k^i - \tilde{f}_k^i|_{C^p(B)} = O(k^{-1/2})$ for every $p \in \mathbb{N}$ and $i \in \{1,2\}$.

Let $\beta : [0,1] \to [0,1]$ be a smooth cut-off function such that $\beta = 1$ over $[0,\frac{1}{2}]$ and $\beta = 0$ over $[\frac{3}{4},1]$, and define, for all $z \in B$ and $i \in \{1,2\}$, $\hat{f}_k^i(z) = \beta(|z|)\tilde{f}_k^i(z) + (1-\beta(|z|))f_k^i(z)$. By construction, the functions $\hat{f}_k^i$ are holomorphic over the ball of radius $\frac{1}{2}$ and differ from f_k^i by $O(k^{-1/2})$.

Going back through the coordinate map, let $\hat{h}_k^i$ be the functions on the neighborhood $U_x = \psi_k(B_{\mathbb{C}^2}(0,\lambda))$ of x which satisfy $\hat{h}_k^i(\psi_k(\lambda z)) = \hat{f}_k^i(z)$ for every $z \in B$. Define $\hat{s}_k^0 = s_k^0$, $\hat{s}_k^1 = \hat{h}_k^1 s_k^0$ and $\hat{s}_k^2 = \hat{h}_k^2 s_k^0$ over U_x, and let σ_k be the global section of $\mathbb{C}^3 \otimes L^k$ which $\forall x \in \mathcal{C}_J(s_k)$ equals $\hat{s}_k$ over U_x and which coincides with s_k away from $\mathcal{C}_J(s_k)$.

Because $\hat{f}_k^i = f_k^i$ near the boundary of B, $\hat{s}_k$ coincides with s_k near the boundary of U_x, and σ_k is therefore a smooth section of $\mathbb{C}^3 \otimes L^k$. For every $p \in \mathbb{N}$, it follows from the bound $|\hat{f}_k^i - f_k^i|_{C^p(B)} = O(k^{-1/2})$ that $|\sigma_k - s_k|_{C^p,g_k} = O(k^{-1/2})$. Moreover, the functions $\hat{f}_k^i$ are holomorphic over $B_{\mathbb{C}^2}(0,\frac{1}{2})$ where they coincide with $\tilde{f}_k^i$, so the functions $\hat{h}_k^i$ are $\tilde{J}_k$-holomorphic over $\psi_k(B_{\mathbb{C}^2}(0,\frac{1}{2}\lambda)) \supset B_{g_k}(x,r')$, and it follows that $\mathbb{P}\sigma_k$ is $\tilde{J}_k$-holomorphic over $B_{g_k}(x,r')$.

Therefore, the almost-complex structures $\tilde{J}_k$ and the sections σ_k satisfy all the required properties, except that the integrability of $\tilde{J}_k$ and the holomorphicity of $\mathbb{P}\sigma_k$ are proved to hold on the r'-neighborhood of $\mathcal{C}_J(s_k)$ rather than on a neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$.

However, the C^p bounds $|\tilde{J}_k - J_k| = O(k^{-1/2})$ and $|\sigma_k - s_k| = O(k^{-1/2})$ imply that $|\mathrm{Jac}_{\tilde{J}_k}(\mathbb{P}\sigma_k) - \mathrm{Jac}_J(\mathbb{P}s_k)| = O(k^{-1/2})$ and $|\mathcal{T}_{\tilde{J}_k}(\sigma_k) - \mathcal{T}_J(s_k)| = O(k^{-1/2})$. Therefore it follows from the transversality properties of s_k that the points of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ lie within g_k-distance $O(k^{-1/2})$ of $\mathcal{C}_J(s_k)$. In particular, if k is large enough, the $\frac{r'}{2}$-neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k)$ is contained in the r'-neighborhood of $\mathcal{C}_J(s_k)$, which ends the proof of Proposition 8 in the case of isolated sections.

In the case of one-parameter families of sections, the argument is similar. One first notices that, because of γ-genericity, there exists $r > 0$ such that, for every $t \in [0,1]$, the set $\mathcal{C}_{J_t}(s_{t,k})$ consists of finitely many points, any two of which are mutually distant of at least $4r$. Therefore, the points of $\mathcal{C}_{J_t}(s_{t,k})$ depend continuously on t, and their number remains constant.

Consider a continuous family $(x_t)_{t\in[0,1]}$ of points of $\mathcal{C}_{J_t}(s_{t,k})$: Lemma 3 provides approximately J_t-holomorphic Darboux maps $\psi_{t,k}$ depending continuously on t on a neighborhood of x_t. By pulling back the standard complex structure of $\mathbb{C}^2$, one obtains integrable almost-complex structures $J'_{t,k}$ over $B_{g_k}(x_t, 2r)$, depending continuously on t and differing from J_t by $O(k^{-1/2})$. As previously, because the set of ω-compatible almost-complex structures is contractible, one can define a continuous family of almost-complex structures $\tilde{J}_{t,k}$ on X by gluing together J_t with the almost-complex structures $J'_{t,k}$ defined over $B_{g_k}(x_t, 2r)$, using a cut-off function at distance r from $\mathcal{C}_{J_t}(s_{t,k})$. By construction, the almost-complex structures $\tilde{J}_{t,k}$ are integrable over the r-neighborhood of $\mathcal{C}_{J_t}(s_{t,k})$, and $|\tilde{J}_{t,k} - J_t|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Next, we perturb $s_{t,k}$ near $x_t \in \mathcal{C}_{J_t}(s_{t,k})$ in order to make the corresponding projective map locally $\tilde{J}_{t,k}$-holomorphic. As before, composing with a rotation of $\mathbb{C}^3$ (constant over X and depending continuously on t) and decreasing r if necessary, we can assume that $s^1_{t,k}(x_t) = s^2_{t,k}(x_t) = 0$ and therefore that $|s^0_{t,k}|$ remains larger than $\frac{\gamma}{2}$ over $B_{g_k}(x_t, r)$. The $\tilde{J}_{t,k}$-holomorphicity of $\mathbb{P}s_{t,k}$ over $B_{g_k}(x_t, r)$ is then equivalent to that of the map $h_{t,k}$ with values in $\mathbb{C}^2$ defined as above.

As previously, there exist constants λ and r' such that $\psi_{t,k}(B_{\mathbb{C}^2}(0, \frac{11}{10}\lambda))$ is contained in $B_{g_k}(x_t, r)$ and $\psi_{t,k}(B_{\mathbb{C}^2}(0, \frac{1}{2}\lambda)) \supset B_{g_k}(x_t, r')$; once again, our goal is to make the functions $f^i_{t,k} : B^+ \to \mathbb{C}$ defined by $f^i_{t,k}(z) = h^i_{t,k}(\psi_{t,k}(\lambda z))$ holomorphic in the usual sense.

Because of the estimates on $\bar{\partial}_{J_t}\psi_{t,k}$ and $\bar{\partial}_{J_t}h_{t,k}$, we have $|\bar{\partial} f^i_{t,k}|_{C^p(B^+)} = O(k^{-1/2})$ $\forall p \in \mathbb{N}$, so Lemma 8 provides holomorphic functions $\tilde{f}^i_{t,k}$ over B which differ from $f^i_{t,k}$ by $O(k^{-1/2})$. By the same cut-off procedure as above, we can thus define functions $\hat{f}^i_{t,k}$ which are holomorphic over $B_{\mathbb{C}^2}(0, \frac{1}{2})$ and coincide with $f^i_{t,k}$ near the boundary of B. Going back through the coordinate maps, we define as previously functions $\hat{h}^i_{t,k}$ and sections $\hat{s}_{t,k}$ over the neighborhood $U_{t,x_t} = \psi_{t,k}(B_{\mathbb{C}^2}(0, \lambda))$ of x_t. Since $\hat{s}_{t,k}$ coincides with $s_{t,k}$ near the boundary of U_{t,x_t}, we can obtain smooth sections $\sigma_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ by gluing $s_{t,k}$ together with the various sections $\hat{s}_{t,k}$ defined near the points of $\mathcal{C}_{J_t}(s_{t,k})$.

As previously, the maps $\mathbb{P}\sigma_{t,k}$ are $\tilde{J}_{t,k}$-holomorphic over the r'-neighborhood of $\mathcal{C}_{J_t}(s_{t,k})$ and satisfy $|\sigma_{t,k} - s_{t,k}|_{C^p,g_k} = O(k^{-1/2})$; therefore the desired result follows from the observation that, for large enough k, $\mathcal{C}_{\tilde{J}_{t,k}}(\sigma_{t,k})$ lies within distance $\frac{r'}{2}$ of $\mathcal{C}_{J_t}(s_{t,k})$.

We now consider the special case where $s_{0,k}$ already satisfies the required conditions, i.e. there exists an almost-complex structure $\bar{J}_{0,k}$ within $O(k^{-1/2})$ of J_0, integrable near $\mathcal{C}_{\bar{J}_{0,k}}(s_{0,k})$, and such that $\mathbb{P}s_{0,k}$ is $\bar{J}_{0,k}$-holomorphic near $\mathcal{C}_{\bar{J}_{0,k}}(s_{0,k})$. Although this is actually not necessary for the result to hold, we also assume, as in the statement of Proposition 8, that $s_{t,k} = s_{0,k}$ and $J_t = J_0$ for every $t \leq \epsilon$, for some $\epsilon > 0$. We want to prove that one can take $\sigma_{0,k} = s_{0,k}$ in the above construction.

We first show that one can assume that $\tilde{J}_{0,k}$ coincides with $\bar{J}_{0,k}$ over a small neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$. For this, remark that $\mathcal{C}_{J_0}(s_{0,k})$ lies within $O(k^{-1/2})$ of $\mathcal{C}_{\bar{J}_{0,k}}(s_{0,k})$, so there exists a constant δ such that, for large enough k, $\bar{J}_{0,k}$ is integrable and $\mathbb{P}s_{0,k}$ is $\bar{J}_{0,k}$-holomorphic over the δ-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$.

Fix points $(x_t)_{t\in[0,1]}$ in $\mathcal{C}_{J_t}(s_{t,k})$, and consider, for all $t \geq \epsilon$, the approximately J_t-holomorphic Darboux coordinates $(z^1_{t,k}, z^2_{t,k})$ on a neighborhood of x_t and the inverse map $\psi_{t,k}$ given by Lemma 3 and which are used to define the almost-complex structures $J'_{t,k}$ and $\tilde{J}_{t,k}$ near x_t. We want to show that one can extend the family $\psi_{t,k}$

to all $t \in [0,1]$ in such a way that the map $\psi_{0,k}$ is $\bar{J}_{0,k}$-holomorphic. The hypothesis that J_t and $s_{t,k}$ are the same for all $t \in [0,\epsilon]$ makes things easier to handle because $J_\epsilon = J_0$ and $x_\epsilon = x_0$.

Since $\bar{J}_{0,k}$ is integrable over $B_{g_k}(x_0,\delta)$ and ω-compatible, there exist local complex Darboux coordinates $Z_k = (Z_k^1, Z_k^2)$ at x_0 which are $\bar{J}_{0,k}$-holomorphic. It follows from the approximate J_0-holomorphicity of the coordinates $z_{\epsilon,k} = (z^1_{\epsilon,k}, z^2_{\epsilon,k})$ and from the bound $|J_0 - \bar{J}_{0,k}| = O(k^{-1/2})$ that, composing with a linear endomorphism of $\mathbb{C}^2$ if necessary, one can assume that the differentials at x_0 of the two coordinate maps, namely $\nabla_{x_0} z_{\epsilon,k}$ and $\nabla_{x_0} Z_k$, lie within $O(k^{-1/2})$ of each other. For all $t \in [0,\epsilon]$, $\check{z}_{t,k} = \frac{t}{\epsilon} z_{\epsilon,k} + (1-\frac{t}{\epsilon}) Z_k$ defines local coordinates on a neighborhood of x_0 ; however, for $t \in (0,\epsilon)$ this map fails to be symplectic by an amount which is $O(k^{-1/2})$. So we apply Moser's argument to $\check{z}_{t,k}$ in order to get local Darboux coordinates $z_{t,k}$ over a neighborhood of x_0 which interpolate between Z_k and $z_{\epsilon,k}$ and which differ from $\check{z}_{t,k}$ by $O(k^{-1/2})$. It is easy to check that, if k is large enough, then the coordinates $z_{t,k}$ are well-defined over the ball $B_{g_k}(x_t, 2r)$. Since $\bar\partial_{J_0} Z_k$ and $\bar\partial_{J_0} z_{\epsilon,k}$ are $O(k^{-1/2})$, and because $z_{t,k}$ differs from $\check{z}_{t,k}$ by $O(k^{-1/2})$, the coordinates defined by $z_{t,k}$ are approximately J_0-holomorphic (in the sense of Lemma 3) for all $t \in [0,\epsilon]$.

Defining $\psi_{t,k}$ as the inverse of the map $z_{t,k}$ for every $t \in [0,\epsilon]$, it follows immediately that the maps $\psi_{t,k}$, which depend continuously on t, are approximately J_t-holomorphic over a neighborhood of 0 for every $t \in [0,1]$, and that $\psi_{0,k}$ is $\bar{J}_{0,k}$-holomorphic.

We can then define $J'_{t,k}$ as previously on $B_{g_k}(x_t, 2r)$, and notice that $J'_{0,k}$ coincides with $\bar{J}_{0,k}$. Therefore, the corresponding almost-complex structures $\tilde{J}_{t,k}$ over X, in addition to all the properties described previously, also satisfy the equality $\tilde{J}_{0,k} = \bar{J}_{0,k}$ over the r-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$.

It follows that, constructing the sections $\sigma_{t,k}$ from $s_{t,k}$ as previously, we have $\sigma_{0,k} = s_{0,k}$. Indeed, since $\mathbb{P}s_{0,k}$ is already $\tilde{J}_{0,k}$-holomorphic over the r-neighborhood of $\mathcal{C}_{J_0}(s_{0,k})$, we get that, in the above construction, $h^1_{0,k}$ and $h^2_{0,k}$ are $\tilde{J}_{0,k}$-holomorphic, and so $f^1_{0,k}$ and $f^2_{0,k}$ are holomorphic. Therefore, by definition of the operator P of Lemma 8, we have $\tilde{f}^1_{0,k} = f^1_{0,k}$ and $\tilde{f}^2_{0,k} = f^2_{0,k}$, which clearly implies that $\sigma_{0,k} = s_{0,k}$.

The same argument applies near $t = 1$ to show that, if $s_{1,k}$ already satisfies the expected properties and if J_t and $s_{t,k}$ are the same for all $t \in [1-\epsilon, 1]$, then one can take $\sigma_{1,k} = s_{1,k}$. This ends the proof of Proposition 8.

4.2. **Holomorphicity at generic branch points.** Our last step in order to obtain $\bar\partial$-tame sections is to ensure, by further perturbation, the vanishing of $\bar\partial_{\tilde{J}_k}(\mathbb{P}s_k)$ over the kernel of $\partial_{\tilde{J}_k}(\mathbb{P}s_k)$ at every branch point.

Proposition 9. *Let $(s_k)_{k \gg 0}$ be γ-generic asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. Assume that there exist ω-compatible almost-complex structures $\tilde{J}_k$ such that $|\tilde{J}_k - J|_{C^p, g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$ and such that, for some constant $c > 0$, $f_k = \mathbb{P}s_k$ is $\tilde{J}_k$-holomorphic over the c-neighborhood of $\mathcal{C}_{\tilde{J}_k}(s_k)$. Then, for all large k, there exist sections σ_k such that the following properties hold : $|\sigma_k - s_k|_{C^p, g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$; σ_k coincides with s_k over the $\frac{c}{2}$-neighborhood of $\mathcal{C}_{\tilde{J}_k}(\sigma_k) = \mathcal{C}_{\tilde{J}_k}(s_k)$; and, at every point of $R_{\tilde{J}_k}(\sigma_k)$, $\bar\partial_{\tilde{J}_k}(\mathbb{P}\sigma_k)$ vanishes over the kernel of $\partial_{\tilde{J}_k}(\mathbb{P}\sigma_k)$.*

Moreover, the same result holds for one-parameter families of asymptotically J_t-holomorphic sections $(s_{t,k})_{t\in[0,1],k\gg 0}$ satisfying the above properties. Furthermore, if $s_{0,k}$ and $s_{1,k}$ already satisfy the properties required of $\sigma_{0,k}$ and $\sigma_{1,k}$, then one can take $\sigma_{0,k} = s_{0,k}$ and $\sigma_{1,k} = s_{1,k}$.

The role of the almost-complex structure J in the statement of this result may seem ambiguous, as the sections s_k are also asymptotically holomorphic and generic with respect to the almost-complex structures $\tilde{J}_k$. The point is that, by requiring that all the almost-complex structures $\tilde{J}_k$ lie within $O(k^{-1/2})$ of a fixed almost-complex structure, one ensures the existence of uniform bounds on the geometry of $\tilde{J}_k$ independently of k.

We now prove Proposition 9 in the case of isolated sections. In all the following, we use the almost complex structure $\tilde{J}_k$ implicitly. Consider a point $x \in R(s_k)$ at distance more than $\frac{3}{4}c$ from $\mathcal{C}(s_k)$, and let K_x be the one-dimensional complex subspace Ker $\partial f_k(x)$ of T_xX. Because $x \notin \mathcal{C}(s_k)$, we have $T_xX = T_xR(s_k) \oplus K_x$. Therefore, there exists a unique 1-form $\theta_x \in T^*_xX \otimes T_{f_k(x)}\mathbb{CP}^2$ such that the restriction of θ_x to $T_xR(s_k)$ is zero and the restriction of θ_x to K_x is equal to $\bar\partial f_k(x)_{|K_x}$.

Because the restriction of $\mathcal{T}(s_k)$ to $R(s_k)$ is transverse to 0 and because x is at distance more than $\frac{3}{4}c$ from $\mathcal{C}(s_k)$, the quantity $|\mathcal{T}(s_k)(x)|$ is bounded from below by a uniform constant, and therefore the angle between $T_xR(s_k)$ and K_x is also bounded from below. So there exists a constant C independent of k and x such that $|\theta_x| \le Ck^{-1/2}$. Moreover, because $\bar\partial f_k$ vanishes over the c-neighborhood of $\mathcal{C}(s_k)$, the 1-form θ_x vanishes at all points x close to $\mathcal{C}(s_k)$; therefore we can extend θ into a section of $T^*X\otimes f_k^*T\mathbb{CP}^2$ over $R(s_k)$ which vanishes over the c-neighborhood of $\mathcal{C}(s_k)$, and which satisfies bounds of the type $|\theta|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Next, use the exponential map of the metric g to identify a tubular neighborhood of $R(s_k)$ with a neighborhood of the zero section in the normal bundle $NR(s_k)$. Given $\delta > 0$ sufficiently small, we define a section χ of $f_k^*T\mathbb{CP}^2$ over the δ-tubular neighborhood of $R(s_k)$ by the following identity : given any point $x \in R(s_k)$ and any vector $\xi \in N_xR(s_k)$ of norm less than δ,

$$\chi(\exp_x(\xi)) = \beta(|\xi|)\,\theta_x(\xi),$$

where the fibers of $f_k^*T\mathbb{CP}^2$ at x and at $\exp_x(\xi)$ are implicitly identified using radial parallel transport, and $\beta : [0,\delta] \to [0,1]$ is a smooth cut-off function equal to 1 over $[0,\frac{1}{2}\delta]$ and 0 over $[\frac{3}{4}\delta,\delta]$. Since χ vanishes near the boundary of the chosen tubular neighborhood, we can extend it into a smooth section over all of X which vanishes at distance more than δ from $R(s_k)$.

Decreasing δ if necessary, we can assume that $\delta < \frac{c}{2}$: it then follows from the vanishing of θ over the c-neighborhood of $\mathcal{C}(s_k)$ that χ vanishes over the $\frac{c}{2}$-neighborhood of $\mathcal{C}(s_k)$. Moreover, because $|\theta|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$ and because the cut-off function β is smooth, χ also satisfies bounds $|\chi|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$.

Fix a point $x \in R(s_k)$: χ is identically zero over $R(s_k)$ by construction, so $\nabla\chi(x)$ vanishes over $T_xR(s_k)$; and, because $\beta \equiv 1$ near the origin and by definition of the exponential map, $\nabla\chi(x)_{|N_xR(s_k)} = \theta_{x|N_xR(s_k)}$. Since $T_xR(s_k)$ and $N_xR(s_k)$ generate T_xX, we conclude that $\nabla\chi(x) = \theta_x$. In particular, restricting to K_x, we get that

$\nabla\chi(x)_{|K_x} = \theta_{x|K_x} = \bar\partial f_k(x)_{|K_x}$. Equivalently, since K_x is a complex subspace of T_xX, we have $\bar\partial\chi(x)_{|K_x} = \bar\partial f_k(x)_{|K_x}$ and $\partial\chi(x)_{|K_x} = 0 = \partial f_k(x)_{|K_x}$.

Recall that, for all $x \in X$, the tangent space to $\mathbb{CP}^2$ at $f_k(x) = \mathbb{P}s_k(x)$ canonically identifies with the space of complex linear maps from $\mathbb{C}s_k(x)$ to $(\mathbb{C}s_k(x))^\perp \subset \mathbb{C}^3 \otimes L^k_x$. This allows us to define $\sigma_k(x) = s_k(x) - \chi(x).s_k(x)$.

It follows from the properties of χ described above that σ_k coincides with s_k over the $\frac{c}{2}$-neighborhood of $\mathcal{C}(s_k)$ and that $|\sigma_k - s_k|_{C^p,g_k} = O(k^{-1/2})$ for all $p \in \mathbb{N}$. Because of the transversality properties of s_k, we get that the points of $\mathcal{C}(\sigma_k)$ lie within distance $O(k^{-1/2})$ of $\mathcal{C}(s_k)$, and therefore if k is large enough that $\mathcal{C}(\sigma_k) = \mathcal{C}(s_k)$.

Let $\tilde f_k = \mathbb{P}\sigma_k$, and consider a point $x \in R(s_k)$: since $\chi(x) = 0$ and therefore $\tilde f_k(x) = f_k(x)$, it is easy to check that $\nabla\tilde f_k(x) = \nabla f_k(x) - \nabla\chi(x)$ in $T^*_xX \otimes T_{f_k(x)}\mathbb{CP}^2$. Therefore, setting $K_x = \text{Ker}\,\partial f_k(x)$ as above, we get that $\partial\tilde f_k(x) = \partial f_k(x) - \partial\chi(x)$ and $\bar\partial\tilde f_k(x) = \bar\partial f_k(x) - \bar\partial\chi(x)$ both vanish over K_x. A first consequence is that $\partial\tilde f_k(x)$ also has rank one, i.e. $x \in R(\sigma_k)$: therefore $R(s_k) \subset R(\sigma_k)$. However, because σ_k differs from s_k by $O(k^{-1/2})$, it follows from the transversality properties of s_k that, for large enough k, $R(\sigma_k)$ is contained in a small neighborhood of $R(s_k)$, and so $R(\sigma_k) = R(s_k)$.

Furthermore, recall that at every point x of $R(\sigma_k) = R(s_k)$ one has $\bar\partial\tilde f_k(x)_{|K_x} = \partial\tilde f_k(x)_{|K_x} = 0$. Therefore $\bar\partial\tilde f_k(x)$ vanishes over the kernel of $\partial\tilde f_k(x)$, and so the sections σ_k satisfy all the required properties.

To handle the case of one-parameter families, remark that the above construction consists of explicit formulae, so it is easy to check that θ, χ and σ_k depend continuously on s_k and $\tilde J_k$. Therefore, starting from one-parameter families $s_{t,k}$ and $\tilde J_{t,k}$, the above construction yields for all $t \in [0,1]$ sections $\sigma_{t,k}$ which satisfy the required properties and depend continuously on t.

Moreover, if $s_{0,k}$ already satisfies the required properties, i.e. if $\bar\partial f_{0,k}(x)_{|K_x}$ vanishes at any point $x \in R(s_{0,k})$, then the above definitions give $\theta \equiv 0$, and therefore $\chi \equiv 0$ and $\sigma_{0,k} = s_{0,k}$; similarly for $t = 1$, which ends the proof of Proposition 9.

4.3. **Proof of the main theorems.** Assuming that Theorem 3 holds, Theorems 1 and 2 follow directly from the results we have proved so far : combining Propositions 1, 4, 5 and 7, one gets, for all large k, asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ which are γ-generic for some constant $\gamma > 0$; Propositions 8 and 9 imply that these sections can be made $\bar\partial$-tame by perturbing them by $O(k^{-1/2})$ (which preserves the genericity properties if k is large enough) ; and Theorem 3 implies that the corresponding projective maps are then approximately holomorphic singular branched coverings.

Let us now prove Theorem 4. We are given two sequences $s_{0,k}$ and $s_{1,k}$ of sections of $\mathbb{C}^3 \otimes L^k$ which are asymptotically holomorphic, γ-generic and $\bar\partial$-tame with respect to almost-complex structures J_0 and J_1, and want to show the existence of a one-parameter family of almost-complex structures J_t interpolating between J_0 and J_1 and of generic and $\bar\partial$-tame asymptotically J_t-holomorphic sections interpolating between $s_{0,k}$ and $s_{1,k}$.

One starts by defining sections $s_{t,k}$ and compatible almost-complex structures J_t interpolating between $(s_{0,k}, J_0)$ and $(s_{1,k}, J_1)$ in the following way : for $t \in [0, \frac{2}{7}]$, let $s_{t,k} = s_{0,k}$ and $J_t = J_0$; for $t \in [\frac{2}{7}, \frac{3}{7}]$, let $s_{t,k} = (3-7t)s_{0,k}$ and $J_t = J_0$; for $t \in [\frac{3}{7}, \frac{4}{7}]$, let $s_{t,k} = 0$ and take J_t to be a path of ω-compatible almost-complex structures from J_0 to J_1 (recall that the space of compatible almost-complex structures is connected) ; for $t \in [\frac{4}{7}, \frac{5}{7}]$, let $s_{t,k} = (7t-4)s_{1,k}$ and $J_t = J_1$; and for $t \in [\frac{5}{7}, 1]$, let $s_{t,k} = s_{1,k}$ and $J_t = J_1$. Clearly, J_t and $s_{t,k}$ depend continuously on t, and the sections $s_{t,k}$ are asymptotically J_t-holomorphic for all $t \in [0,1]$.

Since γ-genericity is a local and C^3-open property, there exists $\alpha > 0$ such that any section differing from $s_{0,k}$ by less than α in C^3 norm is $\frac{\gamma}{2}$-generic, and similarly for $s_{1,k}$. Applying Propositions 1, 4, 5 and 7, we get for all large k asymptotically J_t-holomorphic sections $\sigma_{t,k}$ which are η-generic for some $\eta > 0$, and such that $|\sigma_{t,k} - s_{t,k}|_{C^3,g_k} < \alpha$ for all $t \in [0,1]$.

We now set $s'_{t,k} = s_{0,k}$ for $t \in [0, \frac{1}{7}]$; $s'_{t,k} = (2-7t)s_{0,k} + (7t-1)\sigma_{\frac{2}{7},k}$ for $t \in [\frac{1}{7}, \frac{2}{7}]$; $s'_{t,k} = \sigma_{t,k}$ for $t \in [\frac{2}{7}, \frac{5}{7}]$; $s'_{t,k} = (7t-5)s_{1,k} + (6-7t)\sigma_{\frac{5}{7},k}$ for $t \in [\frac{5}{7}, \frac{6}{7}]$; and $s'_{t,k} = s_{1,k}$ for $t \in [\frac{6}{7}, 1]$. By construction, the sections $s'_{t,k}$ are asymptotically J_t-holomorphic for all $t \in [0,1]$ and depend continuously on t. Moreover, they are $\frac{\gamma}{2}$-generic for $t \in [0, \frac{2}{7}]$ because $s'_{t,k}$ then lies within α in C^3 norm of $s_{0,k}$, and similarly for $t \in [\frac{5}{7}, 1]$ because $s'_{t,k}$ then lies within α in C^3 norm of $s_{1,k}$. They are also η-generic for $t \in [\frac{2}{7}, \frac{5}{7}]$ because $s'_{t,k}$ is then equal to $\sigma_{t,k}$. Therefore the sections $s'_{t,k}$ are η'-generic for all $t \in [0,1]$, where $\eta' = \min(\eta, \frac{\gamma}{2})$.

Next, we apply Proposition 8 to the sections $s'_{t,k}$: since $s'_{0,k} = s_{0,k}$ and $s'_{1,k} = s_{1,k}$ are already $\bar\partial$-tame, and since the families $s'_{t,k}$ and J_t are constant over $[0, \frac{1}{7}]$ and $[\frac{6}{7}, 1]$, one can require of the sections $s''_{t,k}$ given by Proposition 8 that $s''_{0,k} = s'_{0,k} = s_{0,k}$ and $s''_{1,k} = s'_{1,k} = s_{1,k}$. Finally, we apply Proposition 9 to the sections $s''_{t,k}$ to obtain sections $\sigma''_{t,k}$ which simultaneously have genericity and $\bar\partial$-tameness properties. Since $s''_{0,k}$ and $s''_{1,k}$ are already $\bar\partial$-tame, one can require that $\sigma''_{0,k} = s''_{0,k} = s_{0,k}$ and $\sigma''_{1,k} = s''_{1,k} = s_{1,k}$. The sections $\sigma''_{t,k}$ interpolating between $s_{0,k}$ and $s_{1,k}$ therefore satisfy all the required properties, which ends the proof of Theorem 4.

5. Generic tame maps and branched coverings

5.1. **Structure near cusp points.** In order to prove Theorem 3, we need to check that, given any generic and $\bar\partial$-tame asymptotically holomorphic sections s_k of $\mathbb{C}^3 \otimes L^k$, the corresponding projective maps $f_k = \mathbb{P}s_k : X \to \mathbb{CP}^2$ are, at any point of X, locally approximately holomorphically modelled on one of the three model maps of Definition 2. We start with the case of the neighborhood of a cusp point.

Let $x_0 \in X$ be a cusp point of f_k, i.e. an element of $\mathcal{C}_{\tilde J_k}(s_k)$, where $\tilde J_k$ is the almost-complex structure involved in the definition of $\bar\partial$-tameness. By definition, $\tilde J_k$ differs from J by $O(k^{-1/2})$ and is integrable over a neighborhood of x_0, and f_k is $\tilde J_k$-holomorphic over a neighborhood of x_0. Therefore, choose $\tilde J_k$-holomorphic local complex coordinates on X near x_0, and local complex coordinates on $\mathbb{CP}^2$ near $f_k(x_0)$: the map h corresponding to f_k in these coordinate charts is, locally, *holomorphic*. Because the coordinate map on X is within $O(k^{-1/2})$ of being J-holomorphic, we can restrict ourselves to the study of the holomorphic map $h = (h_1, h_2)$ defined over a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$, which satisfies transversality properties following from the genericity of s_k. We need to show that, composing h with holomorphic local diffeomorphisms of the source space $\mathbb{C}^2$ or of the target space $\mathbb{C}^2$, we can get h to be of the form $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$ over a neighborhood of 0.

This statement is a standard result in singularity theory and was first proven by Whitney in [7] (§16–19). Due to a differently formulated definition of cusp points and for the sake of completeness, we provide here the first part of Whitney's argument.

First, because $|\partial f_k|$ is bounded from below and x_0 is a cusp point, the derivative $\partial h(0)$ does not vanish and has rank one. Therefore, composing with a rotation of the target space $\mathbb{C}^2$ if necessary, we can assume that its image is directed along the second coordinate, i.e. Im $(\partial h(0)) = \{0\} \times \mathbb{C}$.

Calling Z_1 and Z_2 the two coordinates on the target space $\mathbb{C}^2$, it follows immediately that the function $z_2 = h^* Z_2$ over the source space has a non-vanishing differential at 0, and can therefore be considered as a local coordinate function on the source space. Choose z_1 to be any linear function whose differential at the origin is linearly independent with $dz_2(0)$, so that (z_1, z_2) define holomorphic local coordinates on a neighborhood of 0 in $\mathbb{C}^2$. In these coordinates, h is of the form $(z_1, z_2) \mapsto (h_1(z_1, z_2), z_2)$ where h_1 is a holomorphic function such that $h_1(0) = 0$ and $\partial h_1(0) = 0$.

Next, notice that, because $\mathrm{Jac}(f_k)$ vanishes transversely at x_0, the quantity $\mathrm{Jac}(h) = \det(\partial h) = \partial h_1/\partial z_1$ vanishes transversely at the origin, i.e.

$$\left(\frac{\partial^2 h_1}{\partial z_1^2}(0), \frac{\partial^2 h_1}{\partial z_1 \partial z_2}(0)\right) \neq (0, 0).$$

Moreover, an argument similar to that of §3.2 shows that locally, because we have arranged for $|\partial h_2|$ to be bounded from below, the ratio between the quantities $\mathcal{T}(s_k)$ and $\hat{\mathcal{T}} = \partial h_2 \wedge \partial \mathrm{Jac}(h)$ is bounded from above and below. In particular, the fact that $x_0 \in \mathcal{C}_{\tilde{J}_k}(s_k)$ implies that the restriction of $\hat{\mathcal{T}}$ to the set of branch points vanishes transversely at the origin.

In our case, $\hat{\mathcal{T}} = dz_2 \wedge \partial(\frac{\partial h_1}{\partial z_1}) = -(\partial^2 h_1/\partial z_1^2)\, dz_1 \wedge dz_2$. Therefore, the vanishing of $\hat{\mathcal{T}}(0)$ implies that $\partial^2 h_1/\partial z_1^2\,(0) = 0$. It follows that $\partial^2 h_1/\partial z_1 \partial z_2\,(0)$ must be non-zero ; rescaling the coordinate z_1 by a constant factor if necessary, this derivative can be assumed to be equal to -1. Therefore, the map h can be written as

$$\begin{aligned} h(z_1, z_2) &= (-z_1 z_2 + \lambda z_2^2 + O(|z|^3), z_2) \\ &= (-z_1 z_2 + \lambda z_2^2 + \alpha z_1^3 + \beta z_1^2 z_2 + \gamma z_1 z_2^2 + \delta z_2^3 + O(|z|^4), z_2) \end{aligned}$$

where λ, α, β, γ and δ are complex coefficients.

We now consider the following coordinate changes : on the target space $\mathbb{C}^2$, define $\psi(Z_1, Z_2) = (Z_1 - \lambda Z_2^2 - \delta Z_2^3, Z_2)$, and on the source space $\mathbb{C}^2$, define $\phi(z_1, z_2) = (z_1 + \beta z_1^2 + \gamma z_1 z_2, z_2)$. Clearly, these two maps are local diffeomorphisms near the origin. Therefore, one can replace h by $\psi \circ h \circ \phi$, which has the effect of killing most terms of the above expansion : this allows us to consider that h is of the form

$$h(z_1, z_2) = (-z_1 z_2 + \alpha z_1^3 + O(|z|^4), z_2).$$

Next, recall that the set of branch points is, in our local setting, the set of points where $\mathrm{Jac}(h) = \partial h_1/\partial z_1 = -z_2 + 3\alpha z_1^2 + O(|z|^3)$ vanishes. Therefore, the tangent direction to the set of branch points at the origin is the z_1 axis, and the transverse vanishing of $\hat{\mathcal{T}}$ at the origin implies that $\frac{\partial}{\partial z_1}\hat{\mathcal{T}}(0) \neq 0$. Using the above formula for $\hat{\mathcal{T}}$, we conclude that $\partial^3 h_1/\partial z_1^3 \neq 0$, i.e. $\alpha \neq 0$.

Rescaling the two coordinates z_1 and Z_1 by a constant factor, we can assume that α is equal to 1. Therefore, we have used all the transversality properties of h to show that, on a neighborhood of x_0, it is of the form

$$h(z_1, z_2) = (-z_1 z_2 + z_1^3 + O(|z|^4), z_2).$$

The uniform bounds and transversality estimates on s_k can be used to show that all the rescalings and transformations we have used are "nice", i.e. they have bounded derivatives and their inverses have bounded derivatives.

It then remains to show that further coordinate changes can kill the higher order terms still present in the expression of h. The idea of Whitney's argument is to use successive coordinate changes in order to reduce to the case where the perturbation term vanishes up to order at least 2 over the parabola $z_2 = 3z_1^2$, which makes it possible to kill all higher order terms by composing h with a well-chosen diffeomorphism of the source space $\mathbb{C}^2$. Details can be found in §16–19 of [7]. One eventually gets that, setting $h_0(z_1, z_2) = (-z_1 z_2 + z_1^3, z_2)$, there exist holomorphic diffeomorphisms Φ and Ψ of $\mathbb{C}^2$ near the origin such that $\Psi \circ h_0 \circ \Phi = h$ over a small neighborhood of 0 in $\mathbb{C}^2$, which is what we wanted to prove.

Moreover, because of the uniform transversality estimates and bounds on the derivatives of s_k, the derivatives of h are uniformly bounded. It follows easily, by going over the argument, that the neighborhood of x_0 over which the map f_k has been shown to be $O(k^{-1/2})$-approximately holomorphically modelled on the map h_0 can be assumed to contain a ball of fixed radius (depending on the bounds and transversality estimates, but independent of x_0 and k).

5.2. **Structure near generic branch points.** We now consider a branch point $x_0 \in R_{\tilde{J}_k}(s_k)$, which we assume to be at distance more than a fixed constant δ from the set of cusp points $\mathcal{C}_{\tilde{J}_k}(s_k)$. We want to show that, over a neighborhood of x_0, $f_k = \mathbb{P}s_k$ is approximately holomorphically modelled on the map $(z_1, z_2) \mapsto (z_1^2, z_2)$.

From now on, we implicitly use the almost-complex structure $\tilde{J}_k$ and write R for the intersection of $R_{\tilde{J}_k}(s_k)$ with the ball $B_{g_k}(x_0, \frac{\delta}{2})$. First note that, since R remains at distance more than $\frac{\delta}{2}$ from the cusp points, the tangent space to R remains everywhere away from the kernel of ∂f_k. Therefore, the restriction of f_k to R is a local diffeomorphism over a neighborhood of x_0, and so $f_k(R)$ is locally a smooth approximately holomorphic submanifold in $\mathbb{CP}^2$. It follows that there exist approximately holomorphic coordinates (Z_1, Z_2) on a neighborhood of $f_k(x_0)$ in $\mathbb{CP}^2$ such that $f_k(R)$ is locally defined by the equation $Z_1 = 0$.

Define the approximately holomorphic function $z_2 = f_k^* Z_2$ over a neighborhood of x_0, and notice that its differential $dz_2 = dZ_2 \circ df_k$ does not vanish, because by construction Z_2 is a coordinate on $f_k(R)$. Therefore, z_2 can be considered as a local complex coordinate function on a neighborhood of x_0. In particular, the level sets of z_2 are smooth and intersect R transversely at a single point.

Take z_1 to be an approximately holomorphic function on a neighborhood of x_0 which vanishes at x_0 and whose differential at x_0 is linearly independent with that of z_2 (e.g. take the two differentials to be mutually orthogonal), so that (z_1, z_2) define approximately holomorphic coordinates on a neighborhood of x_0. From now on we use the local coordinates (z_1, z_2) on X and (Z_1, Z_2) on $\mathbb{CP}^2$.

Because $dz_{2|TR}$ remains away from 0, R has locally an equation of the form $z_1 = \rho(z_2)$ for some approximately holomorphic function ρ (satisfying $\rho(0) = 0$ since $x_0 \in R$). Therefore, shifting the coordinates on X in order to replace z_1 by $z_1 - \rho(z_2)$, one can assume that $z_1 = 0$ is a local equation of R. In the chosen local coordinates, f_k is therefore modelled on an approximately holomorphic map h from a neighborhood of 0 in $\mathbb{C}^2$ with values in $\mathbb{C}^2$, of the form $(z_1, z_2) \mapsto (h_1(z_1, z_2), z_2)$, with the following properties.

First, because $R = \{z_1 = 0\}$ is mapped to $f_k(R) = \{Z_1 = 0\}$, we have $h_1(0, z_2) = 0$ for all z_2. Next, recall that the differential of f_k has real rank 2 at any point of R (because ∂f_k has complex rank 1 and $\bar\partial f_k$ vanishes over the kernel of ∂f_k), so its image is exactly the tangent space to $f_k(R)$. It follows that $\nabla h_1 = 0$ at every point $(0, z_2) \in R$.

Finally, because the chosen coordinates are approximately holomorphic the quantity $\mathrm{Jac}(f_k)$ is within $O(k^{-1/2})$ of $\det(\partial h) = (\partial h_1/\partial z_1)\, \partial z_1 \wedge \partial z_2$ by $O(k^{-1/2})$. Therefore, the transversality to 0 of $\mathrm{Jac}(f_k)$ implies that, along R, the norm of $(\partial^2 h_1/\partial z_1^2, \partial^2 h_1/\partial z_1 \partial z_2)$ remains larger than a fixed constant. However $\partial^2 h_1/\partial z_1 \partial z_2$ vanishes at any point of R because $\partial h_1/\partial z_1\,(0, z_2) = 0$ for all z_2. Therefore the quantity $\partial^2 h_1/\partial z_1^2$ remains bounded away from 0 on R.

The above properties imply that h can be written as

$$h(z_1, z_2) = \left(\alpha(z_2) z_1^2 + \beta(z_2) z_1 \bar z_1 + \gamma(z_2) \bar z_1^2 + \epsilon(z_1, z_2), z_2\right),$$

where α is approximately holomorphic and bounded away from 0, while β and γ are $O(k^{-1/2})$ (because of asymptotic holomorphicity), and $\epsilon(z_1, z_2) = O(|z_1|^3)$ is approximately holomorphic. Moreover, composing with the coordinate change $(Z_1, Z_2) \mapsto (\alpha(Z_2)^{-1} Z_1, Z_2)$ (which is approximately holomorphic and has bounded derivatives because α is bounded away from 0), one reduces to the case where α is identically equal to 1.

We now want to reduce further the problem by removing the β and γ terms in the above expression : for this, we first remark that, given any small enough complex numbers β and γ, there exists a complex number λ, of norm less than $|\beta| + |\gamma|$ and depending smoothly on β and γ, such that

$$\lambda = -\gamma \bar\lambda + \frac{\beta}{2}(1 + |\lambda|^2).$$

Indeed, if $|\beta| + |\gamma| < \frac{1}{2}$ the right hand side of this equation is a contracting map of the unit disc to itself, so the existence of a solution λ in the unit disc follows immediately from the fixed point theorem. Furthermore, using the bound $|\lambda| < 1$ in the right hand side, one gets that $|\lambda| < |\beta| + |\gamma|$. Finally, the smooth dependence of λ upon β and γ follows from the implicit function theorem.

Assuming again that $|\beta| + |\gamma| < \frac{1}{2}$ and defining λ as above, let

$$A = \frac{1 - \bar\lambda^2 \gamma}{1 - |\lambda|^4} \quad \text{and} \quad B = \frac{\gamma - \lambda^2}{1 - |\lambda|^4}.$$

The complex numbers A and B are also smooth functions of β and γ, and it is clear that $|A - 1| = O(|\beta| + |\gamma|)$ and $|B| = O(|\beta| + |\gamma|)$. Moreover, one easily checks that, in the ring of polynomials in z and $\bar z$,

$$A(z + \lambda \bar z)^2 + B(\bar z + \bar\lambda z)^2 = z^2 + 2\frac{\lambda + \gamma\bar\lambda}{1 + |\lambda|^2} z \bar z + \gamma \bar z^2 = z^2 + \beta z \bar z + \gamma \bar z^2.$$

Therefore, if one assumes k to be large enough, recalling that the quantities $\beta(z_2)$ and $\gamma(z_2)$ which appear in the above expression of h are bounded by $O(k^{-1/2})$, there exist $\lambda(z_2)$, $A(z_2)$ and $B(z_2)$, depending smoothly on z_2, such that $|A(z_2)-1| = O(k^{-1/2})$, $|B(z_2)| = O(k^{-1/2})$, $|\lambda(z_2)| = O(k^{-1/2})$ and

$$A(z_2)(z_1+\lambda(z_2)\bar z_1)^2 + B(z_2)(\overline{z_1+\lambda(z_2)\bar z_1})^2 = z_1^2 + \beta(z_2) z_1 \bar z_1 + \gamma(z_2)\bar z_1^2.$$

So, let h_0 be the map $(z_1,z_2)\mapsto(z_1^2,z_2)$, and let Φ and Ψ be the two approximately holomorphic local diffeomorphisms of $\mathbb{C}^2$ defined by $\Phi(z_1,z_2) = (z_1+\lambda(z_2)\bar z_1, z_2)$ and $\Psi(Z_1,Z_2) = (A(Z_2)Z_1 + B(Z_2)\bar Z_1, Z_2)$: then

$$h(z_1,z_2) = \Psi\circ h_0\circ\Phi(z_1,z_2) + (\epsilon(z_1,z_2),0).$$

It follows immediately that $\Psi^{-1}\circ h\circ\Phi^{-1}(z_1,z_2) = (z_1^2 + O(|z_1|^3), z_2)$. Therefore, this new coordinate change allows us to consider only the case where h is of the form $(z_1,z_2)\mapsto(z_1^2+\tilde\epsilon(z_1,z_2),z_2)$, where $\tilde\epsilon(z_1,z_2) = O(|z_1|^3)$.

Because $\tilde\epsilon(z_1,z_2) = O(|z_1|^3)$, the bound $|\tilde\epsilon(z_1,z_2)| < \frac{1}{2}|z_1|^2$ holds over a neighborhood of the origin whose size can be bounded from below independently of k and x_0 by using the uniform estimates on all derivatives. Over this neighborhood, define

$$\phi(z_1,z_2) = z_1\sqrt{1+\frac{\tilde\epsilon(z_1,z_2)}{z_1^2}}$$

for $z_1\neq 0$, where the square root is determined without ambiguity by the condition that $\sqrt{1}=1$. Setting $\phi(0,z_2)=0$, it follows from the bound $|\phi(z_1,z_2)-z_1| = O(|z_1|^2)$ that the function ϕ is C^1. In general ϕ is not C^2, because $\tilde\epsilon$ may contain terms involving $\bar z_1^2 z_1$ or $\bar z_1^3$.

Because $\phi(z_1,z_2) = z_1+O(|z_1|^2)$, the map $\Theta : (z_1,z_2)\mapsto(\phi(z_1,z_2),z_2)$ is a C^1 local diffeomorphism of $\mathbb{C}^2$ over a neighborhood of the origin. As previously, the uniform bounds on all derivatives imply that the size of this neighborhood can be bounded from below independently of k and x_0. Moreover, it follows from the asymptotic holomorphicity of s_k that $\tilde\epsilon$ has antiholomorphic derivatives bounded by $O(k^{-1/2})$, and so $|\bar\partial\phi| = O(k^{-1/2})$. Therefore Θ is $O(k^{-1/2})$-approximately holomorphic, and we have

$$h_0\circ\Theta(z_1,z_2) = h(z_1,z_2),$$

which finally gives the desired result.

5.3. **Proof of Theorem 3.** Theorem 3 follows readily from the above arguments : indeed, consider γ-generic and $\bar\partial$-tame asymptotically holomorphic sections s_k of $\mathbb{C}^3\otimes L^k$, and let $\tilde J_k$ be the almost-complex structures involved in the definition of $\bar\partial$-tameness. We need to show that, at any point $x\in X$, the maps $f_k = \mathbb{P}s_k$ are approximately holomorphically modelled on one of the three maps of Definition 2.

First consider the case where x lies close to a point $y\in\mathcal{C}_{\tilde J_k}(s_k)$. The argument of §5.1 implies the existence of a constant $\delta>0$ independent of k and y such that, over the ball $B_{g_k}(y,2\delta)$, the map f_k is $\tilde J_k$-holomorphically modelled on the cusp covering map $(z_1,z_2)\mapsto(z_1^3-z_1z_2,z_2)$. If x lies within distance δ of y, $B_{g_k}(y,2\delta)$ is a neighborhood of x ; therefore the expected result follows at every point within distance δ of $\mathcal{C}_{\tilde J_k}(s_k)$ from the observation that, because $|\tilde J_k - J| = O(k^{-1/2})$, the relevant coordinate chart on X is $O(k^{-1/2})$-approximately J-holomorphic.

Next, consider the case where x lies close to a point y of $R_{\tilde{J}_k}(s_k)$ which is itself at distance more than δ from $\mathcal{C}_{\tilde{J}_k}(s_k)$. The argument of §5.2 then implies the existence of a constant $\delta' > 0$ independent of k and y such that, over the ball $B_{g_k}(y, 2\delta')$, the map f_k is, in $O(k^{-1/2})$-approximately holomorphic C^1 coordinate charts, locally modelled on the branched covering map $(z_1, z_2) \mapsto (z_1^2, z_2)$. Therefore, if one assumes the distance between x and y to be less than δ', the given ball is a neighborhood of x, and the expected result follows.

So we are left only with the case where x is at distance more than δ' from $R_{\tilde{J}_k}(s_k)$. Assuming k to be large enough, it then follows from the bound $|\tilde{J}_k - J| = O(k^{-1/2})$ that x is at distance more than $\frac{1}{2}\delta'$ from $R_J(s_k)$. Therefore, the γ-transversality to 0 of $\mathrm{Jac}(f_k)$ implies that $|\mathrm{Jac}(f_k)(x)|$ is larger than $\alpha = \min(\frac{1}{2}\delta'\gamma, \gamma)$ (otherwise, the downward gradient flow of $|\mathrm{Jac}(f_k)|$ would reach a point of $R_J(s_k)$ at distance less than $\frac{1}{2}\delta'$ from x).

Recalling that $|\bar{\partial} f_k| = O(k^{-1/2})$, one gets that f_k is a $O(k^{-1/2})$-approximately holomorphic local diffeomorphism over a neighborhood of x. Therefore, choose holomorphic complex coordinates on $\mathbb{CP}^2$ near $f_k(x)$ and pull them back by f_k to obtain $O(k^{-1/2})$-approximately holomorphic local coordinates over a neighborhood of x : in these coordinates, the map f_k becomes the identity map, which ends the proof of Theorem 3.

6. Further remarks

6.1. **Branched coverings of $\mathbb{CP}^2$.** A natural question to ask about the results obtained in this paper is whether the property of being a (singular) branched covering of $\mathbb{CP}^2$, i.e. the existence of a map to $\mathbb{CP}^2$ which is locally modelled at every point on one of the three maps of Definition 2, strongly restricts the topology of a general compact 4-manifold. Since the notion of approximately holomorphic coordinate chart on X no longer has a meaning in this case, we relax Definition 2 by only requiring the existence of a local identification of the covering map with one of the model maps in a smooth local coordinate chart on X. However we keep requiring that the corresponding local coordinate chart on $\mathbb{CP}^2$ be approximately holomorphic, so that the branch locus in $\mathbb{CP}^2$ remains an immersed symplectic curve with cusps. Call such a map a *topological singular branched covering of* $\mathbb{CP}^2$. Then the following holds :

Proposition 10. *Let X be a compact 4-manifold and consider a topological singular covering $f : X \to \mathbb{CP}^2$ branched along a submanifold $R \subset X$. Then X carries a symplectic structure arbitrarily close to $f^*\omega_0$, where ω_0 is the standard symplectic structure of* $\mathbb{CP}^2$.

Proof. The closed 2-form $f^*\omega_0$ on X defines a symplectic structure on $X - R$ which degenerates along R. Therefore, one needs to perturb it by adding a small multiple of a closed 2-form with support in a neighborhood of R in order to make it nondegenerate. This perturbation can be constructed as follows.

Call C the set of cusp points, i.e. the points of R where the tangent space to R lies in the kernel of the differential of f, or equivalently the points around which f is modelled on the map $(z_1, z_2) \mapsto (z_1^3 - z_1 z_2, z_2)$. Consider a point $x \in C$, and work in local coordinates such that f identifies with the model map. In these coordinates, a local equation of R is $z_2 = 3z_1^2$, and the kernel K of the differential of f coincides at every point of R with the subspace $\mathbb{C}\times\{0\}$ of the tangent space ; this complex identification determines a natural orientation of K. Fix a constant $\rho_x > 0$ such that $B_{\mathbb{C}}(0,2\rho_x)\times B_{\mathbb{C}}(0,2\rho_x^2)$ is contained in the local coordinate patch, and choose cut-off functions χ_1 and χ_2 over $\mathbb{C}$ in such a way that χ_1 equals 1 over $B_{\mathbb{C}}(0,\rho_x)$ and vanishes outside of $B_{\mathbb{C}}(0,2\rho_x)$, and that χ_2 equals 1 over $B_{\mathbb{C}}(0,\rho_x^2)$ and vanishes outside of $B_{\mathbb{C}}(0,2\rho_x^2)$. Then, let ψ_x be the 2-form which equals $d(\chi_1(z_1)\,\chi_2(z_2)\,x_1\,dy_1)$ over the local coordinate patch, where x_1 and y_1 are the real and imaginary parts of z_1, and which vanishes over the remainder of X : the 2-form ψ_x coincides with $dx_1\wedge dy_1$ over a neighborhood of x. More importantly, it follows from the choice of the cut-off functions that the restriction of ψ_x to $K=\mathbb{C}\times\{0\}$ is non-negative at every point of R, and positive non-degenerate at every point of R which lies sufficiently close to x.

Similarly, consider a point $x\in R$ away from C and local coordinates such that f identifies with the model map $(z_1,z_2)\mapsto(z_1^2,z_2)$. In these coordinates, R identifies with $\{0\}\times\mathbb{C}$, and the kernel K of the differential of f coincides at every point of R with the subspace $\mathbb{C}\times\{0\}$ of the tangent space. Fix a constant $\rho_x > 0$ such that $B_{\mathbb{C}}(0,2\rho_x)\times B_{\mathbb{C}}(0,2\rho_x)$ is contained in the local coordinate patch, and choose a cut-off function χ over $\mathbb{C}$ which equals 1 over $B_{\mathbb{C}}(0,\rho_x)$ and 0 outside of $B_{\mathbb{C}}(0,2\rho_x)$. Then, let ψ_x be the 2-form which equals $d(\chi(z_1)\,\chi(z_2)\,x_1\,dy_1)$ over the local coordinate patch, where x_1 and y_1 are the real and imaginary parts of z_1, and which vanishes over the remainder of X : as previously, the restriction of ψ_x to $K=\mathbb{C}\times\{0\}$ is non-negative at every point of R, and positive non-degenerate at every point of R which lies sufficiently close to x.

Choose a finite collection of points x_i of R (including all the cusp points) in such a way that the neighborhoods of x_i over which the 2-forms ψ_{x_i} restrict positively to K cover all of R, and define α as the sum of all the 2-forms ψ_{x_i}. Then it follows from the above definitions that the 2-form α is exact, and that at any point of R its restriction to the kernel of the differential of f is positive and non-degenerate. Therefore, the 4-form $f^*\omega_0\wedge\alpha$ is a positive volume form at every point of R.

Now choose any metric on a neighborhood of R, and let d_R be the distance function to R. It follows from the compactness of X and R and from the general properties of the map f that, using the orientation induced by f and the chosen metric to implicitly identify 4-forms with functions, there exist positive constants K, C, C' and M such that the following bounds hold over a neighborhood of R : $f^*\omega_0\wedge f^*\omega_0\geq K d_R$, $f^*\omega_0\wedge\alpha\geq C-C'd_R$, and $|\alpha\wedge\alpha|\leq M$. Therefore, for all $\epsilon>0$ one gets over a neighborhood of R the bound

$$(f^*\omega_0+\epsilon\,\alpha)\wedge(f^*\omega_0+\epsilon\,\alpha)\geq(2\epsilon\,C-\epsilon^2M)+(K-2\epsilon\,C')d_R.$$

If ϵ is chosen sufficiently small, the coefficients $2\epsilon\,C-\epsilon^2M$ and $K-2\epsilon\,C'$ are both positive, which implies that the closed 2-form $f^*\omega_0+\epsilon\,\alpha$ is everywhere nondegenerate, and therefore symplectic. □

Another interesting point is the compatibility of our approximately holomorphic singular branched coverings with respect to the symplectic structures ω on X and ω_0 in $\mathbb{CP}^2$ (as opposed to the compatibility with the almost-complex structures, which has been a major preoccupation throughout the previous sections).

It is easy to check that given a covering map $f : X\to\mathbb{CP}^2$ defined by a section of $\mathbb{C}^3\otimes L^k$, the number of preimages of a generic point is equal to $\frac{1}{4\pi^2}k^2(\omega^2.[X])$, while the homology class of the preimage of a generic line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is Poincaré dual to $\frac{1}{2\pi}k[\omega]$. If we normalize the standard symplectic structure ω_0 on $\mathbb{CP}^2$ in such a way that the symplectic area of a line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is equal to 2π, it follows that the cohomology class of $f^*\omega_0$ is $[f^*\omega_0] = k[\omega]$.

As we have said above, the pull-back $f^*\omega_0$ of the standard symplectic form of $\mathbb{CP}^2$ by the covering map degenerates along the set of branch points, so there is no chance of $(X, f^*\omega_0)$ being symplectic and symplectomorphic to $(X, k\omega)$. However, one can prove the following result which is nearly as good :

Proposition 11. *The 2-forms $\tilde{\omega}_t = tf^*\omega_0 + (1-t)k\omega$ on X are symplectic for all $t \in [0,1)$. Moreover, for $t \in [0,1)$ the manifolds $(X, \tilde{\omega}_t)$ are all symplectomorphic to $(X, k\omega)$.*

This means that $f^*\omega_0$ is, in some sense, a degenerate limit of the symplectic structure defined by $k\omega$: therefore the covering map f behaves quite reasonably with respect to the symplectic structures.

Proof. The 2-forms $\tilde{\omega}_t$ are all closed and lie in the same cohomology class. We have to show that they are non-degenerate for $t < 1$. For this, let x be any point of X and let v be a nonzero tangent vector at x. It is sufficient to prove that there exists a vector $w \in T_xX$ such that $\omega(v,w) > 0$ and $f^*\omega_0(v,w) \geq 0$: then $\tilde{\omega}_t(v,w) > 0$ for all $t < 1$, which implies the non-degeneracy of $\tilde{\omega}_t$.

Recall that, by definition, there exist local approximately holomorphic coordinate maps ϕ over a neighborhood of x and ψ over a neighborhood of $f(x)$ such that locally $f = \psi^{-1} \circ g \circ \phi$ where g is a holomorphic map from a subset of $\mathbb{C}^2$ to $\mathbb{C}^2$. Define $w = \phi_*^{-1}\mathbb{J}_0\phi_* v$, where $\mathbb{J}_0$ is the standard complex structure on $\mathbb{C}^2$: then we have $w = (\phi^*\mathbb{J}_0)v$ and, because g is holomorphic, $f_*w = (\psi^*\mathbb{J}_0)f_*v$.

Because the coordinate maps are $O(k^{-1/2})$-approximately holomorphic, we have $|w - Jv| \leq Ck^{-1/2}|v|$ and $|f_*w - J_0 f_*v| \leq Ck^{-1/2}|f_*v|$, where C is a constant and J_0 is the standard complex structure on $\mathbb{CP}^2$. It follows that $\omega(v,w) \geq |v|^2 - Ck^{-1/2}|v|^2 > 0$, and that $\omega_0(f_*v, f_*w) \geq |f_*v|^2 - Ck^{-1/2}|f_*v|^2 \geq 0$. Therefore, $\tilde{\omega}_t(v,w) > 0$ for all $t \in [0,1)$; since the existence of such a w holds for every nonzero vector v, this proves that the closed 2-forms $\tilde{\omega}_t$ are non-degenerate, and therefore symplectic.

Moreover, these symplectic forms all lie in the cohomology class $[k\omega]$, so it follows from Moser's stability theorem that the symplectic structures defined on X by $\tilde{\omega}_t$ for $t \in [0,1)$ are all symplectomorphic. □

6.2. **Symplectic Lefschetz pencils.** The techniques used in this paper can also be applied to the construction of sections of $\mathbb{C}^2 \otimes L^k$ (i.e. pairs of sections of L^k) satisfying appropriate transversality properties : this is the existence result for Lefschetz pencil structures (and uniqueness up to isotopy for a given value of k) obtained by Donaldson [3].

For the sake of completeness, we give here an overview of a proof of Donaldson's theorem using the techniques described in the above sections. Let (X, ω) be a compact symplectic manifold (of arbitrary dimension $2n$) such that $\frac{1}{2\pi}[\omega]$ is integral, and as before consider a compatible almost-complex structure J, the corresponding metric g, and the line bundle L whose first Chern class is $\frac{1}{2\pi}[\omega]$, endowed with a Hermitian connection of curvature $-i\omega$. The required properties of the sections we wish to construct are determined by the following statement :

Proposition 12. *Let $s_k = (s_k^0, s_k^1)$ be asymptotically holomorphic sections of $\mathbb{C}^2 \otimes L^k$ over X for all large k, which we assume to be η-transverse to 0 for some $\eta > 0$. Let $F_k = s_k^{-1}(0)$ (it is a real codimension 4 symplectic submanifold of X), and define the map $f_k = \mathbb{P}s_k = (s_k^0 : s_k^1)$ from $X - F_k$ to $\mathbb{CP}^1$. Assume furthermore that ∂f_k is η-transverse to 0, and that $\bar\partial f_k$ vanishes at every point where $\partial f_k = 0$. Then, for all large k, the section s_k and the map f_k define a structure of symplectic Lefschetz pencil on X.*

Indeed, F_k corresponds to the set of base points of the pencil, while the hypersurfaces $(\Sigma_{k,u})_{u\in\mathbb{CP}^1}$ forming the pencil are $\Sigma_{k,u} = f_k^{-1}(u) \cup F_k$, i.e. $\Sigma_{k,u}$ is the set of all points where (s_k^0, s_k^1) belongs to the complex line in $\mathbb{C}^2$ determined by u. The transversality to 0 of s_k gives the expected pencil structure near the base points, and the asymptotic holomorphicity implies that, near any point of $X - F_k$ where ∂f_k is not too small, the hypersurfaces $\Sigma_{k,u}$ are smooth and symplectic (and even approximately J-holomorphic).

Moreover, the transversality to 0 of ∂f_k implies that ∂f_k becomes small only in the neighborhood of finitely many points where it vanishes, and that at these points the holomorphic Hessian $\partial\partial f_k$ is large enough and nondegenerate. Because $\bar\partial f_k$ also vanishes at these points, an argument similar to that of §5.2 shows that, near its critical points, f_k behaves like a complex Morse function, i.e. it is locally approximately holomorphically modelled on the map $(z_1, \dots, z_n) \mapsto \sum z_i^2$ from $\mathbb{C}^n$ to $\mathbb{C}$.

The approximate holomorphicity of f_k and its structure at the critical points can be easily shown to imply that the hypersurfaces $\Sigma_{k,u}$ are all symplectic, and that only finitely many of them have isolated singular points, which correspond to the critical points of f_k and whose structure is therefore completely determined.

Therefore, the construction of a Lefschetz pencil structure on X can be carried out in three steps. The first step is to obtain for all large k sections s_k of $\mathbb{C}^2 \otimes L^k$ which are asymptotically holomorphic and transverse to 0 : for example, the existence of such sections follows immediately from the main result of [1]. As a consequence, the required properties are satisfied on a neighborhood of $F_k = s_k^{-1}(0)$.

The second step is to perturb s_k, away from F_k, in order to obtain the transversality to 0 of ∂f_k. For this purpose, one uses an argument similar to that of §2.2, but where Proposition 2 has to be replaced by a similar result for approximately holomorphic functions defined over a ball of $\mathbb{C}^n$ with values in $\mathbb{C}^n$ which has been announced by Donaldson (see [3]). Over a neighborhood of any given point $x \in X - F_k$, composing with a rotation of $\mathbb{C}^2$ in order to ensure the nonvanishing of s_k^0 over a ball centered at x and defining $h_k = (s_k^0)^{-1} s_k^1$, one remarks that the transversality to 0 of ∂f_k is locally equivalent to that of ∂h_k. Choosing local approximately holomorphic coordinates z_k^i, it is possible to write ∂h_k as a linear combination $\sum_{i=1}^n u_k^i \mu_k^i$ of the 1-forms $\mu_k^i = \partial(z_k^i.(s_k^0)^{-1} s_{k,x}^{\mathrm{ref}})$. The existence of $w_k \in \mathbb{C}^n$ of norm less than a given δ ensuring the transversality to 0 of $u_k - w_k$ over a neighborhood of x is then given by the suitable local transversality result, and it follows easily that the section $(s_k^0, s_k^1 - \sum w_k^i z_k^i s_{k,x}^{\mathrm{ref}})$ satisfies the required transversality property over a ball around x. The global result over the complement in X of a small neighborhood of F_k then follows by applying Proposition 3.

An alternate strategy allows one to proceed without proving the local transversality result for functions with values in $\mathbb{C}^n$, if one assumes s_k^0 and s_k^1 to be linear combinations of sections with uniform Gaussian decay (this is not too restrictive since the iterative process described in [1] uses precisely the sections $s^{\mathrm{ref}}_{k,x}$ as building blocks). In that case, it is possible to locally trivialize the cotangent bundle T^*X, and therefore work component by component to get the desired transversality result ; in a manner similar to the argument of [1], one uses Lemma 6 to reduce the problem to the transversality of sections of line bundles over submanifolds of X, and Proposition 6 as local transversality result. The assumption on s_k is used to prove the existence of asymptotically holomorphic sections which approximate s_k very well over a neighborhood of a given point $x \in X$ and have Gaussian decay away from x : this makes it possible to find perturbations with Gaussian decay which at the same time behave nicely with respect to the trivialization of T^*X. This way of obtaining the transversality to 0 of ∂f_k is very technical, so we don't describe the details.

The last step in the proof of Donaldson's theorem is to ensure that $\bar\partial f_k$ vanishes at the points where ∂f_k vanishes, by perturbing s_k by $O(k^{-1/2})$ over a neighborhood of these points. The argument is a much simpler version of §4.2 : on a neighborhood of a point x where ∂f_k vanishes, one defines a section χ of $f_k^* T\mathbb{CP}^1$ by $\chi(\exp_x(\xi)) = \beta(|\xi|)\, \bar\partial f_{k(x)}(\xi)$, where β is a cut-off function, and one uses χ as a perturbation of s_k in order to cancel the antiholomorphic derivative at x.

6.3. **Symplectic ampleness.** We have seen that similar techniques apply in various situations involving very positive bundles over a compact symplectic manifold, such as constructing symplectic submanifolds ([2],[1]), Lefschetz pencils [3], or covering maps to $\mathbb{CP}^2$. In all these cases, the result is the exact approximately holomorphic analogue of a classical result of complex projective geometry. Therefore, it is natural to wonder if there exists a symplectic analogue of the notion of ampleness : for example, the line bundle L endowed with a connection of curvature $-i\,\omega$, when raised to a sufficiently large power, admits many approximately holomorphic sections, and so it turns out that some of these sections behave like generic sections of a very ample bundle over a complex projective manifold.

Let (X, ω) be a compact $2n$-dimensional symplectic manifold endowed with a compatible almost-complex structure, and fix an integer r : it seems likely that any sufficiently positive line bundle over X admits $r+1$ approximately holomorphic sections whose behavior is similar to that of generic sections of a very ample line bundle over a complex projective manifold of dimension n. For example, the zero set of a suitable section is a smooth approximately holomorphic submanifold of X ; two well-chosen sections define a Lefschetz pencil ; for $r = n$, one expects that $n+1$ well-chosen sections determine an approximately holomorphic singular covering $X \to \mathbb{CP}^n$ (this is what we just proved for $n = 2$) ; for $r = 2n$, it should be possible to construct an approximately holomorphic immersion $X \to \mathbb{CP}^{2n}$, and for $r > 2n$ a projective embedding. Moreover, in all known cases, the space of "good" sections is connected when the line bundle is sufficiently positive, so that the structures thus defined are in some sense canonical up to isotopy.

However, the constructions tend to become more and more technical when one gets to the more sophisticated cases, and the development of a general theory of symplectic ampleness seems to be a necessary step before the relations between the approximately holomorphic geometry of compact symplectic manifolds and the ordinary complex projective geometry can be fully understood.

References

[1] D. Auroux, *Asymptotically Holomorphic Families of Symplectic Submanifolds*, Geom. Funct. Anal. **7** (1997) 971–995.
[2] S.K. Donaldson, *Symplectic Submanifolds and Almost-complex Geometry*, J. Differential Geom. **44** (1996) 666–705.
[3] S.K. Donaldson, to appear.
[4] P.A. Griffiths, *Entire Holomorphic Mappings in One and Several Complex Variables*, Ann. Math. Studies, No. 85, Princeton Univ. Press, Princeton, 1976.
[5] R. Paoletti, *Symplectic Subvarieties of Holomorphic Fibrations over Symplectic Manifolds*, preprint.
[6] E. Schmidt, *Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie*, Math. Nachrichten **1** (1948) 81–157.
[7] H. Whitney, *On Singularities of Mappings of Euclidean Spaces I. Mappings of the Plane into the Plane*, Ann. of Math. **62** (1955) 374–410.

Centre de Mathématiques, Ecole Polytechnique, 91128 Palaiseau, France
E-mail address: auroux@math.polytechnique.fr

BRANCHED COVERINGS OF $\mathbb{CP}^2$ AND INVARIANTS OF SYMPLECTIC 4-MANIFOLDS

DENIS AUROUX AND LUDMIL KATZARKOV

Contents

1. Introduction

Symplectic manifolds are an important class of four-dimensional manifolds. The recent work of Seiberg and Witten [21], Taubes [19], Fintushel and Stern [10] has improved drastically our understanding of the topology of four-dimensional symplectic manifolds, based on the use of the Seiberg-Witten invariants.

Recent remarkable results by Donaldson ([8],[9]) open a new direction in conducting investigations of four-dimensional symplectic manifolds by analogy with projective surfaces. He has shown that every four-dimensional symplectic manifold has a structure of symplectic Lefschetz pencil. Using Donaldson's technique of asymptotically holomorphic sections the first author has constructed symplectic maps to $\mathbb{CP}^2$ [3]. In this paper we elaborate on ideas of [3], [8] and [9] in order to adapt the braid monodromy techniques of Moishezon and Teicher from the projective case to the symplectic case.

Our two primary directions are as follows :

(1) To classify, in principle, four-dimensional symplectic manifolds, using braid monodromies. We define new invariants of symplectic manifolds arising from symplectic maps to $\mathbb{CP}^2$, by adapting the braid monodromy technique to the symplectic situation ;

(2) To compute these invariants in some examples.

We will show some computations with these invariants in a sequel of this paper. Here we concentrate on the first direction.

The second author was partially supported by NSF Grant DMS-9700605 and A.P. Sloan research fellowship.

Recall from [3] that a compact symplectic 4-manifold can be realized as an approximately holomorphic branched covering of $\mathbb{CP}^2$. More precisely, let (X, ω) be a compact symplectic 4-manifold, and assume the cohomology class $\frac{1}{2\pi}[\omega]$ to be integral. Fix an ω-compatible almost-complex structure J and the corresponding Riemannian metric g. Let L be a line bundle on X whose first Chern class is $\frac{1}{2\pi}[\omega]$, endowed with a Hermitian metric and a Hermitian connection of curvature $-i\omega$ (more than one such bundle L exists if $H^2(X, \mathbb{Z})$ contains torsion ; any choice will do). Then, for $k \gg 0$, the line bundles L^k admit many approximately holomorphic sections, and the main result of [3] states that for large enough k three suitably chosen sections of L^k determine X as an approximately holomorphic branched covering of $\mathbb{CP}^2$. This branched covering is, in local approximately holomorphic coordinates, modelled at every point of X on one of the holomorphic maps $(x, y) \mapsto (x, y)$ (local diffeomorphism), $(x, y) \mapsto (x^2, y)$ (branched covering), or $(x, y) \mapsto (x^3 - xy, y)$ (cusp). Moreover, the constructed coverings are canonical for large enough k, and their topology is a symplectic invariant (it does not even depend on the chosen almost-complex structure).

Although the concept of approximate holomorphicity mostly makes sense for sequences obtained for increasing values of k, we will for convenience sometimes consider an individual approximately holomorphic map or curve, by which it should be understood that the discussion applies to any map or curve belonging to an approximately holomorphic sequence provided that k is large enough.

The topology of a branched covering of $\mathbb{CP}^2$ is mostly described by that of the image $D \subset \mathbb{CP}^2$ of the branch curve ; this singular curve in $\mathbb{CP}^2$ is symplectic and approximately holomorphic. In the case of a complex curve, the braid group techniques developed by Moishezon and Teicher can be used to investigate its topology : the idea is that, fixing a generic projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$, the monodromy of $\pi_{|D}$ around its critical levels can be used to define a map from $\pi_1(\mathbb{CP}^1 - \text{crit})$ with values in the braid group B_d on $d = \deg D$ strings, called braid monodromy (see e.g. [16],[17],[20]).

The set of critical levels of $\pi_{|D}$, denoted by crit $= \{p_1, \ldots, p_r\}$, consists of the images by π of the singular points of D (generically double points and cusps) and of the smooth points of D where it becomes tangent to the fibers of π ("vertical"). Recall that, denoting by D' a closed disk in $\mathbb{C}$ and by $L = \{q_1, \ldots, q_d\}$ a set of d points in D', the braid group B_d can be defined as the group of equivalence classes of diffeomorphisms of D' which map L to itself and restrict as the identity map on the boundary of D', where two diffeomorphisms are equivalent if and only if they define the same automorphism of $\pi_1(D' - \{q_1, \ldots, q_d\})$. Elements of B_d may also be thought of as motions of d points in the plane.

Since the fibers of π are complex lines, every loop in $\mathbb{CP}^1 - \text{crit}$ induces a motion of the d points in a fiber of $\pi_{|D}$, which after choosing a trivialization of π can be considered as a braid. Since such a trivialization is only available over an affine subset $\mathbb{C} \subset \mathbb{CP}^1$, the braid monodromy should be considered as a group homomorphism from the free group $\pi_1(\mathbb{C} - \text{crit})$ to B_d. Alternately, the braid monodromy can also be encoded by a factorization of the braid Δ_d^2 (the central element in B_d corresponding to a full twist of the d points by an angle of 2π) as a product of powers of half-twists in the braid group B_d (see below).

In any case, it is not clear from the result in [3] that in our case the curve D admits a nice projection to $\mathbb{CP}^1$. It is the aim of Sections 2 and 3 to explain how the proofs of the main results in [3] can be modified in such a way that the existence of a nice projection is guaranteed. The notations and techniques are those of [3].

More precisely, recall that in the result of [3] the branch curves $D = f(R)$ are approximately holomorphic symplectic curves in $\mathbb{CP}^2$ which are immersed everywhere except at a finite number of cusps. We now wish to add the following conditions :

1. $(0:0:1) \notin D$.
2. The curve D is everywhere transverse to the fibers of the projection $\pi : \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$ defined by $\pi(x:y:z) = (x:y)$, except at finitely many points where it becomes nondegenerately tangent to the fibers. A local model in approximately holomorphic coordinates is then $z_2^2 = z_1$ (with projection to the z_1 coordinate).
3. The cusps are not tangent to the fibers of π.
4. D is transverse to itself, i.e. its only singularities besides the cusps are transverse double points, which may have either positive or negative self-intersection number, and the projection of R to D is injective outside of the double points.
5. The "special points", i.e. cusps, double points and tangency points, are all distinct and lie in different fibers of the projection π.
6. In a 1-parameter family of curves obtained from an isotopy of branched coverings as described in [3], the only admissible phenomena are creation or cancellation of a pair of transverse double points with opposite orientations (self-transversality is of course lost at the precise parameter value where the cancellation occurs).

Definition 1. *Approximately holomorphic symplectic curves satisfying these six conditions will be called* quasiholomorphic curves.

We will call quasiholomorphic covering *an approximately holomorphic branched covering $f : X \to \mathbb{CP}^2$ whose branch curve is quasiholomorphic.*

An isotopy of quasiholomorphic coverings *is a continuous one-parameter family of branched coverings, all of which are quasiholomorphic except for finitely many parameter values where a pair of transverse double points is created or removed in the branch curve.*

Clearly the idea behind the definition of a quasiholomorphic covering is to imitate the case of holomorphic coverings. Our main theorem is :

Theorem 1. *For every compact symplectic 4-manifold X there exist quasiholomorphic coverings $f_k : X \to \mathbb{CP}^2$ defined by asymptotically holomorphic sections of the bundle L^k for $k \gg 0$.*

Moreover, we will show in Section 3.2 that, for large enough k, the quasiholomorphic coverings obtained by this procedure are unique up to isotopy (Theorem 5).

From the work of Moishezon and Teicher, we know that the braid monodromy describing a branch curve in $\mathbb{CP}^2$ is given by a *braid factorization.* Namely, the braid monodromy around the point at infinity in $\mathbb{CP}^1$, which is given by the central element Δ_d^2 in B_d (because π determines a line bundle of degree 1 over $\mathbb{CP}^1$), decomposes as the product of the monodromies around the critical levels $p_1, \dots, p_r$ of the projection π. Easy computations in local coordinates show that each of these factors is a power of a half-twist (a half-twist corresponds to the motion of two points being exchanged along a certain path and rotating around each other by a positive half-turn, while the $d-2$ other points remain fixed).

More precisely, the braid monodromy around a point where D is smooth but tangent to the fibers of π is given by a half-twist (the two sheets of the covering $\pi_{|D}$ which come together at the tangency point are exchanged when one moves around the tangency point) ; the braid monodromy around a double point of D with positive self-intersection is the square of a half-twist ; the braid monodromy around a cusp of D is the cube of a half-twist ; finally, the monodromy around a double point with negative self-intersection is the square of a reversed half-twist. Observing that any two half-twists in B_d are conjugate to each other, the braid factorization can be expressed as

$$\Delta_d^2 = \prod_{j=1}^{r} (Q_j^{-1}\, X_1^{r_j}\, Q_j),$$

where X_1 is a positive half-twist exchanging q_1 and q_2, Q_j is any braid, and $r_j \in \{-2, 1, 2, 3\}$. The case $r_j = -2$ corresponds to a negative self-intersection, $r_j = 1$ to a tangency point, $r_j = 2$ to a nodal point, and $r_j = 3$ to a cusp. The braids Q_j are of course only determined up to left multiplication by an element in the commutator of $X_1^{r_j}$.

For example, the standard factorization $\Delta_d^2 = (X_1....X_{d-1})^d$ of Δ_d^2 in terms of the $d-1$ generating half-twists in B_d corresponds to the braid monodromy of a smooth algebraic curve of degree d in $\mathbb{CP}^2$.

With this understood, there are four types of factorizations of Δ_d^2 that we can consider (each class in contained in the next one) :

1) Holomorphic – coming from the braid monodromy of the branch curve of a generic projection of an algebraic surface to $\mathbb{CP}^2$.

2) Geometric – if after complete regeneration, it is (Hurwitz and conjugation) equivalent to the basic factorization $\Delta_d^2 = (X_1 \dots X_{d-1})^d$.

3) Cuspidal – all factors are positive of degree 1, 2 or 3.

4) Cuspidal negative – all factors are of degree -2, 1, 2 or 3.

Moishezon has shown [17] that the geometric factorizations are a much larger class than the holomorphic ones. We do not know examples of cuspidal factorizations that are not geometric. We will prove in Section 4 that cuspidal negative factorizations correspond to symplectic four-manifolds.

The braid factorization describing a curve D with cusps and (possibly negative) nodes makes it possible to compute explicitly the fundamental group of its complement in $\mathbb{CP}^2$, an approach which has led to a series of papers by Moishezon and Teicher in the algebraic case (see e.g. [20]). Consider a generic fiber $\mathbb{C} \subset \mathbb{CP}^2$ of the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$, and call once again $q_1, \dots, q_d$ the d distinct points in which it intersects D. Then, the inclusion of $\mathbb{C} - \{q_1, \dots, q_d\}$ into $\mathbb{CP}^2 - D$ induces a surjective homomorphism on the fundamental groups. Small loops $\gamma_1, \dots, \gamma_d$ around $q_1, \dots, q_d$ in $\mathbb{C}$ generate $\pi_1(\mathbb{CP}^2 - D)$, with relations coming from the cusps, nodes and tangency points of D. These d loops will be called *geometric generators* of $\pi_1(\mathbb{CP}^2 - D)$.

The fundamental group $\pi_1(\mathbb{CP}^2 - D)$ is generated by $\gamma_1, \dots, \gamma_d$, with the relation $\gamma_1 \dots \gamma_d \sim 1$ and one additional relation coming from each of the factors in the braid factorization :

$$\begin{aligned} \gamma_1 * Q_j \sim \gamma_2 * Q_j \qquad & \text{if } r_j = 1, \\ [\gamma_1 * Q_j, \gamma_2 * Q_j] \sim 1 \qquad & \text{if } r_j = \pm 2, \\ (\gamma_1\gamma_2\gamma_1) * Q_j \sim (\gamma_2\gamma_1\gamma_2) * Q_j \qquad & \text{if } r_j = 3, \end{aligned}$$

where $*$ is the right action of B_d on the free group $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) = \langle \gamma_1, \dots, \gamma_d \rangle$, and Q_j and r_j are the braids and exponents appearing in the braid factorization.

In order to describe a map $X \to \mathbb{CP}^2$ we also need a *geometric monodromy representation*, encoding the way in which the various sheets of the covering come together along the branch curve. Recall the following definition [17] :

Definition 2. *A* geometric monodromy representation *associated to the curve $D \subset \mathbb{CP}^2$ is a surjective group homomorphism θ from the free group F_d to the symmetric group S_n of order n, such that the $\theta(\gamma_i)$ are transpositions (thus also the $\theta(\gamma_i * Q_j)$) and*

$\theta(\gamma_1 \dots \gamma_d) = 1,$
$\theta(\gamma_1 * Q_j) = \theta(\gamma_2 * Q_j)$ *if* $r_j = 1,$
$\theta(\gamma_1 * Q_j)$ *and* $\theta(\gamma_2 * Q_j)$ *are distinct and commute if* $r_j = \pm 2,$
$\theta(\gamma_1 * Q_j)$ *and* $\theta(\gamma_2 * Q_j)$ *do not commute (and hence satisfy a relation of the type* $\sigma\tau\sigma = \tau\sigma\tau$*) if* $r_j = 3$.

In this definition, n corresponds to the number of sheets of the covering $X \to \mathbb{CP}^2$; the various conditions imposed on $\theta(\gamma_i * Q_j)$ express the natural requirements that the map $\theta : F_d \to S_n$ should factor through the group $\pi_1(\mathbb{CP}^2 - D)$ and that the branching phenomena should occur in disjoint sheets of the covering for a node and in adjacent sheets for a cusp. Note that the surjectivity of θ corresponds to the connectedness of the covering 4-manifold.

The braid factorization and the geometric monodromy representation are not entirely canonical, because choices were made both when labelling the points $p_1, \dots, p_r$ in the base $\mathbb{CP}^1$ and when labelling the points $q_1, \dots, q_d$ in the fiber of π.

A change in the ordering of the points $q_1, \dots, q_d$ corresponds to the operation called *global conjugation* : all the factors in the braid factorization are simultaneously conjugated by some braid $Q \in B_d$, and the geometric monodromy representation is affected accordingly. More algebraically, let $Q \in B_d$ be any braid, and let $Q_* \in \mathrm{Aut}(F_d)$ be the automorphism of $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ induced by Q. Then, given a pair $(\{(Q_j, r_j)\}_{1 \le j \le r}, \theta)$ consisting of a braid factorization and a geometric monodromy representation, global conjugation by the braid Q leads to the pair $(\{(\tilde{Q}_j, r_j)\}_{1 \le j \le r}, \tilde{\theta})$, where $\tilde{Q}_j = Q_j\, Q^{-1}$ and $\tilde{\theta} = \theta \circ Q_*$.

A change in the ordering of the points $p_1, \dots, p_r$ corresponds to the operation called *Hurwitz equivalence* : the factors in the braid factorization are permuted. A Hurwitz equivalence amounts to a sequence of *Hurwitz moves*, where two consecutive factors A and B in the braid factorization are replaced respectively by ABA^{-1} and A (or B and $B^{-1}AB$, depending on which way the move is performed). The geometric monodromy representation is not affected.

By Theorem 1 we have quasiholomorphic covering maps $f_k : X \to \mathbb{CP}^2$ and, as noted above, the discriminant curves D_k might have negative intersections. Some of these negative intersections are paired with positive ones : in this case, deformations of the curve D_k make it possible to remove a pair of intersection points with opposite orientations, which leads to a new curve D'_k. This operation affects the braid monodromy, and even the fundamental group of the complement is modified : $\pi_1(\mathbb{CP}^2 - D_k)$ is the quotient of $\pi_1(\mathbb{CP}^2 - D'_k)$ by the subgroup generated by the commutator of the two geometric generators which come together at the intersection points.

Applying this procedure we can remove some pairs of positive and negative intersections : this is what we call a *cancellation operation*, which amounts to removing two consecutive factors which are the inverse of each other in the braid factorization (necessarily one of these factors must have degree 2 and the other degree -2). The geometric monodromy representation is not affected.

The opposite operation is the creation of a pair of intersections and corresponds to adding $(Q^{-1} X_1^{-2} Q).(Q^{-1} X_1^2 Q)$ anywhere in the braid factorization. It can only be performed if the new factorization remains compatible with the geometric monodromy representation, i.e. if $\theta(\gamma_1 * Q)$ and $\theta(\gamma_2 * Q)$ are commuting disjoint transpositions.

Definition 3. *We will say that two pairs (F_1, θ_1) and (F_2, θ_2) (where F_i are braid factorizations and θ_i are geometric monodromy representations) are m-equivalent if there exists a sequence of operations which turn one into the other, each operation being either a global conjugation, a Hurwitz move, or a pair cancellation or creation.*

We will prove in Section 3 that the coverings obtained in Theorem 1 are unique up to isotopies of quasiholomorphic coverings. This allows us to define new invariants of symplectic manifolds in Section 4. As a result we get:

Theorem 2. *Every compact symplectic 4-manifold with $\frac{1}{2\pi}[\omega]$ integral is uniquely characterized by the sequence of cuspidal negative braid factorizations and geometric monodromy representations corresponding to the quasiholomorphic coverings of $\mathbb{CP}^2$ canonically obtained for $k \gg 0$, up to m-equivalence.*

If $H^2(X, \mathbb{Z})$ contains torsion, one must either specify a choice of the line bundle L or consider the braid monodromy invariants obtained for all possible choices of L. For general compact symplectic 4-manifolds, a perturbation of ω is required in order to satisfy the integrality condition, so one only obtains a classification up to symplectic deformation (pseudo-isotopy).

Theorem 2 transforms the classification of symplectic four-manifolds into a purely algebraic problem (which is probably quite difficult), namely showing that two words in the braid group (and the accompanying geometric monodromy representations) are m-equivalent.

Remark 1. A different way to state Theorem 2 is to say that 4-dimensional symplectic manifolds are classified up to symplectic deformation (or up to isotopy if one adds the integrality constraint on $\frac{1}{2\pi}[\omega]$) by the sequence of braid factorizations and geometric monodromy representations obtained for $k \gg 0$ up to m-equivalence.

The presence of negative nodes in the branch curves given by Theorem 1 seems to be mostly due to the technique of proof. It seems plausible that these negative nodes can be removed for large k, which gives the following conjecture :

Conjecture 1. *Every compact symplectic 4-manifold with* $\frac{1}{2\pi}[\omega]$ *integral is uniquely characterized by a sequence of cuspidal braid factorizations and geometric monodromy representations corresponding to quasiholomorphic coverings of* $\mathbb{CP}^2$ *canonically obtained for* $k \gg 0$*, up to Hurwitz and conjugation equivalence.*

This conjecture would make easier the algebraic problem raised by Theorem 2.

Conversely, given a cuspidal negative braid factorization and a geometric monodromy representation one can reconstruct a quasiholomorphic curve and a quasiholomorphic covering. A similar result has also been obtained by F. Catanese ; see also the remark in [17], p. 157, for a statement similar to the first part of this result.

Theorem 3. 1) *To every cuspidal negative factorization of* Δ_d^2 *corresponds a quasiholomorphic curve, canonical up to smooth isotopy.*

2) *Let* D *be a quasiholomorphic curve of degree* d *and let* $\theta : F_d \to S_n$ *be a geometric monodromy representation. Then there exists a symplectic 4-manifold* X *which covers* $\mathbb{CP}^2$ *and ramifies at* D*. Moreover the symplectic structure on* X *is canonical up to symplectomorphism, and depends only on the smooth isotopy class of the curve* D*.*

We will also show in §4 that, when (X, ω) is a symplectic 4-manifold and D is the branch curve of a quasiholomorphic covering $X \to \mathbb{CP}^2$ given by three sections of L^k as in Theorem 1, the symplectic structure ω' on X given by assertion 2) of Theorem 3 coincides with $k\omega$ up to symplectomorphism : therefore the construction of Theorem 3 is the exact converse of that of Theorem 2.

We also show (in Section 5) that quasiholomorphic coverings and symplectic Lefschetz pencils are quite closely connected :

Theorem 4. *The quasiholomorphic coverings of* $\mathbb{CP}^2$ *given by three asymptotically holomorphic sections of* L^k *as in Theorem 1 determine symplectic Lefschetz pencils in a canonical way.*

Remark 2. This gives a different proof of Donaldson's theorem of existence of Lefschetz pencil structures on any symplectic 4-manifold with $\frac{1}{2\pi}[\omega]$ integral [8]. Since the Lefschetz pencils we obtain are actually given by two asymptotically holomorphic sections of L^k, they are clearly identical to the ones constructed by Donaldson (up to isotopy).

In Section 5 we will also provide a more topological version of this result, and describe how the monodromy of the Lefschetz pencil can be derived quite easily from that of the branched covering.

Acknowledgments: We are very grateful to F. Bogomolov, M. Gromov and R. Stern for their constant attention to this work. Special thanks to S. Donaldson – without his suggestions this work could not be finished. We were also informed that S. Donaldson and I. Smith have a different approach to some of the above-discussed problems. We would like to thank V. Kulikov and M. Teicher for many discussions and for sharing with us their preprint [15]. We also thank F. Catanese for sharing with us ideas about Theorem 3. Finally, we thank the referee for the careful reading and numerous suggestions improving the exposition.

2. Compatibility of branch curves with a projection to $\mathbb{CP}^1$

We first prove a couple of technical propositions that allow us to extend the results from [3] and prove Theorem 1.

Recall that the main results of [3] are obtained by constructing, for large enough k, sections $s_k = (s_k^0, s_k^1, s_k^2)$ of the vector bundles $\mathbb{C}^3 \otimes L^k$ over X which are asymptotically holomorphic, γ-generic for some $\gamma > 0$, and satisfy a $\bar\partial$-tameness condition. One then shows that these properties imply that the corresponding projective maps $f_k = (s_k^0 : s_k^1 : s_k^2) : X \to \mathbb{CP}^2$ are approximately holomorphic branched coverings. For the sake of completeness we briefly recall the definitions (see [3] for more details) :

Definition 4. *Let $(s_k)_{k\gg 0}$ be a sequence of sections of $C^3 \otimes L^k$ over X. The sections s_k are said to be* asymptotically holomorphic *if they are uniformly bounded in all C^p norms by constants independent of k and if their antiholomorphic derivatives $\bar\partial s_k = (\nabla s_k)^{(0,1)}$ are bounded in all C^p norms by $O(k^{-1/2})$. In these estimates the norms of the derivatives have to be evaluated using the rescaled metrics $g_k = k\, g$ on X.*

Definition 5. *Let s_k be a section of a complex vector bundle E_k, and let $\gamma > 0$ be a constant. The section s_k is said to be* γ-transverse to 0 *if, at any point $x \in X$ where $|s_k(x)| < \gamma$, the covariant derivative $\nabla s_k(x) : T_x X \to (E_k)_x$ is surjective and has a right inverse of norm less than γ^{-1} w.r.t. the metric g_k.*

We will often omit the transversality estimate γ when considering a sequence of sections $(s_k)_{k\gg 0}$: in that case the existence of a uniform transversality estimate which does not depend on k will be implied.

Definition 6. *Let s_k be nowhere vanishing asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$, and fix a constant $\gamma > 0$. Define the projective maps $f_k = \mathbb{P}s_k$ from X to $\mathbb{CP}^2$ as $f_k(x) = (s_k^0(x) : s_k^1(x) : s_k^2(x))$. Define the $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$, and let $R(s_k)$ be the set of points of X where $\mathrm{Jac}(f_k)$ vanishes, i.e. where ∂f_k is not surjective. We say that s_k has the transversality property $\mathcal{P}_3(\gamma)$ if $|s_k| \geq \gamma$ and $|\partial f_k|_{g_k} \geq \gamma$ at every point of X, and if $\mathrm{Jac}(f_k)$ is γ-transverse to 0.*

Assume that s_k satisfies $\mathcal{P}_3(\gamma)$: if k is large enough this implies that $R(s_k)$ is a smooth symplectic submanifold in X. At a point of $R(s_k)$, ∂f_k has complex rank one, so we can consider the quantity $\mathcal{T}(s_k) = \partial f_k \wedge \partial \mathrm{Jac}(f_k)$ as a section of a line bundle over $R(s_k)$.

We say that s_k is γ-generic *if it satisfies $\mathcal{P}_3(\gamma)$ and if $\mathcal{T}(s_k)$ is γ-transverse to 0 over $R(s_k)$. We then define the set of cusp points $\mathcal{C}(s_k)$ as the set of points of $R(s_k)$ where $\mathcal{T}(s_k) = 0$.*

Definition 7. *Let s_k be γ-generic asymptotically J-holomorphic sections of $\mathbb{C}^3 \otimes L^k$. We say that the sections s_k are* $\bar\partial$-tame *if there exist ω-compatible almost complex structures $\tilde J_k$, such that $|\tilde J_k - J| = O(k^{-1/2})$, $\tilde J_k$ coincides with J away from the cusp points, and $\tilde J_k$ is integrable over a small neighborhood of $\mathcal{C}(s_k)$, with the following properties :*

(1) *the map $f_k = \mathbb{P}s_k$ is $\tilde J_k$-holomorphic over a small neighborhood of $\mathcal{C}(s_k)$;*

(2) *at every point of $R(s_k)$, the antiholomorphic derivative $\bar\partial(\mathbb{P}s_k)$ vanishes over the kernel of $\partial(\mathbb{P}s_k)$.*

Note that γ-genericity is an open condition, and therefore stable under small perturbations (up to decreasing γ). The existence of γ-generic sections of $\mathbb{C}^3 \otimes L^k$ follows from Propositions 1, 4, 5 and 7 of [3] ; $\bar{\partial}$-tameness is then enforced by a small perturbation ($O(k^{-1/2})$), the aim of which is to cancel the antiholomorphic derivatives of the projective map $f_k = \mathbb{P}s_k$ at the points of the branch curve $R(s_k) \subset X$ and at the cusp points $\mathcal{C}(s_k) \subset X$. This process yields asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ which simultaneously have genericity and $\bar{\partial}$-tameness properties, and therefore gives rise to branched coverings.

For the enhanced result we wish to obtain here, the beginning of the proof is the same : one first constructs asymptotically holomorphic sections which are γ-generic exactly as in [3]. However, we need to add an extra transversality requirement, in order to prepare the ground for obtaining the properties 1, 2 and 3 of a quasiholomorphic covering (see Introduction).

Proposition 1. *Let $(s_k)_{k \gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$, and fix a constant $\epsilon > 0$. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3, g_k} \leq \epsilon$ and that the sections (σ_k^0, σ_k^1) of $\mathbb{C}^2 \otimes L^k$ are η-transverse to 0 over X. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0, 1]$.*

This is precisely a restatement of the main result of [2], applied to the case of the asymptotically holomorphic sections (s_k^0, s_k^1) of $\mathbb{C}^2 \otimes L^k$. The bound by ϵ is stated here in C^3 norm rather than C^1 norm, but as explained in [3] this is not relevant since such bounds automatically hold in all C^p norms (see the statement of Lemma 2 in [3] : because the uniform decay properties of the sections $s_{k,x}^{\mathrm{ref}}$ hold in C^3 norm, the size of the perturbation is controlled in C^3 norm as well). □

By choosing ϵ much smaller than γ and applying this result to γ-generic sections, one can therefore add the extra requirement that, decreasing γ if necessary, the sections (s_k^0, s_k^1) are γ-transverse to 0 for large enough k. This transversality result means that, wherever s_k^0 and s_k^1 are both smaller in norm than $\frac{\gamma}{3}$, the differential of (s_k^0, s_k^1) is surjective and larger than γ. Moreover, by the definition of γ-genericity the section s_k remains everywhere larger in norm than γ, so at such points one has $|s_k^2| \geq \frac{\gamma}{3}$. It is then easy to check that, because the derivatives of s_k^2 are uniformly bounded, there exists of a constant $\tilde{\gamma} \in (0, \frac{\gamma}{3})$, independent of k, such that, at any point of X where s_k^0 and s_k^1 are smaller than $\tilde{\gamma}$, the map $f_k = \mathbb{P}s_k$ is a local diffeomorphism. As a consequence, the points where s_k^0 and s_k^1 are smaller than $\tilde{\gamma}$ cannot belong to the set of branch points $R(s_k)$; therefore, because $|s_k^2|$ is uniformly bounded over X, there exists a constant $\tilde{\tilde{\gamma}} > 0$ such that $D(s_k) = f_k(R(s_k))$ remains at distance more than $\tilde{\tilde{\gamma}}$ from $(0:0:1)$. By requiring all perturbations in the following steps of the proof to be sufficiently small (in comparison with $\tilde{\tilde{\gamma}}$), one can ensure that such a condition continues to hold in all the rest of the proof ; this already gives the required property 1, and more importantly makes it possible to obtain another transversality condition which is vital to obtain properties 2 and 3.

At all points where s_k^0 and s_k^1 do not vanish simultaneously, including (by the above argument) a neighborhood of $R(s_k)$, define $\phi_k = (s_k^0 : s_k^1)$ (ϕ_k is a function with values in $\mathbb{CP}^1$). What we wish to require is that the restriction to $TR(s_k)$ of

$\partial\phi_k$ be transverse to 0 over $R(s_k)$, with some uniform estimates ; alternately this can be expressed in the following terms :

Definition 8. *A section $s_k \in \Gamma(\mathbb{C}^3 \otimes L^k)$ is said to be γ-transverse to the projection if the quantity $\mathcal{K}(s_k) = \partial\phi_k \wedge \partial\mathrm{Jac}(f_k)$ is γ-tranverse to 0 over $R(s_k)$ (as a section of a line bundle).*

Another equivalent criterion (up to a change in the constants), by Lemma 6 of [3], is the γ-transversality to 0 of the quantity $\mathrm{Jac}(f_k) \oplus \mathcal{K}(s_k)$ (as a section of a rank 2 bundle) over a neighborhood of $R(s_k)$; this allows us to use the globalization principle described in Proposition 3 of [3] in order to obtain the required property by applying successive local perturbations (note that the property we have just defined is local and C^3-open in the terminology of [3]). The argument below is very close to that in §3.2 of [3], except that the property first needs to be established by hand near the cusp points before the machinery of [3] can be applied to obtain transversality everywhere else.

Proposition 2. *Let δ and γ be two constants such that $0 < \delta < \frac{\gamma}{4}$, and let $(s_k)_{k\gg 0}$ be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$ which are γ-generic and such that (s_k^0, s_k^1) is γ-transverse to 0. Then there exists a constant $\eta > 0$ such that, for all large enough values of k, there exist asymptotically holomorphic sections σ_k of $\mathbb{C}^3 \otimes L^k$ such that $|\sigma_k - s_k|_{C^3,g_k} \le \delta$ and that the sections σ_k are γ-transverse to the projection. Moreover, the same statement holds for families of sections indexed by a parameter $t \in [0,1]$.*

Proof. **Step 1.** We first define, near a point $x \in X$ which lies in a small neighborhood of $R(s_k)$, an equivalent expression for $\mathcal{K}(s_k)$ in local coordinates. First, composing with a rotation in $\mathbb{C}^2$ (constant over X, and acting on the first two components s_k^0 and s_k^1), we can assume that $s_k^1(x) = 0$ and therefore $|s_k^0(x)| \ge \tilde\gamma$ for some constant $\tilde\gamma > 0$ independent of k (because of the transversality to 0 of (s_k^0, s_k^1)). Consequently s_k^0 remains bounded away from 0 over a ball of fixed radius around x. It follows that over this small ball we can consider, rather than f_k, the map

$$h_k(y) = (h_k^1(y), h_k^2(y)) = \Big(\frac{s_k^1(y)}{s_k^0(y)}, \frac{s_k^2(y)}{s_k^0(y)}\Big).$$

In this setup, $\mathrm{Jac}(f_k)$ can be replaced by $\mathrm{Jac}(h_k) = \partial h_k^1 \wedge \partial h_k^2$, and ϕ_k can be replaced by $s_k^1/s_k^0 = h_k^1$. Therefore, set

$$\hat{\mathcal{K}}(s_k) = \partial h_k^1 \wedge \partial\mathrm{Jac}(h_k).$$

The same argument as in §3.2 of [3] proves that the transversality to 0 of $\mathcal{K}(s_k)$ over a small ball $B_{g_k}(x,r) \cap R(s_k)$ is equivalent, up to a change in constants, to that of $\hat{\mathcal{K}}(s_k)$: the key remark is that the ratio between $\mathrm{Jac}(f_k)$ and $\mathrm{Jac}(h_k)$ is the jacobian of the map $\iota : (z_1, z_2) \mapsto [1 : z_1 : z_2]$ from $\mathbb{C}^2$ to $\mathbb{CP}^2$ (which is a quasi-isometry over a neighborhood of $h_k(x)$), and therefore has bounded derivatives and remains bounded both from below and above over a neighborhood of x ; and similarly for the ratio between $\partial\phi_k$ and ∂h_k^1, which is the jacobian of the locally quasi-isometric map $\iota' : z_1 \mapsto [1 : z_1]$ from $\mathbb{C}$ to $\mathbb{CP}^1$.

Therefore, in order to be able to apply Proposition 3 of [3] to obtain the desired result, we only need to show that there exist constants p, c and $c' > 0$ such that, if k is large enough and if $B_{g_k}(x,c) \cap R(s_k) \ne \emptyset$, then by adding to s_k a perturbation smaller than δ and with gaussian decay away from x it is possible to ensure the η-transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ over $B_{g_k}(x,c) \cap R(s_k)$, where $\eta = c'\delta(\log \delta^{-1})^{-p}$.

Step 2. In this step we wish to obtain transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ over a neighborhood of the set of cusp points $\mathcal{C}(s_k)$. For this, recall that, by the assumption of γ-genericity, the quantity $\mathcal{T}(s_k) = \partial f_k \wedge \partial \mathrm{Jac}(f_k)$, which by definition vanishes at the cusp points, is γ-transverse to 0 over $R(s_k)$. It follows that at any cusp point $x \in \mathcal{C}(s_k)$, using the notations of Step 1, at least one of the two quantities $\partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ and $\partial h_k^2 \wedge \partial \mathrm{Jac}(h_k)$, which both vanish at x, has a derivative along $R(s_k)$ larger than some constant γ' (independent of k). So there are two cases : the first possibility is that $\hat{\mathcal{K}}(s_k) = \partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ has a derivative at x along $R(s_k)$ bounded away from zero by γ'. In that case, i.e. when the derivative $\partial h_k(x)$ has a sufficiently large first component, or equivalently when the limit tangent space to $D(s_k)$ at the cusp point lies sufficiently away from the direction of the fibers of π, no perturbation of s_k is necessary to achieve the required property over a small neighborhood of x. Note by the way that this geometric criterion is consistent with the observation that the transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ at the cusp points precisely corresponds to the required property 3, i.e. the cusps not being tangent to the fibers of the projection to $\mathbb{CP}^1$.

The second case corresponds to the situation where the cusp of $D(s_k)$ at $f_k(x)$ is nearly tangent to the fiber of π. In that case, a perturbation of s_k is necessary in order to move the direction of the cusp away from the fiber and achieve the required transversality property. The norm of $\partial h_k^1(x)$ can be assumed to be as small as needed (smaller than any given fixed constant independent of k, since if it were larger the cusp at x would actually satisfy the first alternative for a suitable choice of γ' and no perturbation would be necessary). The transversality properties of s_k then imply that $\partial h_k^2(x)$ is bounded away from 0 by a fixed constant, and so does the restriction to $R(s_k)$ of $\partial(\partial h_k^2 \wedge \partial \mathrm{Jac}(h_k))(x)$.

Consider local approximately holomorphic Darboux coordinates (z_k^1, z_k^2) on a neighborhood of x as given by Lemma 3 of [3], and let $s_{k,x}^{\mathrm{ref}}$ be an approximately holomorphic section of L^k with gaussian decay away from x as given by Lemma 2 of [3]. Let λ be the polynomial function of degree 3 in z_k^1, z_k^2 and their complex conjugates obtained by keeping the degree 1, 2 and 3 terms of the Taylor series expansion of $h_k^2\, s_k^0/s_{k,x}^{\mathrm{ref}}$ at x : λ vanishes at x, and the function $\tilde{\lambda} = \lambda s_{k,x}^{\mathrm{ref}}/s_k^0$ has the property that $\partial\tilde{\lambda} = \partial h_k^2 + O(|z|^3)$, where $|z|$ is a notation for the norm of (z_k^1, z_k^2) or equivalently up to a constant factor the g_k-distance to x. Moreover the asymptotic holomorphicity of s_k implies that the antiholomorphic terms in λ are bounded by $O(k^{-1/2})$, which makes $\lambda s_{k,x}^{\mathrm{ref}}$ an admissible perturbation as its antiholomorphic derivatives are bounded by $O(k^{-1/2})$. We now study the effect of replacing s_k by $s_k + wQ$, where $w \in \mathbb{C}$ is a small coefficient and $Q = (0, \lambda s_{k,x}^{\mathrm{ref}}, 0)$.

We first look at how this perturbation of s_k affects $R(s_k)$ and the cusp point x. Adding wQ to s_k amounts to adding $(w\partial\tilde{\lambda}, 0)$ to $(\partial h_k^1, \partial h_k^2)$, and therefore adding $w\Delta$ to $\mathrm{Jac}(h_k)$, where $\Delta = \partial\tilde{\lambda} \wedge \partial h_k^2 = O(|z|^3)$. It follows in particular that x still belongs to $R(s_k + wQ)$, and even the tangent spaces to $R(s_k + wQ)$ and $R(s_k)$ coincide at x. Since for small w the submanifold $R(s_k + wQ)$ is a small deformation of $R(s_k)$, it can locally be seen as a section of TX over $R(s_k)$. Recall that $\mathrm{Jac}(h_k)$ is γ'-transverse to 0 over a ball of fixed g_k-radius around x for some $\gamma' > 0$ independent of k : therefore, restricting to a smaller ball (whose size remains independent of k and x) if necessary, the derivative $\nabla \mathrm{Jac}(h_k)$ admits everywhere a right inverse $\rho : \Lambda^{2,0}T^*X \to TX$. It is then easy to see that $R(s_k + wQ)$ is obtained by shifting $R(s_k)$ by an amount equal to $-\rho(w\Delta) + O(|w\Delta|^2)$. It follows that the value of $\hat{\mathcal{K}}(s_k + wQ)$ at a point of $R(s_k + wQ)$ differs from the value of $\hat{\mathcal{K}}(s_k)$ at the corresponding point of $R(s_k)$ by an amount

$$\Theta(w) = w\,\partial\tilde{\lambda} \wedge \partial \mathrm{Jac}(h_k) + w\,\partial h_k^1 \wedge \partial\Delta - \nabla(\hat{\mathcal{K}}(s_k)).\rho(w\Delta) + O(w^2|z|^2).$$

Recall that $\partial\tilde{\lambda} - \partial h_k^2 = O(|z|^3)$, $\Delta = O(|z|^3)$ and $\partial\Delta = O(|z|^2)$: therefore

$$\Theta(w) = w\,\partial h_k^2 \wedge \partial \mathrm{Jac}(h_k) + O(|z|^2).$$

Recall that the restriction to $T_xR(s_k)$ of $\partial(\partial h_k^2 \wedge \partial \mathrm{Jac}(h_k))(x)$ is bounded away from 0 by a fixed constant : therefore, a suitable choice of the complex number w ensures both that the perturbation wQ added to s_k is much smaller than δ in C^3 norm, and that the derivative

$$\partial(\hat{\mathcal{K}}(s_k + wQ))_{|TR(s_k+wQ)}(x) = \partial(\hat{\mathcal{K}}(s_k))_{|TR(s_k)}(x) + \partial(\Theta(w))_{|TR(s_k)}(x)$$

has norm bounded from below by a certain constant independently of k.

Because of the uniform bounds on all derivatives of s_k, the quantity $\partial(\hat{\mathcal{K}}(s_k + wQ))_{|TR(s_k+wQ)}$ remains bounded from below over the intersection of $R(s_k + wQ)$ with a ball of fixed g_k-radius centered at x. It follows that the restriction of $\mathcal{K}(s_k + wQ)$ to $R(s_k + wQ)$ is transverse to 0 over a neighborhood of x. Checking more carefully the dependence of the estimates on the size of the maximum allowable perturbation δ, one gets that there exist constants c and $c' \in (0,1)$ (independent of x, k and δ) such that a perturbation of s_k smaller than δ in C^3 norm and with gaussian decay away from x can be used to achieve the $c\delta$-transversality to 0 of $\mathcal{K}(s_k)_{|R(s_k)}$ over the ball $B_{g_k}(x, c'\delta)$.

This result is quite different from what is required to apply Proposition 3 of [3] (in particular the size of the ball on which transversality is achieved is not independent of δ) ; however a similar globalization argument can be applied, as we wish to cover only a neighborhood of the set of cusp points rather than all of X. As in the usual argument, the key observation is the existence of a constant $D > 0$ (independent of k and δ) such that, if two cusp points x and x' are mutually g_k-distant of more than D, then the perturbation applied at x becomes much smaller than $\frac{1}{2}c\delta$ in C^3 norm over a neighborhood of x' (this is because the perturbations we use have uniform gaussian decay properties). Therefore, as the required transversality property is local and C^3-open, it is possible to simultaneously add to s_k the perturbations corresponding to several cusp points x_i which lie sufficiently far apart from each other, without any risk of interference between the perturbations : denoting by σ_k the perturbed section, $\frac{1}{2}c\delta$-transversality to 0 holds for $\mathcal{K}(\sigma_k)_{|R(\sigma_k)}$ over the union of all balls $B_{g_k}(x_i, c'\delta)$.

Moreover the perturbation applied at x_i preserves the property of x_i being a cusp point, so the positions of the cusp points are only affected by the perturbations coming from the *other* points : therefore, because of the γ-genericity properties of s_k, the cusp point x_i' of the perturbed section σ_k which corresponds to the cusp point x_i of the original section s_k lies at g_k-distance from x_i bounded by a fixed multiple of $c\delta$. In particular, decreasing the value of c if necessary to make it much smaller than c' (and increasing D consequently) one may assume that the cusp points x_i are moved by less than $\frac{1}{2}c'\delta$, so that $\frac{1}{2}c\delta$-transversality to 0 holds for $\mathcal{K}(\sigma_k)_{|R(\sigma_k)}$ over the union of all balls $B_{g_k}(x'_i, \frac{1}{2}c'\delta)$.

Now notice that, because the sections s_k are γ-generic, there exists a constant $r > 0$ independent of k such that any two points of $\mathcal{C}(s_k)$ are mutually g_k-distant of more than r (cusps are isolated). It follows that there exists an integer N independent of k such that the set of cusps can be partitioned into at most N finite subsets $\mathcal{C}_j(s_k)$, $1 \le j \le N$, such that any two points in a given subset are mutually distant of more than $D + 2$. We can then proceed by induction : in the first step one starts from $s_{k,0} = s_k$ and perturbs it by less than $\frac{1}{2}\delta$ over a neighborhood of $\mathcal{C}_1(s_{k,0})$ in order to achieve $\frac{1}{4}c\delta$-transversality to 0 of $\mathcal{K}(s_{k,1})_{|R(s_{k,1})}$ (where $s_{k,1}$ is the perturbed section) over the $\frac{1}{4}c'\delta$-neighborhood of $\mathcal{C}_1(s_{k,1})$ (where the partition of $\mathcal{C}(s_{k,0})$ in N subsets is implicitly transferred to $\mathcal{C}(s_{k,1})$).

In the $(j+1)$-th step one starts from the section $s_{k,j}$ constructed at the previous step, which satisfies the property that $\mathcal{K}(s_{k,j})_{|R(s_{k,j})}$ is $(\frac{1}{4}c)^j\delta$-transverse to 0 over the $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j})$. A perturbation smaller than $\frac{1}{2}(\frac{1}{4}c)^j\delta$ at the points of $\mathcal{C}_{j+1}(s_{k,j})$ can be used to obtain a section $s_{k,j+1}$ such that $\mathcal{K}(s_{k,j+1})_{|R(s_{k,j+1})}$ is $(\frac{1}{4}c)^{j+1}\delta$-transverse to 0 over the $\frac{1}{4}c'(\frac{1}{4}c)^j\delta$-neighborhood of $\mathcal{C}_{j+1}(s_{k,j+1})$. Moreover, since the perturbation was chosen small enough and by the assumption on $s_{k,j}$, this transversality property also holds over the $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j})$. Since the cusp points of $s_{k,j+1}$ differ from those of $s_{k,j}$ by a distance of at most a fixed multiple of $(\frac{1}{4}c)^j\delta$, which is much less than $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$ by an assumption made on c and c' ($c \ll c'$, see above), the $\frac{1}{4}c'(\frac{1}{4}c)^j\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j+1})$ is contained in the $\frac{1}{4}c'(\frac{1}{4}c)^{j-1}\delta$-neighborhood of $\bigcup_{i\le j}\mathcal{C}_i(s_{k,j})$. Therefore $s_{k,j+1}$ satisfies the hypotheses needed for the following step of the inductive process, and the construction can be carried out until all cusp points have been taken care of.

The only point which one has to check carefully is that the points of $\mathcal{C}_{j+1}(s_{k,j})$ are indeed mutually distant of more than D (otherwise one cannot proceed as claimed above). However $s_{k,j}$ differs from $s_{k,0}$ by at most $\sum_{i<j}\frac{1}{2}(\frac{1}{4}c)^i\delta$, which is less than δ since $c < 1$. Therefore the cusp points of $s_{k,j}$ differ from those of $s_{k,0}$ by a g_k-distance which is at most a fixed multiple of δ, i.e. less than 1 if one takes δ sufficiently small in the statement of Proposition 2 (decreasing the size of the maximum allowable perturbation is obviously not a restriction). It follows immediately that, since the points of $\mathcal{C}_{j+1}(s_{k,0})$ are mutually distant of at least $D+2$, those of $\mathcal{C}_{j+1}(s_{k,j})$ are mutually distant of at least D, and the inductive argument given above is indeed valid.

This ends Step 2, as we have shown that a perturbation of s_k smaller than δ can be used to ensure the η-transversality to 0 of $\mathcal{K}(s_k)_{|R(s_k)}$ over the c''-neighborhood of $\mathcal{C}(s_k)$, where $\eta = (\frac{1}{4}c)^N\delta$ and $c'' = \frac{1}{4}c'(\frac{1}{4}c)^{N-1}\delta$.

Step 3. In this step we wish to obtain the transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ everywhere. As observed at the end of Step 1, we only need to show that there exist constants p, c and $c' > 0$ independent of δ such that, for large enough k, given any point $x \in X$, if $B_{g_k}(x,c) \cap R(s_k) \neq \emptyset$ then by adding to s_k a perturbation smaller than δ and with gaussian decay away from x it is possible to ensure the η-transversality to 0 of $\hat{\mathcal{K}}(s_k)_{|R(s_k)}$ over $B_{g_k}(x,c) \cap R(s_k)$, where $\eta = c'\delta(\log\delta^{-1})^{-p}$.

By the result of Step 2, and restricting oneself to a choice of c smaller than half the constant c'' introduced in Step 2, one actually needs to obtain this result only in the case where x lies at distance more than $\frac{1}{2}c''$ from $\mathcal{C}(s_k)$.

Recall that the γ-genericity of s_k and the assumption that x lies away from the cusp points imply that the quantity $\mathcal{T}(s_k) = \partial f_k \wedge \partial \mathrm{Jac}(f_k)$, which is γ-transverse to 0, is bounded away from 0 at x. With the notations of Step 1, it follows that at least one of the two quantities $\partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ and $\partial h_k^2 \wedge \partial \mathrm{Jac}(h_k)$ has norm bounded from below by a fixed constant α at x (depending only on γ and the uniform bounds on s_k). Therefore, two cases can occur : the first possibility is that $\hat{\mathcal{K}}(s_k) = \partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ has norm more than α at x, and therefore remains larger than $\frac{\alpha}{2}$ over a ball of fixed radius around x as its derivatives are uniformly bounded. In that case, one gets $\frac{\alpha}{2}$-transversality to 0 over a ball of fixed g_k-radius around x without any perturbation.

The other case, which is the one where we need to perturb s_k to obtain transversality, is the one where $\partial h_k^1 \wedge \partial \mathrm{Jac}(h_k)$ is small (i.e. $D(s_k)$ is nearly tangent at $f_k(x)$ to the fiber of π). In that case, however, the quantity $\mathcal{X}(s_k) = \partial h_k^2 \wedge \partial \mathrm{Jac}(h_k)$ is bounded from below by $\frac{\alpha}{2}$ over a neighborhood of x.

As in Step 2, consider local approximately holomorphic Darboux coordinates (z_k^1, z_k^2) on a neighborhood of x as given by Lemma 3 of [3], and let $s_{k,x}^{\mathrm{ref}}$ be an approximately holomorphic section of L^k with gaussian decay away from x as given by Lemma 2 of [3]. Let λ be the polynomial function of degree 3 in z_k^1, z_k^2 and their complex conjugates obtained by keeping the degree 1, 2 and 3 terms of the Taylor series expansion of $h_k^2\, s_k^0/s_{k,x}^{\mathrm{ref}}$ at x : λ vanishes at x, and the function $\tilde{\lambda} = \lambda s_{k,x}^{\mathrm{ref}}/s_k^0$ has the property that $\partial\tilde{\lambda} = \partial h_k^2 + O(|z|^3)$, where $|z|$ is a notation for the norm of (z_k^1, z_k^2) or equivalently up to a constant factor the g_k-distance to x. Moreover the asymptotic holomorphicity of s_k implies that the antiholomorphic terms in λ are bounded by $O(k^{-1/2})$, which makes $\lambda s_{k,x}^{\mathrm{ref}}$ an admissible perturbation as its antiholomorphic derivatives are bounded by $O(k^{-1/2})$. We now study the effect of replacing s_k by $s_k + wQ$, where $w \in \mathbb{C}$ is a small coefficient and $Q = (0, \lambda s_{k,x}^{\mathrm{ref}}, 0)$.

As in Step 2, one computes that $R(s_k + wQ)$ is obtained by shifting $R(s_k)$ by an amount equal to $-\rho(w\Delta) + O(|w\Delta|^2)$, where ρ is a right inverse of $\nabla \mathrm{Jac}(h_k)$ and $\Delta = \partial\tilde{\lambda} \wedge \partial h_k^2 = O(|z|^3)$. It follows that the value of $\hat{\mathcal{K}}(s_k + wQ)$ at a point of $R(s_k + wQ)$ differs from the value of $\hat{\mathcal{K}}(s_k)$ at the corresponding point of $R(s_k)$ by an amount

$$\Theta(w) = w\, \partial\tilde{\lambda} \wedge \partial \mathrm{Jac}(h_k) + w\, \partial h_k^1 \wedge \partial\Delta - \nabla(\hat{\mathcal{K}}(s_k)).\rho(w\Delta) + O(w^2|z|^2).$$

As $\nabla(\hat{\mathcal{K}}(s_k))$ and ρ are approximately holomorphic, one has $\Theta(w) = w\Theta^0 + O(|w|^2) + O(k^{-1/2})$, where

$$\Theta^0 = \partial\tilde{\lambda} \wedge \partial \mathrm{Jac}(h_k) + \partial h_k^1 \wedge \partial\Delta - \nabla(\hat{\mathcal{K}}(s_k)).\rho(\Delta).$$

Recalling that $\partial\tilde{\lambda} - \partial h_k^2 = O(|z|^3)$, $\Delta = O(|z|^3)$ and $\partial\Delta = O(|z|^2)$, one gets

$$\Theta^0 = \partial h_k^2 \wedge \partial \mathrm{Jac}(h_k) + O(|z|^2) = \mathcal{X}(s_k) + O(|z|^2).$$

In particular, it follows from the initial assumption $|\mathcal{X}(s_k)(x)| \geq \alpha$ that Θ^0 remains larger than $\frac{\alpha}{2}$ over a ball of fixed radius centered at x.

We now proceed as in Section 3.2 of [3] : first use Lemma 7 of [3] to find an approximately holomorphic map $\theta_k : D^+ \to R(s_k)$ (where D^+ is the disk of radius

$\frac{11}{10}$ in $\mathbb{C}$), satisfying the estimates given in the statement of the lemma, whose image is contained in a neighborhood of x over which Θ^0 remains larger than $\frac{\alpha}{2}$, and such that the image of the unit disk D contains $R(s_k) \cap B_{g_k}(x, r')$ for some fixed constant $r' > 0$. Define over D^+ the complex valued function

$$v_k(z) = \frac{\hat{\mathcal{K}}(s_k)(\theta_k(z))}{\Theta^0(\theta_k(z))}.$$

Because Θ^0 is bounded from below over $\theta_k(D^+)$, the function v_k satisfies the hypotheses of Proposition 6 of [3] (or equivalently Proposition 3 of [2]) provided that k is sufficiently large. Therefore, if C_0 is a constant larger than $|Q|_{C^3,g_k}$, and if k is large enough, there exists $w \in \mathbb{C}$, with $|w| \le \frac{\delta}{C_0}$, such that $v_k + w$ is ϵ-transverse to 0 over the unit disk D in $\mathbb{C}$, where $\epsilon = \frac{\delta}{C_0} \log((\frac{\delta}{C_0})^{-1})^{-p}$.

Multiplying again by Θ^0 and recalling that $\theta_k(D) \supset R(s_k) \cap B_{g_k}(x, r')$, we get that the restriction to $R(s_k)$ of $\hat{\mathcal{K}}(s_k) + w\Theta^0$ is ϵ'-transverse to 0 over $R(s_k) \cap B_{g_k}(x, r')$, for some ϵ' differing from ϵ by at most a constant factor. Recall that $\Theta(w) = w\Theta^0 + O(|w|^2) + O(k^{-1/2})$, and note that $|w|^2$ is at most of the order of δ^2 while ϵ' is of the order of $\delta \log(\delta^{-1})^{-p}$: so, if δ is small enough and k is large enough, $\hat{\mathcal{K}}(s_k) + \Theta(w)$ differs from $\hat{\mathcal{K}}(s_k) + w\Theta^0$ by less than $\frac{\epsilon'}{2}$ and is therefore $\frac{\epsilon'}{2}$-transverse to 0 over $R(s_k) \cap B_{g_k}(x, r')$.

The perturbation wQ is smaller than δ, and therefore moves $R(s_k)$ by at most $O(\delta)$. So, if δ is chosen small enough, one can safely assume that the points of $R(s_k)$ are shifted by a distance less than $\frac{r'}{2}$, and therefore that the point of $R(s_k)$ corresponding to any given point in $R(s_k + wQ) \cap B_{g_k}(x, \frac{r'}{2})$ lies in $B_{g_k}(x, r')$. It then follows immediately from the definition of $\Theta(w)$ that $\hat{\mathcal{K}}(s_k + wQ)_{|R(s_k+wQ)}$ is ϵ''-transverse to 0 over $R(s_k + wQ) \cap B_{g_k}(x, \frac{r'}{2})$ for some $\epsilon'' > 0$ differing from ϵ' by at most a constant factor.

This is precisely what we set out to prove, and it is then easy to combine Lemma 6 and Proposition 3 of [3] in order to show that the local perturbations of s_k which give transversality near a given point x can be fitted together to obtain a transversality result over all X. The proof of Proposition 2 in the case of isolated sections is therefore complete.

Step 4. We now consider the case of one-parameter families of sections, where the argument still works similarly : we are now given sections $s_{t,k}$ depending continuously on a parameter $t \in [0, 1]$, and try to perform the same construction as above for each value of t, in such a way that everything depends continuously on t.

The argument of Step 1 carries over to the case of 1-parameter families without any change ; however one has to be very careful when carrying out the argument of Step 2. As explained in Section 4.1 of [3], the transversality properties of $s_{t,k}$ imply that the cusp points (i.e. the points of $\mathcal{C}_{J_t}(s_{t,k})$) depend continuously on t and that their number remains constant (actually, the g_k-distance between two cusp points remains uniformly bounded from below independently of k and t). Without loss of generality, we can assume the maximum allowable perturbation size δ to be much smaller than the constant γ' introduced in Step 2 (minimum size of the derivative at x_t along $R(s_{t,k})$ of $\partial h_k \wedge \partial \mathrm{Jac}(h_k)$ as given by the transversality estimates on $\mathcal{T}(s_{t,k})$). Moreover, let us assume for now that when t varies over $[0, 1]$, the cusp points move by no more than the unit distance in g_k norm (i.e. two cusp points which are far from each other at $t = 0$ retain this property for all $t \in [0, 1]$).

Let $(x_t)_{t\in[0,1]}$ be a continuous path of points of $\mathcal{C}_{J_t}(s_{t,k})$, and let Ω be the set of all t such that the derivative at x_t of $\hat{\mathcal{K}}(s_{t,k})$ along $R(s_{t,k})$ is smaller than γ' (i.e. all t such that a perturbation is necessary in order to ensure the required transversality property). For $t \in \Omega$, the same construction as in Step 2 still works, since the technical results from [3] are also valid in the case of 1-parameter families : therefore one can define, for all $t \in \Omega$, approximately J_t-holomorphic sections Q_t of $\mathbb{C}^3 \otimes L^k$ and complex numbers w_t smaller than δ, which depend continuously on t, in such a way that $s_{t,k} + w_t Q_t$ satisfies the desired transversality property on a neighborhood of x_t. We need to define a valid perturbation for all $t \in [0, 1]$ rather than only for $t \in \Omega$: for this, define $\beta : [0, \gamma'] \to [0, 1]$ to be a smooth cut-off function which equals 1 over $[0, \frac{1}{2}\gamma']$ and vanishes over $[\frac{3}{4}\gamma', \gamma']$, and set $\nu(t)$ to be the norm of $\nabla\hat{\mathcal{K}}(s_{t,k})_{|R(s_{t,k})}(x_t)$. Set

$$\tau_{t,k} = \beta(\nu(t))\, w_t Q_t$$

for $t \in \Omega$ and $\tau_{t,k} = 0$ for $t \notin \Omega$: this section of $\mathbb{C}^3 \otimes L^k$ is approximately holomorphic for all t (as the multiplicative coefficient is just a constant number for any given t), and depends continuously on t by construction. When $\nu(t)$ is less than $\frac{1}{2}\gamma'$, the perturbation $\tau_{t,k}$ coincides with $w_t Q_t$, so adding it to $s_{t,k}$ does indeed provide the expected transversality properties. For all other values of t, the bound $|\nabla\hat{\mathcal{K}}(s_{t,k})_{|R(s_{t,k})}(x_t)| \geq \frac{1}{2}\gamma'$ implies that $s_{t,k}$ already satisfies the required transversality property over a neighborhood of x_t : so it follows from the fact that the required property is C^3-open that, if δ is sufficiently small compared to γ', then the transversality property still holds (with a slightly decreased transversality estimate) for the perturbed section $s_{t,k} + \tau_{t,k}$. Therefore, we have established the transversality to 0 of $\hat{\mathcal{K}}(s_{t,k} + \tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ over a neighborhood of x_t for all $t \in [0, 1]$.

Recall that we have made the assumption that when t varies over $[0, 1]$, the cusp points move by no more than the unit distance in g_k norm : this is necessary in order to apply the globalization process described in Step 2. Indeed, this ensures that, if one partitions $\mathcal{C}(s_{0,k})$ into a fixed number N of subsets $\mathcal{C}_i(s_{0,k})$ whose points are mutually distant of at least $D+4$, then for all $t \in [0, 1]$ the corresponding partition of $\mathcal{C}(s_{t,k})$ into subsets $\mathcal{C}_i(s_{t,k})$ $(1 \leq i \leq N)$ still has the property that any two points of $\mathcal{C}_i(s_{t,k})$ are distant of at least $D + 2$. Therefore, the globalization argument of Step 2 can be applied for all $t \in [0, 1]$: as previously, successive perturbations make it possible to ensure the expected transversality property for all $t \in [0, 1]$ first near the points of the first subset, then near the points of the second subset, and so on until after N steps all the cusp points have been handled.

We now consider the general case, where the variations of the cusp points with t are no longer bounded. In that case, a simple compactness argument allows one to find a sequence of numbers $0 = t_0 < t_1 < \ldots < t_{2I-1} < t_{2I} = 1$ such that, over each interval $[t_i, t_{i+1}]$, the cusp points move by a g_k-distance no greater than $\frac{1}{2}$ (the length $2I$ of the sequence cannot be controlled a priori). Over each of the intervals $[t_i, t_{i+1}]$ the previous argument can be applied. In particular, in a first step we can find, for all $t \in T_1 = \bigcup_{i<I}[t_{2i}, t_{2i+1}]$, sections $\tau_{t,k}$ of $\mathbb{C}^3 \otimes L^k$, smaller

than $\frac{\delta}{2}$, depending continuously on t, and such that $\mathcal{K}(s_{t,k}+\tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ is η-transverse to 0 over a neighborhood of the cusp points for all $t \in T_1$ and for some constant $\eta > 0$ independent of k. It is then clearly possible to define asymptotically holomorphic sections $\tau_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ for $t \notin T_1$ in such a way that the sections $\tau_{t,k}$ are for all $t \in [0,1]$ smaller than $\frac{\delta}{2}$ and depend continuously on t (e.g. by using cut-off functions away from T_1). Let $s'_{t,k} = s_{t,k} + \tau_{t,k}$: these sections depend continuously on t, and $\mathcal{K}(s'_{t,k})_{|R(s'_{t,k})}$ is η-transverse to 0 over a neighborhood of the cusp points for all $t \in T_1$.

Because the cusp points of $s'_{t,k}$ differ from those of $s_{t,k}$ by $O(\delta)$, it can be ensured (decreasing δ if necessary) that the cusp points of $s'_{t,k}$ move by a g_k-distance no greater than 1 over each interval $[t_{2i+1}, t_{2i+2}]$. Therefore the above procedure can be applied again : one can find, for all $t \in T_2 = \bigcup_{i<I}[t_{2i+1}, t_{2i+2}]$, sections $\tau'_{t,k}$ of $\mathbb{C}^3 \otimes L^k$, much smaller than η, depending continuously on t, and such that $\mathcal{K}(s'_{t,k}+\tau'_{t,k})_{|R(s'_{t,k}+\tau'_{t,k})}$ is transverse to 0 over a neighborhood of the cusp points for all $t \in T_2$. As previously, one can define asymptotically holomorphic sections $\tau'_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ for all $t \notin T_2$ in such a way that the sections $\tau'_{t,k}$ are for all $t \in [0,1]$ much smaller than η and depend continuously on t. Let $s''_{t,k} = s'_{t,k} + \tau'_{t,k}$: these sections depend continuously on t, and $\mathcal{K}(s''_{t,k})_{|R(s''_{t,k})}$ is transverse to 0 over a neighborhood of the cusp points not only for all $t \in T_2$ by construction, but also for all $t \in T_1$ because they differ from $s'_{t,k}$ by less than η and transversality to 0 is an open property. This ends the proof that the construction of Step 2 can be carried out in the case of one-parameter families of sections.

We now consider the construction of Step 3, in order to complete the proof of Proposition 2 for one-parameter families of sections. The argument is then similar to the one at the end of Section 3.2 of [3]. We have to show that, near any point $x \in X$, one can perturb $s_{t,k}$ to ensure that, for all t such that x lies in a neighborhood of $R(s_{t,k})$, $\mathcal{K}(s_{t,k})_{|R(s_{t,k})}$ is transverse to 0 over the intersection of $R(s_{t,k})$ with a ball centered at x : Proposition 3 of [3] also applies to one-parameter families of sections and is then sufficient to conclude. As observed at the beginning of Step 3, because we already know how to ensure the required transversality property over a neighborhood of the cusp points, it is sufficient to restrict oneself to those values of t such that x lies away from $\mathcal{C}(s_{t,k})$. Even more, one only needs to handle the case where the quantity $\hat{\mathcal{K}}(s_{t,k})$ is small at x (because, as explained in Step 3, the transversality property otherwise holds near x without perturbing $s_{t,k}$).

When all these conditions hold, the argument of Step 3 can be used to provide the required transversality property over a neighborhood of x for all suitable values of t, because all the technical results involved in the construction, namely Lemma 2, Lemma 3, Lemma 7 and Proposition 6 of [3], also apply to the case of one-parameter families of sections. More precisely, there exist constants c, c', c'', α and $\alpha' > 0$ with the following properties : let $\Omega \subset [0,1]$ be the set of all t such that $B_{g_k}(x, 2c) \cap R(s_{t,k}) \neq \emptyset$, $B_{g_k}(x, \frac{1}{2}c'') \cap \mathcal{C}(s_{t,k}) = \emptyset$ and $|\mathcal{K}(s_{t,k})(x)| < 2\alpha$. Let $\check{\Omega} \subset [0,1]$ be the set of all t such that either $B_{g_k}(x,c) \cap R(s_{t,k}) = \emptyset$, $B_{g_k}(x,c'') \cap \mathcal{C}(s_{t,k}) \neq \emptyset$ or $|\mathcal{K}(s_{t,k})(x)| > \alpha$. Then for all $t \in \check{\Omega}$ the restriction to $R(s_{t,k})$ of $\mathcal{K}(s_{t,k})$ is α'-transverse to 0 over $B_{g_k}(x,c) \cap R(s_{t,k})$ (this comes from trivial remarks and from having already obtained the transversality property near the cusp points) ; and, provided that k is large enough, one can by the argument of Step 3 construct, for all $t \in \Omega$, sections Q_t of $\mathbb{C}^3 \otimes L^k$ and complex numbers w_t smaller than δ, depending continuously on t, and such that $\mathcal{K}(s_{t,k} + w_t Q_t)_{|R(s_{t,k}+w_t Q_t)}$ is η-transverse to 0 over $B_{g_k}(x, c) \cap R(s_{t,k} + w_t Q_t)$, where $\eta = c'\delta(\log \delta^{-1})^{-p}$.

It is clear that Ω and $\check{\Omega}$ cover $[0, 1]$. Let $\beta : [0, 1] \to [0, 1]$ be a continuous function which equals 1 outside of $\check{\Omega}$ and vanishes outside of Ω (such a β can e.g. be constructed using cut-off functions and distance functions), and let $\tau_{t,k}$ be the sections of $\mathbb{C}^3 \otimes L^k$ defined by $\tau_{t,k} = \beta(t) w_t Q_t$ for all $t \in \Omega$, and $\tau_{t,k} = 0$ for all $t \notin \Omega$. Then it is easy to check that the sections $s_{t,k} + \tau_{t,k}$, which depend continuously on t and differ from $s_{t,k}$ by at most δ, satisfy the required transversality property over $B_{g_k}(x, \frac{1}{2}c)$ for all $t \in [0, 1]$. Indeed, for $t \in \check{\Omega}$, one notices that $s_{t,k} + \tau_{t,k}$ differs from $s_{t,k}$ by at most δ, and therefore the α'-transversality to 0 of $\mathcal{K}(s_{t,k})_{|R(s_{t,k})}$ over $B_{g_k}(x, c) \cap R(s_{t,k})$ implies the $\frac{1}{2}\alpha'$-transversality to 0 of $\mathcal{K}(s_{t,k} + \tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ over $B_{g_k}(x, \frac{c}{2}) \cap R(s_{t,k} + \tau_{t,k})$, provided that δ is sufficiently small compared to α' (decreasing δ if necessary is clearly not a problem). Meanwhile, for $t \notin \check{\Omega}$, one has $\tau_{t,k} = w_t Q_t$, so the η-transversality to 0 of $\mathcal{K}(s_{t,k} + \tau_{t,k})_{|R(s_{t,k}+\tau_{t,k})}$ over $B_{g_k}(x, \frac{c}{2}) \cap R(s_{t,k} + \tau_{t,k})$ follows immediately from the construction.

Therefore the required transversality property can be ensured locally by small perturbations for one-parameter families of sections as well, which allows us to complete the proof of Proposition 2 by the usual globalization argument (recall that Lemma 6 and Proposition 3 of [3] also apply to one-parameter families of sections). □

3. Existence and uniqueness of quasiholomorphic coverings

3.1. **Self-transversality and proof of Theorem 1.** In this subsection we give a proof of Theorem 1. Propositions 1 and 2, together with the results in Sections 2 and 3 of [3], allow us to construct, for some constant $\gamma > 0$ and for all large k, asymptotically holomorphic sections (or 1-parameter families of sections) whose first two components are γ-transverse to 0 and with the additional properties of being γ-generic and γ-transverse to the projection to $\mathbb{CP}^1$. We now consider further perturbation in order to obtain $\bar{\partial}$-tameness (see Definition 7), enhanced by a similar condition of tameness with respect to the projection (ensuring the second property stated in the introduction), and simple self-transversality conditions (properties 4, 5 and 6 in the introduction). The procedure is the following.

Step 1. We first use Proposition 8 of [3] in order to obtain the correct picture over a neighborhood of $\mathcal{C}(s_k)$: namely, the existence of perturbed almost-complex structures $\tilde{J}_k$, which differ from J by $O(k^{-1/2})$, are integrable near the cusp points and enable us to perturb the sections s_k by $O(k^{-1/2})$ to make them holomorphic over a neighborhood of the set of cusp points (the same result also holds for 1-parameter families of sections).

Step 2. We now add the property of tameness with respect to the projection :

Definition 9. *Let s_k be asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^k$, transverse to the projection. Let $\mathbb{T}(s_k)$ be the (finite) set of points of $R(s_k) - \mathcal{C}(s_k)$ where $\mathcal{K}(s_k)$ vanishes (these points will be called "tangency points"). We say that s_k is tamed by the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$ if $\bar{\partial}(\mathbb{P}s_k)$ vanishes at every point of $\mathbb{T}(s_k)$.*

Note that, since the g_k-distance between a tangency point and a cusp point is bounded from below (because of the transversality estimates), it doesn't actually matter whether one works with J or $\tilde{J}_k$, as they coincide outside of a small neighborhood of the cusp points whose size can be chosen freely (see Section 4.1 of [3]).

We now show that, by adding to s_k a perturbation of size $O(k^{-1/2})$, one can ensure tameness with respect to the projection. Indeed, let x be a point of $\mathbb{T}(s_k)$, and let $f_k = \mathbb{P}s_k$. Choose a constant $\delta > 0$ smaller than half the g_k-distance between any two tangency points and than half the g_k-distance between any tangency point and any cusp point (these distances are uniformly bounded from below because of the transversality estimates). Define a section χ of $f_k^*T\mathbb{CP}^2$ over $B_{g_k}(x,\delta)$ by the following identity : given any vector $\xi \in T_xX$ of norm less than δ,

$$\chi(\exp_x(\xi)) = \beta(|\xi|)\,\bar\partial f_k(x).\xi,$$

where $\beta : [0,\delta] \to [0,1]$ is a smooth cut-off function equal to 1 over $[0, \frac{1}{2}\delta]$ and 0 over $[\frac{3}{4}\delta, \delta]$, and where the fibers of $f_k^*T\mathbb{CP}^2$ at x and at $\exp_x(\xi)$ are implicitly identified using radial parallel transport. Repeating the same process at any point of $\mathbb{T}(s_k)$, one similarly defines χ over the δ-neighborhood of $\mathbb{T}(s_k)$. Moreover, since χ vanishes near the boundary of $B_{g_k}(x,\delta)$, it can be extended into a smooth global section of $f_k^*T\mathbb{CP}^2$ which vanishes outside of the δ-neighborhood of $\mathbb{T}(s_k)$.

Recall that $\forall x \in X$ the tangent space to $\mathbb{CP}^2$ at $f_k(x) = \mathbb{P}s_k(x)$ is canonically identified with the space of complex linear maps from $\mathbb{C}s_k(x)$ to $(\mathbb{C}s_k(x))^\perp \subset \mathbb{C}^3 \otimes L_x^k$. This allows us to define $\sigma_k(x) = s_k(x) - \chi(x).s_k(x)$. It follows from the construction of χ that σ_k remains equal to s_k outside the δ-neighborhood of $\mathbb{T}(\sigma_k) = \mathbb{T}(s_k)$ and differs from s_k by $O(k^{-1/2})$; therefore σ_k satisfies the same holomorphicity and transversality properties as s_k provided that k is large enough. Moreover, σ_k is tamed by the projection to $\mathbb{CP}^1$, since at any point $x \in \mathbb{T}(s_k)$ one has $\bar\partial(\mathbb{P}\sigma_k)(x) = \bar\partial f_k(x) - \bar\partial\chi(x) = 0$.

The construction clearly applies to one-parameter families without any change, as the above construction is completely explicit and can be applied for all $t \in [0,1]$ in order to obtain χ_t and $\sigma_{t,k}$ depending continuously on t and satisfying for all t the properties described above. Moreover it is easy to check that, if $s_{0,k}$ is already tamed by the projection, then the construction yields $\sigma_{0,k} = s_{0,k}$, and similarly for $t = 1$.

Step 3. Without losing the previous properties, we now perturb s_k in order to ensure that the images in $\mathbb{CP}^2$ of the cusp points and tangency points are all mutually disjoint, and lie in different fibers of the projection to $\mathbb{CP}^1$.

Wherever s_k^0 and s_k^1 are not both zero, let $\phi_k(x) = (s_k^0(x) : s_k^1(x)) \in \mathbb{CP}^1$. One easily checks by a standard transversality argument that it is possible to choose for all $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ an element $w_x \in T_{\phi_k(x)}\mathbb{CP}^1$ of norm smaller than $k^{-1/2}$, in such a way that the points $\exp_{\phi_k(x)}(w_x)$ are all different in $\mathbb{CP}^1$. Moreover, the differential at the identity of the action of $\mathrm{SU}(2)$ on $\mathbb{CP}^1$ yields a surjective map from $\mathfrak{su}(2)$ to $T_{\phi_k(x)}\mathbb{CP}^1$, so one can actually find elements $u_x \in \mathfrak{su}(2)$ of norm $O(k^{-1/2})$ and such that the infinitesimal action of u_x at $\phi_k(x)$ coincides with w_x.

Fix a constant $\delta > 0$ smaller than the g_k-distance between any two points of $\mathbb{T}(s_k) \cup \mathcal{C}(s_k)$, and let $\beta : [0,\delta] \to [0,1]$ be a smooth cut-off function equal to 1 over $[0, \frac{1}{2}\delta]$ and 0 over $[\frac{3}{4}\delta, \delta]$: then let χ be the $\mathrm{SU}(2)$-valued map defined over the ball of g_k-radius δ around any point $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ by the formula

$$\chi(y) = \exp\left(\beta(\mathrm{dist}(x, y))\, u_x\right).$$

As $\chi(y)$ becomes the identity near the boundary of $B_{g_k}(x, \delta)$, one can extend χ into a map from X to SU(2) by setting $\chi(y) = \mathrm{Id}$ for all y at distance more than δ from $\mathbb{T}(s_k) \cup \mathcal{C}(s_k)$. Finally, let $\sigma_k = \chi.s_k$, where SU(2) acts canonically on the first two components (s_k^0, s_k^1) and acts trivially on the third component s_k^2.

By construction σ_k differs from s_k by $O(k^{-1/2})$, so all asymptotic holomorphicity and genericity properties of s_k are preserved by the perturbation provided that k is large enough. Moreover, over a ball of radius $\frac{\delta}{2}$ around any point $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ the map $\mathbb{P}\sigma_k$ differs from $\mathbb{P}s_k$ by the mere rotation $\exp(u_x)$: therefore the cusp points and tangency points of σ_k are exactly the same as those of s_k, and the properties of holomorphicity near the cusp points and of tameness with respect to the projection to $\mathbb{CP}^1$ are satisfied by σ_k as well. Finally, given any point $x \in \mathbb{T}(s_k) \cup \mathcal{C}(s_k)$ the projection to $\mathbb{CP}^1$ of $\sigma_k(x)$ is $\exp(u_x).\phi_k(x) = \exp_{\phi_k(x)}(w_x)$, so the images in $\mathbb{CP}^1$ of the various cusp and tangency points are by construction all different, as desired.

The same result also holds for one-parameter families of sections. Indeed, as the dimension of $\mathbb{CP}^1$ is strictly more than 1, the space of admissible choices for the elements w_x of $T_{\phi_k(x)}\mathbb{CP}^1$ is always connected ; so one easily defines, for all $t \in [0, 1]$ and for all $x \in \mathbb{T}(s_{t,k}) \cup \mathcal{C}(s_{t,k})$, tangent vectors $w_{t,x}$ such that the same properties as above hold for all t, and such that along any continuous path $(x_t)_{t\in[0,1]}$ of cusp or tangency points the quantity w_{t,x_t} depends continuously on t. The tangent vectors $w_{t,x}$ can then be lifted continuously to elements in $\mathfrak{su}(2)$, and the same construction as above yields sections $\sigma_{t,k}$ which depend continuously on t and satisfy the desired properties for all $t \in [0, 1]$. Moreover, if $s_{0,k}$ already satisfies the required property, then one can clearly choose the vectors $w_{t,x}$ in such a way that all $w_{0,x}$ are zero, and therefore one gets $\sigma_{0,k} = s_{0,k}$; similarly for $t = 1$.

Step 4. Without losing the previous properties, we now perturb s_k in order to ensure that the curve $D(s_k) = f_k(R(s_k))$ is *transverse to itself*, i.e. that all its self-intersection points are transverse double points (requirement 4 of the introduction) and no self-intersection occurs in the same fiber as a cusp point or a tangency point.

For this, we simply remark that there exists a section u of $f_k^*T\mathbb{CP}^2$ over $R(s_k)$ (i.e. a small deformation of $D(s_k)$ in $\mathbb{CP}^2$), smaller than $k^{-1/2}$ in C^3 norm, and which vanishes identically near the cusp and tangency points, such that the deformed curve $\{\exp_{f_k(x)}(u(x)),\ x \in R(s_k)\}$ is transverse to itself. This follows from elementary results in transversality theory.

Use the exponential map to identify a tubular neighborhood of $R(s_k)$ with a neighborhood of the zero section in the normal bundle $NR(s_k)$. Moreover, let θ be the section of $T^*X \otimes f_k^*T\mathbb{CP}^2$ over $R(s_k)$, vanishing at the cusp points, such that at any point $x \in R(s_k) - \mathcal{C}(s_k)$ the 1-form θ_x satisfies the properties $\theta_{x|TR(s_k)} = 0$ and $\theta_{x|K_x} = -(\nabla u \circ p)_{|K_x}$, where $K_x = \mathrm{Ker}\ \partial f_k(x)$ and p is the orthogonal projection to $TR(s_k)$.

Fix a constant $\delta > 0$ sufficiently small, and define a section χ of $f_k^*T\mathbb{CP}^2$ over the δ-tubular neighborhood of $R(s_k)$ by the following identity : given any point $x \in R(s_k)$ and any vector $\xi \in N_xR(s_k)$ of norm less than δ,

$$\chi(\exp_x(\xi)) = \beta(|\xi|)\,(u(x) + \theta_x(\xi)),$$

where $\beta : [0,\delta] \to [0,1]$ is a smooth cut-off function which equals 1 over $[0, \frac{1}{2}\delta]$ and vanishes over $[\frac{3}{4}\delta, \delta]$, and where the fibers of $f_k^* T\mathbb{CP}^2$ at x and at $\exp_x(\xi)$ are implicitly identified using radial parallel transport. Since χ vanishes near the boundary of the chosen tubular neighborhood it can be extended into a smooth section over all of X which vanishes away from $R(s_k)$.

We can then define $\sigma_k = s_k + \chi.s_k$, where the action of χ on s_k is as explained in Step 2. The section σ_k differs from s_k by $O(k^{-1/2})$, so all asymptotic holomorphicity and genericity properties of s_k are preserved by the perturbation provided that k is large enough. Moreover the perturbation vanishes identically over a neighborhood of $\mathbb{T}(s_k) \cup \mathcal{C}(s_k)$, so the cusp and tangency points of σ_k coincide with those of s_k, and the properties we have obtained in Steps 1–3 above are not affected by the perturbation and remain valid for σ_k.

We now show that the curve $D(\sigma_k)$ is transverse to itself : indeed, we first notice that $R(s_k) \subset R(\sigma_k)$, because at any point $x \in R(s_k)$ one has

$$\nabla(\mathbb{P}\sigma_k)(x) = \nabla(\mathbb{P}s_k)(x) + \nabla\chi(x) = \nabla(\mathbb{P}s_k)(x) + \nabla u(x) \circ p + \theta_x,$$

and therefore $\nabla(\mathbb{P}\sigma_k)$ and $\nabla(\mathbb{P}s_k)$ coincide over the complex subspace $K_x \subset T_xX$, so that $\partial(\mathbb{P}\sigma_k)$ vanishes over K_x, and therefore $\mathrm{Jac}(\mathbb{P}\sigma_k)$ vanishes at x, and $x \in R(\sigma_k)$. Because σ_k is close to s_k, $R(\sigma_k)$ is contained in a neighborhood of $R(s_k)$, so it is easy to prove that $R(\sigma_k) = R(s_k)$. Moreover, at a point $x \in R(s_k)$ one has $\chi(x) = u(x)$, so the curve $D(\sigma_k)$ is obtained from $D(s_k)$ by applying the deformation u : therefore $D(\sigma_k)$ is by construction transverse to itself.

In the case of one-parameter families of sections, elementary transversality theory implies that one can still find, for all $t \in [0,1]$, sections u_t of $f_{t,k}^* T\mathbb{CP}^2$ over $R(s_{t,k})$, depending continuously on t and vanishing identically near the cusps and tangency points, which can be used as perturbations to ensure a generic behavior of the curves $D(s_{t,k})$. The only additional generic phenomenon that we must allow is the creation or cancellation of a pair of transverse double points with opposite orientations ; apart from this phenomenon the curves $D(s_{t,k})$ are isotopic to each other. Once the sections u_t are obtained, the rest of the construction is explicit, so defining θ_t, χ_t and $\sigma_{t,k}$ as above for all $t \in [0,1]$ yields the desired result. Moreover, if the curve $D(s_{0,k})$ is already transverse to itself then one can safely choose $u_0 = 0$, which yields $\sigma_{0,k} = s_{0,k}$; similarly for $t = 1$.

Step 5. We finally use Proposition 9 of [3] in order to construct sections σ_k of $\mathbb{C}^3 \otimes L^k$, differing from s_k by $O(k^{-1/2})$, and such that at any point of $R(\sigma_k)$ the derivative $\bar\partial(\mathbb{P}\sigma_k)$ vanishes over the kernel of $\partial(\mathbb{P}\sigma_k)$. The construction of this perturbation is described in Section 4.2 of [3]. It is very important to observe that $R(\sigma_k) = R(s_k)$ as stated in [3] ; because σ_k coincides with s_k over $R(s_k)$ one also has $D(\sigma_k) = D(s_k)$. So this last perturbation, whose aim is to ensure that the constructed sections are $\bar\partial$-tame and therefore define approximately holomorphic branched coverings of $\mathbb{CP}^2$, does not affect the branch curve in $\mathbb{CP}^2$ and therefore preserves the various properties of $R(s_k)$ and $D(s_k)$ obtained in the previous steps.

The sections σ_k of $\mathbb{C}^3 \otimes L^k$ we have constructed at this point satisfy all the required properties : indeed, for sufficiently large k they are asymptotically holomorphic and generic because they differ from the original sections by $O(k^{-1/2})$; they are $\bar\partial$-tame by construction (the property of holomorphicity near the cusp points obtained in Step 1 was not affected by the later perturbations) ; therefore by Theorem 3 of [3] the corresponding projective maps are approximately holomorphic singular branched coverings. Moreover, the first two components of σ_k are transverse to 0 (this open property is preserved by all our perturbations provided that k is large enough), so $(0:0:1)$ does not belong to the branch curve $D(\sigma_k)$, which is the first requirement stated in the introduction. Because our sections are γ-transverse to the projection for some constant $\gamma > 0$ (see Definition 8 and Proposition 2), there are only finitely many tangency points, and since the sections σ_k are tamed by the projection (because of the construction carried out in Step 2) the local model at the tangency points is as stated in the second requirement of the introduction.

The third requirement also follows directly from the property of γ-transversality to the projection (see the beginning of Step 2 in the proof of Proposition 2 for the geometric interpretation of $\mathcal{K}(s_k)$ near a cusp point). The fourth requirement, namely the self-transversality of $D(\sigma_k)$, has been obtained in Step 4 and is not affected by the perturbation of Step 5. Moreover, the images in $\mathbb{CP}^1$ of the cusp and tangency points are all disjoint, as obtained in Step 3 (this property is preserved by the perturbations carried out in Steps 4 and 5), and the same property for double points has been achieved in Step 4, so the fifth requirement stated in the introduction holds as well. Therefore we have shown that the construction of branched covering maps described in [3] can be improved in order to obtain branched coverings whose branch curve satisfies the additional requirements stated in the introduction. This proves Theorem 1.

3.2. **Uniqueness up to isotopy.** In the next section we will use Theorem 1 to define invariants of the symplectic four-manifolds. We need the following result of uniqueness up to isotopy.

Theorem 5. *For large enough k, the coverings constructed following the procedure described above are unique, up to isotopies of quasiholomorphic coverings (see Definition 1).*

This is a straightforward analogue of the result of uniqueness up to isotopy obtained in [3], except that we must allow the cancellation of pairs of transverse double points with opposite orientations. More precisely, consider sections $s_{0,k}$ and $s_{1,k}$ ($k \gg 0$) which define quasiholomorphic coverings (the almost-complex structures J_0 and J_1 for which the approximate holomorphicity properties hold need not be the same). Imitating the argument in Section 4.3 of [3], interpolating one-parameter families of almost-complex structures J_t and asymptotically J_t-holomorphic sections $s_{t,k}$ of $\mathbb{C}^3 \otimes L^k$ can be constructed for all large k in such a way that the sections $s_{t,k}$ satisfy the required transversality properties for all $t \in [0,1]$: namely the sections $s_{t,k}$ are γ-generic for some constant $\gamma > 0$, their first two components are transverse to 0, and they are transverse to the projection to $\mathbb{CP}^1$.

Without loss of generality we may assume that $J_t = J_0$ and $s_{t,k} = s_{0,k}$ for all t in some interval $[0, \epsilon]$, and similarly that $J_t = J_1$ and $s_{t,k} = s_{1,k}$ for $t \in [1-\epsilon, 1]$. This makes it possible to perform Step 1 of §3.1 in such a way that $s_{0,k}$ and $s_{1,k}$ are not affected by the perturbation (see the statement of Proposition 8 of [3]). Because $s_{0,k}$ and $s_{1,k}$ already satisfy all the expected properties, it is then possible to carry out Steps 2–5 of §3.1 in such a way that $s_{0,k}$ and $s_{1,k}$ are not modified by the successive perturbations. The result of this construction is a one-parameter family of branched covering maps interpolating between the covering maps $\mathbb{P}s_{0,k}$ and $\mathbb{P}s_{1,k}$; moreover all these covering maps are quasiholomorphic, except for finitely many values of t which correspond to the cancellation or creation of a pair of transverse double points in the branch curve (for these values of t requirement 4 no longer holds and needs to be replaced by requirement 6 of the introduction).

4. New invariants of symplectic four-manifolds

As a consequence of Theorems 1 and 5, for large k the topology of the branch curves $D(s_k) \subset \mathbb{CP}^2$ and of the corresponding branched covering maps is, up to cancellations and creations of pairs of double points, a topological invariant of the symplectic manifold (X, ω).

As explained in the introduction, the topology of a quasiholomorphic curve $D \subset \mathbb{CP}^2$ of degree d is described by its braid monodromy, which can be expressed as a group homomorphism $\rho : \pi_1(\mathbb{C} - \text{crit}) \to B_d$, where crit $= \{p_1, \dots, p_r\}$ consists of the projections of the nodes, cusps and tangency points of the curve D. If one does not want to restrict the description to an affine subset $\mathbb{C} \subset \mathbb{CP}^1$, it is also possible to consider the *reduced braid group* $B'_d = B_d/\langle \Delta_d^2 \rangle$ and view the braid monodromy as a map $\bar{\rho} : \pi_1(\mathbb{CP}^1 - \text{crit}) \to B'_d$; as soon as $d > 2$ one can easily recover ρ from $\bar{\rho}$, since the images by $\bar{\rho}$ of loops around each of the points p_j can be lifted in only one way from B'_d to B_d as powers of half-twists (this follows from easy degree considerations in B_d). More importantly, the braid monodromy can be expressed as a factorization of the full twist Δ_d^2 in B_d. This factorization is of the form

$$\Delta_d^2 = \prod_{j=1}^{r} (Q_j^{-1} \, X_1^{r_j} \, Q_j),$$

where r_j is equal to -2 for a negative self-intersection, 1 for a tangency point, 2 for a nodal point, and 3 for a cusp. For a given curve D any two factorizations representing the braid monodromy of D are Hurwitz and conjugation equivalent (see e.g. [17]).

Consider two symplectic 4-manifolds X_1 and X_2, and let $f_k^i : X_i \to \mathbb{CP}^2$, $i \in \{1, 2\}$, $k \gg 0$ be the maps given by Theorem 1, with discriminant curves D_k^i. Assume that D_k^1 and D_k^2 have the same degree d_k. Denote by F_k^i the braid factorizations in B_{d_k} describing these curves, and by θ_k^i the corresponding geometric monodromy representations (see the introduction). Recall from the introduction that (F_k^1, θ_k^1) and (F_k^2, θ_k^2) are said to be m-equivalent if they differ by a sequence of global conjugations, Hurwitz moves, and node cancellations or creations. The above considerations and the uniqueness result obtained in the previous section (Theorem 5) imply the following corollary :

Corollary 1. *For any compact symplectic 4-manifold with $\frac{1}{2\pi}[\omega]$ integral, the sequence of braid factorizations and geometric monodromy representations describing the coverings obtained in Theorem 1 is, up to m-equivalence, an invariant of the symplectic structure.*

In other words, given two symplectic manifolds X_1 and X_2, if the corresponding sequences of braid factorizations and geometric monodromy representations are not m-equivalent for large k then X_1 and X_2 are not symplectomorphic.

The above invariants can be used to distinguish symplectic manifolds. There is a technique developed by Moishezon and Teicher for doing that in some cases ; unfortunately the fact that there might be negative intersections complicates everything. Two approaches are possible :

1) If the negative intersections cannot be removed then we have :

Corollary 2. *In the situation above, if the sequences of minimal numbers of negative half-twists in the factorizations F_k^1 and F_k^2 are different for large k then X_1 and X_2 are not symplectomorphic.*

Remark 3. In this statement we have to take the minimal numbers of negative half twists among the results of all possible sequences of node cancellations and creations. For example it may happen that creating pairs of nodes allows cancellations which were not possible initially.

Also note that all cancellation procedures are not equivalent : namely, there might exist examples of positive cuspidal factorizations which are m-equivalent but not Hurwitz and conjugation equivalent.

It will be interesting to find and study examples of symplectic manifolds that can be told apart by the minimal numbers of negative half-twists in their braid factorizations. Another interesting question is to study which properties of projective surfaces remain valid for symplectic coverings of $\mathbb{CP}^2$ that correspond to cuspidal positive factorizations – e.g. can they have arbitrary fundamental group ?

2) In case the negative intersections can be removed then we get Conjecture 1. We now outline a possible approach to the elimination of negative intersections, using the symplectic Lefschetz pencil structure associated to a quasiholomorphic covering (see Section 5 below).

It seems possible to define a finite dimensional space E_t of approximately holomorphic sections of L^k over each fiber C_t of the symplectic Lefschetz pencil. These spaces determine a vector bundle E over $\mathbb{CP}^1$. Each space E_t contains a divisor F_t consisting of all sections of L^k such that two critical levels of the corresponding projective map come together. A section σ of E determines a $\mathbb{CP}^2$-valued map, and the nodes of the corresponding branch curve are given by the intersections of σ with F. Our aim is therefore to find an approximately holomorphic section σ which always intersects F positively.

It seems that, whatever the chosen connection on the bundle E, it should be possible by computing the index of the $\bar{\partial}$ operator to prove that E admits holomorphic sections. However, it appears that it is not possible to find a connection on E which makes the divisor F pseudo-holomorphic : therefore the holomorphicity of the section σ is not sufficient to ensure positive intersection.

On the other hand, it seems relatively easy to find a connection on E for which the divisor F is approximately holomorphic. Unfortunately this does not guarantee positive intersection with the section σ unless one manages to obtain some uniform transversality estimates, and the techniques developped in this paper fall short of applying to this situation.

The prospect of being able to remove all negative nodes and obtain Conjecture 1 is very appealing for many reasons. Among these, one can note that the fundamental group $\pi_1(\mathbb{CP}^2 - D_k)$ becomes a symplectic invariant in this situation.

Remark 4. It is an interesting question to try to relate the braid monodromies obtained from the same manifold X for different degrees k. One can actually show using techniques similar to Sections 2 and 3 that, if $N \geq 2$ is any integer and if k is large enough, then the branch curve D_{Nk} can be obtained from D_k in the following way.

Consider the Veronese map $V_N : \mathbb{P}^2 \to \mathbb{P}^2$ of degree N^2, and let $R(V_N)$ be the corresponding smooth branch curve in the source $\mathbb{P}^2$. We can realize the covering $f_k : X \to \mathbb{P}^2$ in such a way that the branch curve D_k is transverse to $R(V_N)$. The covering $f_{Nk} : X \to \mathbb{P}^2$ can then be seen as a small perturbation of $V_N \circ f_k$. The branch curve of $V_N \circ f_k$ in X is the union of the branch curve of f_k and of $f_k^{-1}(R(V_N))$, so a perturbation is necessary to remove its singularities and obtain the generic covering f_{Nk}. The curve D_{Nk} can then be seen as a small deformation of the union of $V_N(D_k)$ and $\deg(f_k)$ copies of the branch curve of the Veronese map. This construction will be described in detail in a separate paper [4].

We will now prove Theorem 3, namely that any cuspidal negative factorization together with a geometric monodromy representation can be used to reconstruct a symplectic manifold. We start with part 1) of the statement.

Proof. Let $\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d$ be the representation corresponding to the given cuspidal negative factorization of Δ_d^2, and let C' be the universal covering of $\mathbb{C} - \{p_1, \dots, p_r\}$.

Recall that elements of B_d are equivalence classes of diffeomorphisms of the disk D' inducing the identity on the boundary of D' and preserving a set of d points $\{q_1, \dots, q_d\} \subset D'$: therefore it is possible, at least from a purely topological point of view, to construct the cross-product R of C' and D' above ρ, i.e. the quotient of $C' \times D'$ by the relations

$$\gamma(z, w) \sim (\gamma z, \rho(\gamma) w) \qquad \forall \gamma \in \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}),$$

where z and w are the coordinates on C' and D' respectively. Define Γ as the curve in R consisting of all points (z, q_i).

By construction, R is a disk bundle over $\mathbb{C} - \{p_1, \dots, p_r\}$ containing a curve Γ whose braid monodromy is precisely given by ρ.

Since the monodromy of ρ around infinity is Δ_d^2 we can extend R to a $\mathbb{P}^1$-bundle R' over $\mathbb{P}^1 - \{p_1, \dots, p_r\}$. In order to extend R' over all of $\mathbb{P}^1$ we need to define the geometry near the singular fibers. If the fiber corresponds to an element of degree 1 in B_d (half-twist), we can arrange that, in a suitable local trivialization of R' and choosing a local coordinate $z \in \mathbb{C} - \{0\}$ in the base, the two sheets of Γ exchanged by the half-twist correspond to the two square roots of z in suitable local coordinates on the fiber D'. Similarly, the two sheets should correspond to $\pm z$ respectively in the case of an element of degree 2 in the braid factorization, to the two square roots of z^3 in the case of an element of degree 3, and to $\pm \bar{z}$ for an element of degree -2.

With this geometric picture it is now possible to glue in the missing fibers. Moreover we can arrange that all points $q_1, ..., q_d$ lie close to the origin in D', i.e. that the curve Γ is contained in a neighboorhood of the zero section in R'. We can also arrange that, near the singular fibers, the local models described above hold in local approximately holomorphic complex coordinates. With such a choice of complex structure, we get the Hirzebruch surface F^1, as well as a curve $\Gamma' \subset F^1$ with the prescribed singularities and admitting a projection to $\mathbb{P}^1$, simply obtained as the closure of Γ in F^1.

It now follows from the construction that Γ' is a quasiholomorphic curve. Indeed, recall that Γ' lies in a neighborhood of the zero section in F^1 : this neighborhood can be made as small as desired, and the curve Γ' can be made as horizontal as desired (except near its tangency points), simply by rescaling the vertical coordinate in F^1. More explicitly, this vertical rescaling results from the automorphism of F^1 described in each fiber $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by the linear transformation $z \mapsto \lambda z$, where $\lambda > 0$ is a small enough constant.

After this rescaling process, which clearly does not affect the topology of the curve Γ', the properties expected of a quasiholomorphic curve still hold near the singular fibers, as it follows from the choice of the local models made above and from the observation that the rescaling diffeomorphism preserves the complex structure. Moreover, the tangent space to Γ' is almost horizontal everywhere except near the tangency points, and therefore Γ' is symplectic (because its tangent space at every point lies very close to a complex subspace – near a tangency point this follows from the local model, and at other points from the almost horizontality property of Γ'). Finally we need to observe that Γ' remains away from the infinity section in F^1, so we can blow down and recover a curve in $\mathbb{CP}^2$. This construction is clearly canonical up to isotopy. □

Remark 5. One can also try to prove assertion 1) of Theorem 3 in the following way. To every cuspidal negative factorization corresponds a representation $\bar{\rho}$: $\pi_1(\mathbb{P}^1 - \text{crit}) \to B'_d$. This representation defines a bundle $S \to \mathbb{P}^1 - \{p_1, ...p_r\}$ with a fiber $S_t - \Delta_t$, where S_t is the configuration space of d points in $\mathbb{C}$ and Δ_t is its diagonal. This is a bundle which is flat w.r.t. the nonabelian Gauss-Manin connection [18]. The bundle S admits a section s, defined by the braid factorization of the full twist (see [17]). The section s defines a covering of $\mathbb{P}^1 - \{p_1, ...p_r\}$ by a curve Γ. By construction this curve is in F^1, and one can proceed similarly to the above argument in order to complete the proof. This second approach presents an interesting way to look at the construction : if one can show that for $k \gg 0$ the section s is pseudoholomorphic and has nice intersection properties then we obtain Conjecture 1.

We now turn to the second part of Theorem 3, namely reconstructing a symplectic 4-manifold from a quasiholomorphic curve and a geometric monodromy representation. Note by the way that geometric monodromy representations are a very restrictive class of maps from F_d to S_n : the existence of such a representation is a non-trivial constraint on the braid factorization, and in many cases the geometric monodromy representation is unique up to conjugation (see [7] and [14]).

Proof. By definition, the geometric monodromy representation $\theta : F_d \to S_n$ factors through $\pi_1(\mathbb{CP}^2 - D)$ and therefore makes it possible to define a smooth four-dimensional manifold X (unique up to diffeomorphism). The projection $X \to \mathbb{P}^2$ is given everywhere by one of the three local models given in [3] for branched coverings (local diffeomorphism, branched covering of order 2, or cusp). Moreover these local models hold in orientation preserving coordinates on X and approximately holomorphic coordinates on $\mathbb{P}^2$ (because the curve D is approximately holomorphic).

Therefore the existence of a symplectic structure on X follows immediately from Proposition 10 of [3].

In order to show that this symplectic structure is canonically determined up to symplectomorphism we need to recall the argument more in detail. Proposition 10 of [3] is based on the following two observations. First, the local models describing the map $f : X \to \mathbb{CP}^2$ at any point of its branch set $R \subset X$ make it possible to construct an exact 2-form α on X such that, at any point $x \in R$, the restriction of α_x to the (2-dimensional) kernel K_x of the differential of f is nonzero and compatible with the natural orientation of K_x (in other words, α induces a volume form on K_x). Next, one observes that, calling ω_0 the standard symplectic form on $\mathbb{CP}^2$, and given any exact 2-form α which induces a volume form on K_x $\forall x \in R$, the 2-form $f^*\omega_0 + \epsilon\alpha$ is symplectic for any small enough $\epsilon > 0$.

Although the construction of the 2-form α in [3] is far from being canonical, the uniqueness of the resulting symplectic structure follows from a straightforward argument : to start with, note that, because the 2-form α is exact, Moser's theorem implies that for a fixed α the symplectic structure does not depend on the chosen value of ϵ (provided it is small enough). Therefore we can fix ϵ as small as needed and just need to consider the dependence on α. For this, let α_0 and α_1 be two exact 2-forms which induce volume forms on K_x at every point of R, and let $\alpha_t = t\alpha_1 + (1-t)\alpha_0$. Then, for all $t \in [0,1]$, the 2-form α_t is exact and induces a volume form on K_x $\forall x \in R$. It follows easily that, for small enough $\epsilon > 0$, the 2-forms $f^*\omega_0 + \epsilon\alpha_t$ are symplectic for all $t \in [0,1]$. Since the forms α_t are exact, if follows from Moser's theorem that $(X, f^*\omega_0 + \epsilon\alpha_0)$ is symplectomorphic to $(X, f^*\omega_0 + \epsilon\alpha_1)$. Therefore the symplectic structure on X is canonical.

Moreover, the symplectic structure does not depend either on the choice of D inside its isotopy class : indeed, let $(D_t)_{t\in[0,1]}$ be a family of quasiholomorphic curves and fix a geometric monodromy representation θ. It is clear that the corresponding branched covers are all diffeomorphic. Moreover, for any $t_0 \in [0,1]$ we can find an exact 2-form α_{t_0} which induces a volume form on K_x at every point of the branch curve of the covering f_{t_0}. However, this non-degeneracy condition is open, so there exists an open subset U_{t_0} in $[0,1]$ such that α_{t_0} induces a volume form at every point of the branch curve of the covering f_t for any $t \in U_{t_0}$. The compactness of $[0,1]$ implies that finitely many subsets $U_{t_1}, \dots, U_{t_q}$ cover $[0,1]$; for every t in $[0,1]$ a proper linear combination of $\alpha_{t_1}, \dots, \alpha_{t_q}$ can be defined in such a way as to obtain exact 2-forms which still have the required property but depend continuously on t. Once this is done, the above construction yields a family ω_t of symplectic forms on X which depend continuously on t and all lie in the same cohomology class. The desired uniqueness result is then a direct consequence of Moser's theorem. □

Note that, when X is already known to admit a symplectic form ω and $f : X \to \mathbb{CP}^2$ is a branched covering given by sections of L^k as in Theorem 1, the symplectic structure we construct is actually symplectomorphic to $k\omega$. Indeed, in this case we have $[f^*\omega_0] = k[\omega]$. Therefore, the 2-form $\alpha = k\omega - f^*\omega_0$ is exact. Since the local models for the covering map hold in approximately holomorphic coordinates, the restriction of ω to K_x is positive at any point x of R, so that α induces a volume form on K_x : therefore the canonical symplectic structure given by Theorem 3 can be chosen to be $f^*\omega_0 + \epsilon\alpha$ for any small $\epsilon > 0$. However it is known from Proposition 11 of [3] that the 2-forms $f^*\omega_0 + \epsilon\alpha$ are symplectic for all $\epsilon \in (0,1]$ and define the same structure up to symplectomorphism. In particular, for $\epsilon = 1$ one has $f^*\omega_0 + \alpha = k\omega$, so the symplectic form of Theorem 3 coincides with $k\omega$ up to symplectomorphism : therefore (X, ω) can be recovered from its braid monodromy invariants.

As a consequence, the manifold (X, ω) is uniquely characterized by its braid monodromy invariants ; this observation and Corollary 1 imply Theorem 2.

Remark 6. To obtain a symplectic structure on X we could also use the topological Lefschetz pencil $X \to \mathbb{P}^1$ corresponding to the branched covering (see Theorem 6 in §5) : the existence of a symplectic structure on X then follows from the results of Gompf (see also [1]). The fact that the curve D is quasiholomorphic implies that all Dehn twists in the Lefschetz pencil have the same orientation.

For braid factorizations which are not cuspidal negative, we do not get a quasiholomorphic curve and as consequence we cannot build the symplectic form on X as above. Of course the manifold X might still be symplectic and admit a different quasiholomorphic covering to $\mathbb{P}^2$.

Finally, the procedure of constructing invariants can be generalized in higher dimensions and using projections to higher-dimensional projective spaces. Of course describing the properties of the branch set and finding an analogue of the braid factorizations presents a real challenge.

In the 6-dimensional setting and still considering maps to $\mathbb{CP}^2$ given by three sections of L^k, we should get the following picture.

Given any compact 6-dimensional symplectic manifold X, three suitably chosen asymptotically holomorphic sections of L^k for $k \gg 0$ determine a map f_k from the complement of a finite set $B_k \subset X$ to $\mathbb{CP}^2$ which behaves like a generic projection of a complex projective 3-fold to $\mathbb{CP}^2$.

In particular, the generic fibers of f_k are smooth symplectic curves in X which fill X and intersect each other at the points of B_k (the base points of the family of curves). Moreover, there exists a singular symplectic curve D_k in $\mathbb{CP}^2$ which parametrizes the singular fibers of f_k. At a generic point $p \in D_k$, the fiber $f_k^{-1}(p)$ is a singular symplectic curve where one loop is pinched into a point, and the monodromy of the family of curves around D_k at p is given by a positive Dehn twist along the corresponding geometric vanishing cycle, exactly as in a 4-dimensional Lefschetz fibration.

Conjecture 2. *For large enough k the topological data arising from these structures provides symplectic invariants : to every 6-dimensional symplectic manifold corresponds a canonical sequence of braid factorizations (characterizing the curves D_k) and maps from $\pi_1(\mathbb{CP}^2 - D_k)$ to* Map_g *(characterizing the family of symplectic curves), with suitable properties (in particular every geometric generator of $\pi_1(\mathbb{CP}^2 - D_k)$ is mapped to a Dehn twist).*

Conversely, given a braid factorization with suitable properties and a map from $\pi_1(\mathbb{CP}^2 - D_k) \to$ Map_g *which sends geometric generators to Dehn twists, we should be able to reconstruct a symplectic 6-manifold.*

In this setup, let L be a generic line in $\mathbb{CP}^2$, and let $W_k = f_k^{-1}(L)$: W_k is a symplectic hypersurface in X realizing the class $\frac{k}{2\pi}[\omega]$ as in Donaldson's construction ; the restriction to L of the family of curves coincides with the Lefschetz pencil structure obtained by Donaldson's construction on W_k. We will consider the above conjecture in a separate paper.

5. Projections to $\mathbb{CP}^2$ and Lefschetz pencils

5.1. **Quasiholomorphic coverings and Lefschetz pencils.** In this section we prove Theorem 4, namely the fact that quasiholomorphic coverings determine Lefschetz pencils.

Proof. Let $s_k = (s_k^0, s_k^1, s_k^2) \in \Gamma(\mathbb{C}^3 \otimes L^k)$ be the sections which determine the covering map $f_k = \mathbb{P}(s_k)$, and let ϕ_k be the $\mathbb{CP}^1$-valued map determined by s_k^0 and s_k^1 outside of $f_k^{-1}(0:0:1)$. We use the notations and definitions of Section 2. By assumption, the section s_k is the result of the procedure described in Sections 2 and 3 for achieving Theorem 1, and therefore satisfies all the transversality properties introduced in Section 2, as well as the tameness properties described in Section 3.1.

We claim that s_k^0 and s_k^1 define a structure of symplectic Lefschetz pencil on X. For this we need to check that, for some $\gamma > 0$, (s_k^0, s_k^1) is γ-tranverse to 0 as a section of $\mathbb{C}^2 \otimes L^k$, that $\partial\phi_k$ is γ-transverse to 0 as well, and that $\bar{\partial}\phi_k$ vanishes at the points where $\partial\phi_k = 0$. By Proposition 12 of [3], these three properties imply that s_k^0 and s_k^1 define a Lefschetz pencil (see also [8]) : the first property yields the expected structure at the base points of the pencil, and the two other conditions imply that ϕ_k is a complex Morse function.

The transversality to 0 of (s_k^0, s_k^1) is granted by the construction carried out to prove Theorem 1 : more precisely this property, which is one of the transversality properties required at the very beginning of Section 3.1, is achieved in Proposition 1. The other transversality properties which one requires in this construction are γ-genericity (Definition 6) and γ-transversality to the projection (Definition 8) : we now show that these properties imply the transversality to 0 of $\partial\phi_k$ with a transversality estimate decreased by at most a constant factor. In other words, we show that the (2,0)-Hessian $\partial\partial\phi_k$ is non-degenerate (and has determinant bounded from below) at any point where $\partial\phi_k$ is small.

Consider a point $p \in X$ where $|\partial\phi_k|$ is smaller than γ/C for some suitable constant C. To start with, note that since ∂f_k is uniformly bounded $\partial\phi_k$ cannot be smaller than γ/C unless the (2,0)-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$ is smaller than γ. Because of the genericity property, $\mathrm{Jac}(f_k)$ is γ-transverse to 0, and it follows immediately that p must lie very close to the branch set $R(s_k)$. In particular, if C is chosen large enough there exists a point $p' \in R(s_k)$ which lies sufficiently close to p in order to ensure that $|\partial\phi_k(p')|$ is also much smaller than γ. This in turn implies that the quantity $\mathcal{K}(s_k) = \partial\phi_k \wedge \partial\mathrm{Jac}(f_k)$ is smaller than γ at p' ; since $\mathcal{K}(s_k)$ is γ-transverse to 0 over $R(s_k)$ (see Definition 8), p' must lie very close to a point $q \in R(s_k)$ where $\mathcal{K}(s_k)$ vanishes, i.e. either a cusp or a tangency point (see Definition 9). Moreover, cusp points are characterized by the transverse vanishing of $\partial f_k \wedge \partial\mathrm{Jac}(f_k)$, so, as noted at the beginning of Step 2 in the proof of Proposition 2, the transverse vanishing of $\mathcal{K}(s_k)$ at the cusps implies that $\partial\phi_k$ cannot be too small at a cusp point (in other words, the cusps are not close to being tangent to the fibers of the projection to $\mathbb{CP}^1$). Therefore q is a tangency point, i.e. $\partial\phi_k(q) = 0$.

Because s_k is tamed by the projection $\pi : \mathbb{CP}^2 - \{pt\} \to \mathbb{CP}^1$ we also have $\bar{\partial}\phi_k(q) = 0$ (see Definition 9). Therefore the image of $df_k(q)$ is exactly the tangent space to the fiber of π through $f_k(q)$. Let Z_1 and Z_2 be local complex coordinates on

$\mathbb{CP}^2$ at $f_k(q)$ chosen in such a way that the projection π is given by $(Z_1, Z_2) \mapsto Z_1$ locally : it is then easy to check that $z_2 = f_k^* Z_2$ has nonvanishing derivative at q and that one can find a complex-valued function z_1 such that (z_1, z_2) are approximately holomorphic local complex coordinates on X. In these local coordinates the map f_k is given by

$$f_k(z_1, z_2) = (a_{k,q} z_1^2 + b_{k,q} z_1 z_2 + c_{k,q} z_2^2 + O(k^{-1/2}|z|^2) + O(|z|^3), z_2).$$

One then has $\partial\phi_k = (2a_{k,q}z_1 + b_{k,q}z_2)\, dz_1 + (b_{k,q}z_1 + 2c_{k,q}z_2)\, dz_2 + O(k^{-1/2}|z|) + O(|z|^2)$ and $\mathrm{Jac}(f_k) = 2a_{k,q}z_1 + b_{k,q}z_2 + O(k^{-1/2}|z|) + O(|z|^2)$, and therefore

$$\mathcal{K}(s_k) = \partial\phi_k \wedge \partial\mathrm{Jac}(f_k) = (b_{k,q}^2 - 4a_{k,q}c_{k,q}) z_2\, dz_1 \wedge dz_2 + O(k^{-1/2}|z|) + O(|z|^2).$$

The transverse vanishing of $\mathcal{K}(s_k)$ at q therefore implies that $b_{k,q}^2 - 4a_{k,q}c_{k,q}$ is bounded away from 0. However this quantity is exactly the determinant of the Hessian $\partial\partial\phi_k$ at q, so $\partial\phi_k$ vanishes transversely at q. Since the point p lies close to q, the (2,0)-Hessian of ϕ_k at p is nondegenerate as well. This establishes the γ'-transversality to 0 of $\partial\phi_k$ for some constant $\gamma' > 0$ (independently of k).

We also know that $\bar\partial f_k$ vanishes at every tangency point, i.e. at every point where $\partial\phi_k$ vanishes (this follows from the property of tameness with respect to the projection, see Definition 9) : this immediately implies that $\bar\partial\phi_k$ vanishes at all points where $\partial\phi_k$ vanishes. The properties of s_k^0 and s_k^1 are therefore sufficient to ensure by Proposition 12 of [3] that they define a symplectic Lefschetz pencil. □

Even when a branched covering is not determined by three approximately holomorphic sections of a line bundle, it is still possible to recover a Lefschetz pencil. This can actually be carried out in a setting more general than that of quasiholomorphic curves : starting with a braid factorization consisting of factors of degrees ranging from -3 to $+3$, it is possible to construct a curve $D \subset \mathbb{CP}^2$ which realizes this factorization and whose only singularities are nodes and cusps (with either positive or negative orientation), and which is transverse to the projection to $\mathbb{CP}^1$ except at finitely many points where a local model in complex coordinates is either $x^2 = y$ (when the degree is $+1$) or $x^2 = \bar y$ (when the degree is -1). Given a geometric monodromy representation $\theta : F_d \to S_n$, we can then construct a 4-manifold X which covers $\mathbb{CP}^2$ and ramifies at D (in general this manifold is not symplectic because we allow factors of degree -1 in the braid factorization). In this very general setting we have :

Theorem 6. *To every covering of $\mathbb{CP}^2$ ramified at a curve given by a factorization of Δ^2 into elements of degrees -3 to 3 there corresponds a topological Lefschetz pencil whose singular fibers are given by the elements of degree ± 1 in the braid factorization. Moreover, if there are no elements of degree -1 then the Lefschetz pencil is chiral and therefore admits a symplectic structure.*

The easiest way to prove this result is to use local models in order to show that the composition of the $\mathbb{CP}^2$-valued covering map with the projection to $\mathbb{CP}^1$ defines a Lefschetz pencil. To start with, the branch curve in $\mathbb{CP}^2$ does not hit the point $(0:0:1)$ (the pole of the projection to $\mathbb{CP}^1$) ; this implies that the topological structure near the base points (i.e. the preimages of $(0:0:1)$) is exactly that of a pencil, because the covering map is a local diffeomorphism at each of these points. Therefore one just needs to check that the $\mathbb{CP}^1$-valued map obtained by projection of the covering map has isolated critical points and that the topological structure at these points is as expected. For this, observe that, when one restricts to the preimage of any small ball in $\mathbb{CP}^1$, the branch curve in $\mathbb{CP}^2$ behaves exactly like in the quasiholomorphic case (after reversing the orientation in the case of a negative tangency point or a negative cusp) : this implies that, like in the proof of Theorem 4, the only critical points of the map to $\mathbb{CP}^1$ correspond to the tangency points. At a positive tangency point (i.e. an element of degree +1 in the braid factorization) the behavior is that of a complex Morse function, by local identification with the quasiholomorphic model ; while at a negative tangency point (i.e. an element of degree -1 in the braid factorization) the picture is mirrored and one needs to reverse the orientation of $\mathbb{CP}^1$ in order to recover the correct local model. In any case one gets a topological Lefschetz pencil, and in the absence of negative tangency points this pencil structure is compatible with the orientation. □

5.2. **Braid groups and mapping class groups.** This observation that branched coverings determine Lefschetz pencils can also be made at the more algebraic level of monodromy factorizations. Indeed, let us consider a negative cuspidal braid factorization of Δ_d^2 in B_d, or equivalently the corresponding braid monodromy morphism $\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d$. Denote by m the number of factors of degree 1 (we will assume that these correspond to the points $p_1, \dots, p_m$) ; the argument also applies to the more general factorizations described in Theorem 6, in which case one also needs to add the elements of degree -1. Let $D \subset \mathbb{CP}^2$ be the curve determined by this braid factorization, and let us consider a geometric monodromy representation $\theta : F_d \to S_n$ (recall from the introduction that θ factors through the natural surjection from $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ to $\pi_1(\mathbb{CP}^2 - D)$).

Because a branched covering determines a Lefschetz pencil, the monodromies ρ and θ of the branched covering should determine a monodromy representation $\psi : \pi_1(\mathbb{C} - \{p_1, \dots, p_m\}) \to M_g$, where M_g is the mapping class group of a Riemann surface of genus $g = 1 - n + (d/2)$, which describes the topology of the Lefschetz pencil. The way in which ψ is related to ρ and θ can be described as follows ; the reader may also refer to the work of Birman and Wajnryb [5] for a detailed investigation of the case $n = 3$.

First, consider the set $\mathcal{C}_n(q_1, \dots, q_d)$ of all simple n-fold coverings of $\mathbb{CP}^1$ branched at $q_1, \dots, q_d$ whose sheets are labelled by integers between 1 and n. We just think of coverings in combinatorial terms, i.e. up to isotopy, so this set is actually finite : more precisely $\mathcal{C}_n(q_1, \dots, q_d)$ is the set of all surjective group homomorphisms $F_d \to S_n$ which map each of the generators $\gamma_1, \dots, \gamma_d$ of F_d to a transposition and map their product $\gamma_1 \cdots \gamma_d$ to the identity element in S_n. In particular, the given geometric monodromy representation $\theta : F_d \to S_n$ determines an n-fold branched covering of $\mathbb{CP}^1$, i.e. θ is an element of $\mathcal{C}_n(q_1, \dots, q_d)$.

Observe that the braid group B_d acts naturally on $\mathcal{C}_n(q_1, \dots, q_d)$. Indeed, recall that braids can be considered as equivalence classes of diffeomorphisms of the disk preserving the set $\{q_1, \dots, q_d\}$; therefore, given a braid $Q \in B_d$, one can choose a diffeomorphism ϕ representing it, and extend it to a diffeomorphism $\bar{\phi}$ of $\mathbb{CP}^1$ which is the identity outside of the disk. The action of the braid Q on $\mathcal{C}_n(q_1, \dots, q_d)$ is given by the map which to a given covering $f : \Sigma_g \to \mathbb{CP}^1$ associates the covering $\bar{\phi} \circ f$. It can be easily checked that the topology of the resulting covering does not depend on the choice of $\bar{\phi}$ in its equivalence class. Alternately, viewing a braid as a motion of the branch points $q_1, \dots, q_d$ in the plane, the above-described action of the braid Q on $\mathcal{C}_n(q_1,\dots,q_d)$ simply corresponds to the natural transformation that occurs when the branch points are moved along the given trajectories.

We now describe the action of B_d on $\mathcal{C}_n(q_1,\dots,q_d)$ in terms of morphisms from F_d to S_n. Recall that the braid group B_d acts on the free group $F_d=\pi_1(\mathbb{C}-\{q_1,\dots,q_d\})$, and denote by $Q_*:F_d\to F_d$ the automorphism induced by a braid $Q\in B_d$. Then, it can be easily checked that the action of Q on $\mathcal{C}_n(q_1,\dots,q_d)$ simply corresponds to composition with Q_* : the action of the braid Q on the covering described by $\theta:F_d\to S_n$ yields the covering described by $\theta\circ Q_*:F_d\to S_n$.

We now define the subgroup $B^0_d(\theta)$ of B_d as the stabilizer of θ for this action, i.e. the set of all braids Q such that $\theta\circ Q_*=\theta$. These braids are exactly those which preserve the covering structure defined by θ. Note by the way that $B^0_d(\theta)$ is clearly a subgroup of finite index in B_d.

Whenever $Q\in B^0_d(\theta)$, its action on the covering determined by θ can be thought of as an element $\theta_*(Q)$ of the mapping class group M_g, describing how the Riemann surface Σ_g is affected when the branch points $q_1,\dots,q_d$ are moved along the braid Q. More precisely, choose as above a diffeomorphism ϕ of the disk representing Q and extend it as a diffeomorphism $\bar\phi$ of $\mathbb{CP}^1$ preserving the branch points. It is then possible to lift $\bar\phi$ via the branched covering as a diffeomorphism of the surface Σ_g, whose class in the mapping class group does not depend on the choice of ϕ in its equivalence class. This element in M_g is precisely $\theta_*(Q)$. Viewing the braid Q as a motion of the branch points, the transformation $\theta_*(Q)$ can also be described in terms of the monodromy that arises when the points $q_1,\dots,q_d$ are moved along their trajectories. The map $\theta_*:B^0_d(\theta)\to M_g$ is naturally a group homomorphism.

Remark 7. A more abstract definition of θ_* is as follows. Denote by $\mathcal{X}_d$ the space of configurations of d distinct points in the plane. The set of all n-fold coverings of $\mathbb{CP}^1$ with d branch points and such that no branching occurs above the point at infinity can be thought of as a covering $\tilde{\mathcal{X}}_{d,n}$ above $\mathcal{X}_d$, whose fiber above the configuration $\{q_1,\dots,q_d\}$ identifies with $\mathcal{C}_n(q_1,\dots,q_d)$. The braid group B_d identifies with the fundamental group of $\mathcal{X}_d$, and the action of B_d on $\mathcal{C}_n(q_1,\dots,q_d)$ described above is exactly the same as the action of $\pi_1(\mathcal{X}_d)$ by deck transformations of $\tilde{\mathcal{X}}_{d,n}$. The subgroup $B^0_d(\theta)$ is then the set of all the loops in $\mathcal{X}_d$ whose lift at the point $p_\theta\in\tilde{\mathcal{X}}_{d,n}$ corresponding to the covering described by θ is a closed loop in $\tilde{\mathcal{X}}_{d,n}$.

There exists a natural (tautologically defined) bundle $\mathcal{Y}_{d,n}$ over $\tilde{\mathcal{X}}_{d,n}$ whose fiber is a Riemann surface of genus g. Given an element Q of $B^0_d(\theta)$, it lifts to $\tilde{\mathcal{X}}_{d,n}$ as a loop based at the point p_θ, and the monodromy of the fibration $\mathcal{Y}_{d,n}$ around this loop is precisely the mapping class group element $\theta_*(Q)$.

It is easy to check that the image of the braid monodromy homomorphism $\rho:\pi_1(\mathbb{C}-\{p_1,\dots,p_r\})\to B_d$ is contained in $B^0_d(\theta)$: this is because the geometric monodromy representation θ factors through $\pi_1(\mathbb{CP}^2-D)$, on which the action of the elements of $\mathrm{Im}\,\rho$ is clearly trivial. Therefore, we can define the composed map

$$\psi:\pi_1(\mathbb{C}-\{p_1,\dots,p_r\})\xrightarrow{\ \rho\ }B^0_d(\theta)\xrightarrow{\ \theta_*\ }M_g.$$

The group homomorphism ψ is naturally the monodromy of the Lefschetz pencil corresponding to ρ and θ. Because the only singular fibers of the Lefschetz pencil are those which correspond to elements of degree 1 in the braid factorization, this map actually factors through the canonical surjection map $\pi_1(\mathbb{C}-\{p_1,\dots,p_r\})\to$

$\pi_1(\mathbb{C} - \{p_1, \dots, p_m\})$, thus yielding the ordinary description of the monodromy of a Lefschetz pencil as a factorization of the identity into a product of positive Dehn twists in the mapping class group.

We now describe how the images of the various factors in the braid factorization by the map θ_* can be computed explicitly. Such an explicit description makes it very easy to recover the monodromy of the Lefschetz pencil out of the braid factorization and the geometric monodromy representation.

Proposition 3. *The elements of degree ± 2 and 3 in the braid factorization (i.e. the nodes and cusps) lie in the kernel of the map $\theta_* : B_d^0(\theta) \to M_g$.*

Proof. This result is a direct consequence of the fact that the cusps and nodes in the branch curve do not correspond to singular fibers of the Lefschetz pencil. From a more topological point of view, the argument is as follows. Consider a braid $Q \in B_d$ which arises as an element of degree ± 2 or 3 in the braid factorization. Since Q is a power of a half-twist, it can be realized by a diffeomorphism ϕ of the disk D' whose support is contained in a small neighborhood U of an arc in $D' - \{q_1, \dots, q_d\}$ joining two of the branch points, say q_i and q_j. As explained above the element $\theta_*(Q)$ in M_g is obtained by extending ϕ to the sphere and lifting it via the branched covering $f : \Sigma_g \to \mathbb{CP}^1$. In particular, $\theta_*(Q)$ can be represented by a diffeomorphism of Σ_g whose support is contained in $f^{-1}(U)$.

In the case of a node ($r_j = \pm 2$), the transpositions in S_n corresponding to loops around the two branch points are disjoint, and therefore $f^{-1}(U)$ consists of $n-2$ components : two of these components are double covers of the disk U branched at one point (q_i for one, q_j for the other), and f restricts to each of the $n-4$ other components as an homeomorphism. Therefore, $f^{-1}(U)$ is topologically a disjoint union of $n-2$ disks contained in the surface Σ_g ; since no non-trivial element of the mapping class group can have support contained in a union of disks, we conclude that $\theta_*(Q)$ is trivial, i.e. $Q \in \mathrm{Ker}\ \theta_*$.

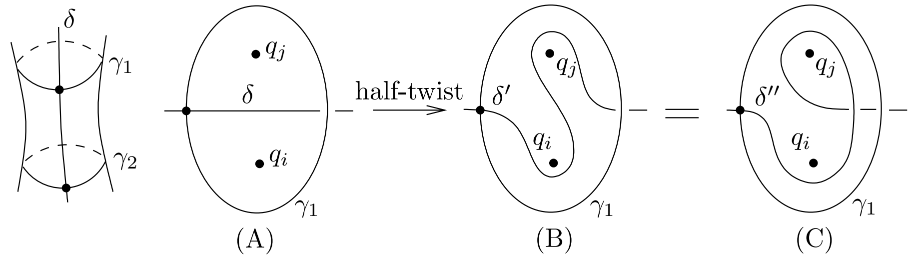
In the case of a cusp ($r_j = 3$), the transpositions in S_n corresponding to loops around the two branch points are adjacent, and $f^{-1}(U)$ consists of $n-2$ components : one of these components is a triple cover of the disk U branched at two points, and f restricts to each of the $n-3$ other components as an homeomorphism. By the same argument as above, $f^{-1}(U)$ is still topologically a disjoint union of disks in Σ_g, and therefore $Q \in \mathrm{Ker}\ \theta_*$. □

We now turn to the case where Q is an element of degree 1 in the braid factorization. We keep the same notation as above, letting U be a embedded disk containing the two branch points q_i and q_j as well as the path joining them along which the half-twist is performed. As previously, the mapping class group element $\theta_*(Q)$ can be represented by a diffeomorphism whose support is contained in $f^{-1}(U)$. However, since the transpositions in S_n arising in the picture are now equal to each other, $f^{-1}(U)$ contains a topologically non-trivial component, namely a double cover of U branched at the two points q_i and q_j, which is homeomorphic to a cylinder. Since $\theta_*(Q)$ is necessarily trivial in the other components of $f^{-1}(U)$, we can restrict ourselves to this cylinder and assume that $n = 2$.

Denote by γ the (oriented) boundary of U, and by γ_1 and γ_2 its two lifts to the cylinder $f^{-1}(U)$, which are precisely the two components of its boundary.

Proposition 4. *The image of the half-twist Q by $\theta_* : B^0_d(\theta) \to M_g$ is the positive Dehn twist along γ_1 (or γ_2).*

Proof. Without loss of generality we can restrict ourselves to a neighborhood of $f^{-1}(U)$, and assume that f is a two-fold covering. The mapping class group element $\theta_*(Q)$ is supported in the cylinder $f^{-1}(U)$, and therefore it acts trivially on all loops in Σ_g which admit a representative disjoint from $f^{-1}(U)$. It is then easy to check that $\theta_*(Q)$ is necessarily a power of the Dehn twist along γ_1 (or equivalently γ_2). This transformation is therefore completely determined by the way in which it affects an arc δ joining γ_1 to γ_2 across the cylinder. The projections of γ_1 and δ, as well as their intersection point and the two branch points, are as represented below (situation (A)).



The half-twist Q has the effect of moving the curve δ to the new curve δ' represented in (B). Observing that the lift of a small loop going twice around q_j is homotopically trivial in $f^{-1}(U)$, the arc δ' is homotopic to the curve δ'' represented in (C), which can be easily seen to differ from δ by a positive Dehn twist along γ_1. Therefore the transformation $\theta_*(Q) \in M_g$ is the positive Dehn twist along γ_1. ☐

Example. Let X be a smooth algebraic surface of degree 3 in $\mathbb{CP}^3$, and let us consider a generic projection of $\mathbb{CP}^3 - \{pt\}$ to $\mathbb{CP}^2$. This makes X a 3-fold cover of $\mathbb{CP}^2$ branched along a curve C of degree 6 with 6 cusps (there are no nodes in this case). For a generic projection to $\mathbb{CP}^1$ the curve C has 12 tangency points, and the corresponding braid group factorization in B_6 has been computed by Moishezon in [16]. For all $1 \le j < k \le 6$, let $Z_{jk} = X_{k-1}.\cdots.X_{j+1}.X_j.X_{j+1}^{-1}.\cdots.X_{k-1}^{-1}$ be the half-twist along the segment which joins q_j and q_k in D^2 when the points $q_1, \dots, q_6$ are placed along a circle : then the braid group factorization is given by

$$\Delta_6^2 = \left(Z_{35}Z_{46}Z_{13}Z_{24}Z_{12}^3Z_{34}^3Z_{56}^3\right)^2 Z_{35}Z_{46}Z_{13}Z_{24},$$

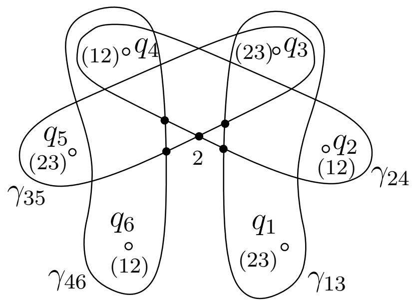
and the corresponding geometric monodromy representation

$$\theta : \pi_1(D^2 - \{q_1, \dots, q_6\}) \to S_3$$

maps the geometric generators around $q_1, \dots, q_6$ to the transpositions (23), (12), (23), (12), (23) and (12) respectively.

The corresponding Lefschetz pencil has 3 base points and consists of elliptic curves ; after blowing up X three times it becomes the standard elliptic fibration of $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ over $\mathbb{CP}^1$ with 12 singular fibers. Its monodromy is therefore expected to be given by the word $(D_aD_b)^6 = 1$ in the mapping class group $M_1 = SL(2,\mathbb{Z})$, where D_a and D_b are the Dehn twists along the two generators a and b of $\pi_1(T^2)$. We now check that this is indeed consistent with what one obtains from the above braid monodromy.

We know that the braids Z_{12}^3, Z_{34}^3 and Z_{56}^3 lie in the kernel of θ_*, by Proposition 3. Moreover, by Proposition 4 the other elements which appear in the braid factorization are mapped to Dehn twists along suitable curves γ_{35}, γ_{46}, γ_{13} and γ_{24} in T^2. The projections of these curves to $\mathbb{P}^1$ are as shown in the diagram below ; their only intersections are the five points indicated by solid circles, and all these intersections happen in the second sheet of the covering.



γ_{13} and γ_{24} have intersection number $+1$, so they generate $\pi_1(T^2) = \mathbb{Z}^2$ and will be referred to as respectively a and b. One then easily checks that $\gamma_{35} = b - a$ and $\gamma_{46} = -a$ (we use additive notation). Note that we don't have to worry about orientations as the positive Dehn twists D_γ and $D_{-\gamma}$ are the same for any loop γ.

It follows from these computations that the braid factorization given above is mapped by θ_* to the factorization $(D_{b-a}D_aD_aD_b)^3$ in M_1. A Hurwitz operation changes $D_{b-a}D_a$ into D_aD_b, so the Lefschetz pencil monodromy we have just obtained is indeed Hurwitz equivalent to the expected factorization $(D_aD_b)^6$.

We end with a couple of general remarks.

Remark 8. There should exist intrinsic restrictions on braid monodromies coming from the very structure of the braid group, in a manner quite similar to the restrictions on the monodromy of a symplectic Lefschetz pencil coming from the structure of the mapping class group. Since a braid factorization and a geometric monodromy representation determine a word in the mapping class group, every known restriction on the monodromy of Lefschetz fibrations should yield a corresponding restriction on the braid group factorizations for which a geometric monodromy representation exists.

For example, it is known [1] that the image of the monodromy of a symplectic Lefschetz fibration cannot be contained in the Torelli group. It is also known that there does not exist any non-trivial element in the fundamental group of the generic fiber Σ_g which remains fixed by the monodromy of the Lefschetz fibration ([12], [11]). It is an interesting question to study how these restrictions translate on the level of braid factorizations. Another related question is to look for any specific constraints on the separating vanishing cycles of a symplectic Lefschetz fibration coming from the underlying braid factorization.

Remark 9. Both the braid factorizations arising from branched coverings and the mapping class group factorizations arising in the Lefschetz pencil situation are quite difficult to use directly. In the Lefschetz pencil situation Donaldson has introduced the idea of dealing with an invariant which would be easier to handle although containing less information. This invariant arises by considering cylinders joining the geometric vanishing cycles and coning them to get immersed Lagrangian -2-spheres in the symplectic manifold. Using the correspondence described above between branched coverings and Lefschetz pencils, we can see these cylinders as corresponding to all possible degenerations of the branch curve where two tangency points come together and form a double point. Hopefully these degenerations or other related structures might help in deriving a more usable invariant from braid monodromies.

6. Examples

We now consider the examples defined by Moishezon in [17]. These examples are obtained by putting together several geometric projections of the Veronese surface to $\mathbb{CP}^2$ and applying certain twists to the corresponding braid factorization. These twists are performed in such a way that the braid factorization remains geometric, so that one obtains new manifolds as branched coverings of $\mathbb{CP}^2$ ramified along the curves constructed by this procedure (see [17]). Specializing to the case of Veronese maps of degree 3, we obtain a infinite sequence of smooth four-dimensional manifolds $X_{3,i}$.

Proposition 5. *The manifolds $X_{3,i}$ are all homeomorphic.*

Proof. It was remarked by Moishezon in [17] that all $X_{3,i}$ are simply connected. We will show that $X_{3,i}$ are not spin and therefore their homeomorphism type is determined by their signature and Euler characteristic.

We now compute the signature and Euler characteristic of the manifolds $X_{p,i}$ obtained by Moishezon by twisting Veronese maps of degree p. All $X_{p,i}$ are p^2-sheeted coverings of $\mathbb{CP}^2$ ramified at curves $D_{p,i}$ of degree d_p with κ_p cusps and ν_p nodes, where $d_p = 9p(p-1)$ and

$$\kappa_p = 27(p-1)(4p-5), \quad \nu_p = \frac{27}{2}(p-1)((p-1)(3p^2-14)+2)$$

(these values are computed in [17]). We get immediately that the genus g_p of $D_{p,i}$ is given by $2g_p - 2 = d_p^2 - 3d_p - 2(\kappa_p + \nu_p) = 27(p-1)(5p-6)$.

Let us denote by $f_{p,i}$ the covering map, and consider the homology class $L = f_{p,i}^*(H) \in H_2(X_{p,i}, \mathbb{Z})$ given by the pull-back of the hyperplane. Also, call K the canonical class of $X_{p,i}$, and let $R \subset X_{p,i}$ be the set of branch points of the covering $f_{p,i}$. Because we are in a quasiholomorphic situation we can consider R (or a small perturbation of it) as the zero set of the $(2,0)$-Jacobian $\mathrm{Jac}(f_{p,i})$, which is an approximately holomorphic section of $\Lambda^{2,0}T^*X \otimes f_{p,i}^* \det T\mathbb{CP}^2$, a line bundle over $X_{p,i}$ whose first Chern class is $3L + K$. It follows that $[R] = 3L + K$.

We can now express the quantities d_p, κ_p and $2g_p - 2$ in terms of the classes L and K. To start with, note that the degree of the covering $f_{p,i}$ is given by $\deg f_{p,i} = L.L$. Next, $d_p = [D_{p,i}].H = [R].L = 3L.L + K.L$. Moreover, R is a smooth connected symplectic curve, so its genus is given by the adjunction formula : $2g_p - 2 = [R].[R] + K.[R] = 9L.L + 9K.L + 2K.K$. Finally, the cusps are the points where $\partial f_{p,i} \wedge \partial \mathrm{Jac}(f_{p,i})$ vanishes and $\partial \mathrm{Jac}(f_{p,i})$ does not vanish ; a quick computation of the Euler classes yields that $\kappa_p = 12L.L + 9K.L + 2K.K - e_{p,i}$, where $e_{p,i}$ is the Euler-Poincaré characteristic of $X_{p,i}$. Comparing these values with those from [17] one gets the equations

$$\begin{cases} L.L & = & p^2 \\ 3L.L + K.L & = & 9p(p-1) \\ 9L.L + 9K.L + 2K.K & = & 27(p-1)(5p-6) \\ 12L.L + 9K.L + 2K.K - e_{p,i} & = & 27(p-1)(4p-5) \end{cases}$$

This yields

$$\begin{array}{ll} L.L = p^2 & K.L = 6p^2 - 9p \\ K.K = 36p^2 - 108p + 81 & e_{p,i} = 30p^2 - 54p + 27. \end{array}$$

In the case $p = 3$ this implies that $K^2 = 81$, and $e(X_{3,i}) = 135$. Therefore we conclude that the signature is $\sigma(X_{3,i}) = -63$ and hence the manifolds $X_{3,i}$ are not spin. Since they have the same Euler characteristic and signature they are all homeomorphic. □

It follows from [17] that the fundamental groups $\pi_1(\mathbb{CP}^2 - D_{3,i})$ are all different, although the curves $D_{3,i}$ are in the same homology class and have the same numbers of cusps and nodes.

This situation is a generalization of the well-known phenomenon of Zariski pairs. Of course there are finitely many non-isotopic holomorphic curves of a given degree with given numbers of nodes and cusps, so only finitely many of the curves $D_{3,i}$ are holomorphic.

On the other hand, as a consequence from Theorem 3 we get that all smooth four-manifolds $X_{3,i}$ are symplectic. It is then natural to ask the following :

Question : Are the manifolds $X_{3,i}$ symplectomorphic ?

We expect the answer to this question to be negative, because the braid factorizations computed by Moishezon are quite different. If the manifolds $X_{3,i}$ are not symplectomorphic, then other natural questions arise : are these manifolds diffeomorphic ? Do they have the same Seiberg-Witten invariants ?

If the Seiberg-Witten invariants cannot tell apart the manifolds $X_{3,i}$, then the only way to show that the Moishezon manifolds are not symplectomorphic might be to use the invariants arising from symplectic branched coverings or symplectic Lefschetz pencils.

References

[1] J. Amorós, F. Bogomolov, L. Katzarkov, T. Pantev, I. Smith, *Symplectic Lefschetz fibrations with arbitrary fundamental groups*, preprint, math.GT/9810042.

[2] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[3] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$, Invent. Math., to appear.

[4] D. Auroux, L. Katzarkov, *The degree doubling formula for braid monodromies and Lefschetz pencils*, in preparation.

[5] J. Birman, B. Wajnryb, *3-fold branched coverings and the mapping class group of a surface*, Geometry and Topology (College Park, 1983/84), Lecture Notes in Math. **1167**, Springer, 1985, 24–46.

[6] F. Catanese, *Connected components of the moduli spaces*, J. Diff. Geom. **24** (1986), 395–399.

[7] F. Catanese, *(Some) old and new results on algebraic surfaces*, First European Congress of Mathematics, Vol. I (Paris, 1992), Progr. Math. **119**, Birkhäuser, 1994, 445–490.

[8] S. Donaldson, *Lefschetz pencils on symplectic manifolds*, preprint (1999).

[9] S. Donaldson, *Lefschetz fibrations in symplectic geometry*, Documenta Math., Extra Volume ICM 1998, II, 309–314.

[10] R. Fintushel, R. Stern, *Knots, links and 4-manifolds*, Invent. Math. **134** (1998), 363–400.
[11] I. Smith, *Geometric monodromy and the hyperbolic disc*, preprint (1999), submitted to Math. Res. Lett.
[12] L. Katzarkov, T. Pantev, C. Simpson, *Non-abelian open orbit theorems*, preprint (1998).
[13] V. Kulikov, *On Zariski's pairs of m-th canonical discriminant curves*, preprint, math.AG/9807154.
[14] V. Kulikov, *On a Chisini conjecture*, preprint, math.AG/9803144.
[15] V. Kulikov, M. Teicher, *Braid monodromy factorization and diffeomorphism types*, preprint (1999), math.AG/9905129.
[16] B. Moishezon, *Stable branch curves and braid monodromies*, Algebraic Geometry (Chicago, 1980), Lecture Notes in Math. **862**, Springer, 1981, 107–192.
[17] B. Moishezon, *The arithmetic of braids and a statement of Chisini*, Geometric Topology (Haifa, 1992), Contemp. Math. **164**, Amer. Math. Soc., Providence, 1994, 151–175.
[18] C.T. Simpson, *Moduli of representations of the fundamental groups of a smooth projective variety II*, IHES - Publications Mathématiques No. **80** (1994), 5–79.
[19] C.H. Taubes, *The geometry of the Seiberg-Witten invariants*, Surveys in Diff. Geom., Vol. III (Cambridge, 1996), Int. Press, Boston, 1998, 299–339.
[20] M. Teicher, *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math., **62** (part 1), Amer. Math. Soc., 1997, 127–150.
[21] E. Witten, *Monopoles and 4-manifolds*, Math. Res. Lett. **1** (1994), 769–796.

Centre de Mathématiques, Ecole Polytechnique, 91128 Palaiseau, France
E-mail address: auroux@math.polytechnique.fr

UC Irvine, Irvine, CA 92612
E-mail address: lkatzark@math.uci.edu

SYMPLECTIC MAPS TO PROJECTIVE SPACES AND SYMPLECTIC INVARIANTS

DENIS AUROUX

Abstract. After reviewing recent results on symplectic Lefschetz pencils and symplectic branched covers of $\mathbb{CP}^2$, we describe a new construction of maps from symplectic manifolds of any dimension to $\mathbb{CP}^2$ and the associated monodromy invariants. We also show that a dimensional induction process makes it possible to describe any compact symplectic manifold by a series of words in braid groups and a word in a symmetric group.

1. Introduction

Let (X^{2n}, ω) be a compact symplectic manifold. We will throughout this text assume that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ is integral. This assumption makes it possible to define a complex line bundle L over X such that $c_1(L) = \frac{1}{2\pi}[\omega]$. We also endow X with a compatible almost-complex structure J, and endow L with a Hermitian metric and a Hermitian connection of curvature $-i\omega$.

The line bundle L should be thought of as a symplectic version of an ample line bundle over a complex manifold. Indeed, although the lack of integrability of J prevents the existence of holomorphic sections, it was observed by Donaldson in [8] that, for large k, the line bundles $L^{\otimes k}$ admit many approximately holomorphic sections.

Observe that all results actually apply as well to the case where $\frac{1}{2\pi}[\omega]$ is not integral, with the only difference that the choice of the line bundle L is less natural : the idea is to perturb ω into a symplectic form ω' whose cohomology class is rational, and then work with a suitable multiple of ω'. One chooses an almost-complex structure J' which simultaneously is compatible with ω' and satisfies the positivity property $\omega(v, J'v) > 0$ for all tangent vectors. All the objects that we construct are then approximately J'-holomorphic, and therefore symplectic with respect to not only ω' but also ω.

Donaldson was the first to show in [8] that, among the many approximately holomorphic sections of $L^{\otimes k}$ for $k \gg 0$, there is enough flexibility in order to obtain nice transversality properties ; this makes it possible to imitate various classical topological constructions from complex algebraic geometry in the symplectic category. Let us mention in particular the construction of smooth symplectic submanifolds ([8], see also [2] and [15]), symplectic Lefschetz pencils ([10], see also [9]), branched covering maps to $\mathbb{CP}^2$ ([3],[5]), Grassmannian embeddings and determinantal submanifolds ([15]).

Intuitively, the main reason why the approximately holomorphic framework is suitable to imitate results from algebraic geometry is that, for large values of k, the increasing curvature of $L^{\otimes k}$ provides access to the geometry of X at very small scale ; as one zooms into X, the geometry becomes closer and closer to a standard complex model, and the lack of integrability of J becomes negligible.

The introduction of approximately holomorphic sections was motivated in the first place by the observation that, if suitable transversality properties are satisfied, then every geometric object that can be defined from these sections automatically becomes symplectic. Therefore, in order to perform a given construction using such sections, the strategy is always more or less the same : starting with a sequence of approximately holomorphic sections of $L^{\otimes k}$ for all $k \gg 0$, the goal is to perturb them in order to ensure uniform transversality properties that will guarantee the desired topological features.

For example, the required step in order to construct symplectic submanifolds is to obtain bounds of the type $|\nabla s_k|_{g_k} > \eta$ along the zero set of s_k for a fixed constant $\eta > 0$ independent of k, while approximate holomorphicity implies a bound of the type $|\bar{\partial} s_k|_{g_k} = O(k^{-1/2})$ everywhere (see §3.1). Here $g_k = kg$ is a rescaled metric which dilates everything by a factor of $k^{1/2}$ in order to adapt to the decreasing "characteristic scale" imposed by the increasing curvature $-ik\omega$ of the line bundles $L^{\otimes k}$. The desired topological picture, similar to the complex algebraic case, emerges for large k as an inequality of the form $|\bar{\partial} s_k| \ll |\partial s_k|$ becomes satisfied at every point of the zero set : this can easily be shown to imply that the zero set of s_k is smooth, approximately pseudo-holomorphic, and symplectic. Indeed, the surjectivity of ∇s_k implies the smoothness of the zero set, while the fact that $|\bar{\partial} s_k| \ll |\partial s_k|$ implies that the tangent space to the zero set, given by the kernel of $\nabla s_k = \partial s_k + \bar{\partial} s_k$, is very close to the complex subspace $\mathrm{Ker}(\partial s_k)$, hence its symplecticity (see also [8]).

The starting points for the construction, in all cases, are the existence of very localized approximately holomorphic sections of $L^{\otimes k}$ concentrated near any given point $x \in X$, and an effective transversality result for approximately holomorphic functions defined over a ball in $\mathbb{C}^n$ with values in $\mathbb{C}^r$ due to Donaldson (see [8] for the case $r = 1$ and [10] for the general case). These two ingredients imply that a small localized perturbation can be used to ensure uniform transversality over a small ball. Combining this local result with a globalization argument ([8], see also [3] and [15]), one obtains transversality everywhere.

The interpretation of the construction of submanifolds as an effective transversality result for sections extends verbatim to the more sophisticated constructions (Lefschetz pencils, branched coverings) : in these cases the transversality properties also concern the covariant derivatives of the sections, and this can be thought of as an effective analogue in the approximately holomorphic category of the standard generalized transversality theorem for jets.

This is especially clear when looking at the arguments in [15], [10] or [3] : the perturbative argument is now used to obtain uniform transversality of the holomorphic parts of the 1-jets or 2-jets of the sections with respect to certain closed submanifolds in the space of holomorphic jets. Successive perturbations are used to obtain transversality to the various strata describing the possible singular models ; one uses that each stratum is smooth away from lower dimensional strata, and that transversality to these lower dimensional strata is enough to imply transversality to the higher dimensional stratum near its singularities.

An extra step is necessary in the constructions : recall that desired topological properties only hold when the antiholomorphic parts of the derivatives are much smaller than the holomorphic parts. In spite of approximate holomorphicity, this can be a problem when the holomorphic part of the jet becomes singular. Therefore, a small perturbation is needed to kill the antiholomorphic part of the jet near the singularities ; this perturbation is in practice easy to construct. The reader is referred to [10] and [3] for details.

Although no general statement has yet been formulated and proved, it is completely clear that a very general result of uniform transversality for jets holds in the approximately holomorphic category. Therefore, the observed phenomenon for Lefschetz pencils and maps to $\mathbb{CP}^2$, namely the fact that near every point $x \in X$ the constructed maps are given in approximately holomorphic coordinates by one of the standard local models for generic holomorphic maps, should hold in all generality, independently of the dimensions of the source and target spaces. This approach will be developed in a forthcoming paper [4].

In the remainder of this paper we focus on the topological monodromy invariants that can be derived from the various available constructions. In Section 2 we study symplectic Lefschetz pencils and their monodromy, following the results of Donaldson [10] and Seidel [16]. In Section 3 we describe symplectic branched covers of $\mathbb{CP}^2$ and their monodromy invariants, following [3] and [5] ; we also discuss the connection with 4-dimensional Lefschetz pencils. In Section 4 we extend this framework to the higher dimensional case, and investigate a new type of monodromy invariants arising from symplectic maps to $\mathbb{CP}^2$. We finally show in Section 5 that a dimensional induction process makes it possible to describe a compact symplectic manifold of any dimension by a series of words in braid groups and a word in a symmetric group.

Acknowledgement. The author wishes to thank Ludmil Katzarkov, Paul Seidel and Bob Gompf for stimulating discussions, as well as Simon Donaldson for his interest in this work.

2. Symplectic Lefschetz pencils

Let (X^{2n},ω) be a compact symplectic manifold as above, and let s_0, s_1 be suitably chosen approximately holomorphic sections of $L^{\otimes k}$. Then X is endowed with a structure of *symplectic Lefschetz pencil*, which can be described as follows.

For any $\alpha \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, define $\Sigma_\alpha = \{x \in X,\ s_0 + \alpha s_1 = 0\}$. Then the submanifolds Σ_α are symplectic hypersurfaces, smooth except for finitely many values of the parameter α ; for these parameter values Σ_α contains a singular point (a normal crossing when $\dim X = 4$). Moreover, the submanifolds Σ_α fill all of X, and they intersect transversely along a codimension 4 symplectic submanifold $Z = \{x \in X,\ s_0 = s_1 = 0\}$, called the set of *base points* of the pencil.

Define the projective map $f = (s_0 : s_1) : X - Z \to \mathbb{CP}^1$, whose level sets are precisely the hypersurfaces Σ_α. Then f is required to be a *complex Morse function*, i.e. its critical points are isolated and non-degenerate, with local model $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$ in approximately holomorphic coordinates.

The following result due to Donaldson holds :

Theorem 2.1 (Donaldson [10])**.** *For $k \gg 0$, two suitably chosen approximately holomorphic sections of $L^{\otimes k}$ endow X with a structure of symplectic Lefschetz pencil, canonical up to isotopy.*

This result is proved by obtaining uniform transversality with respect to the strata $s_0 = s_1 = 0$ (of complex codimension 2) and $\partial f = 0$ (of complex codimension n) in the space of holomorphic 1-jets of sections of $\mathbb{C}^2 \otimes L^{\otimes k}$, by means of the techniques described in the introduction. A small additional perturbation ensures the compatibility requirement that $\bar{\partial} f$ vanishes at the points where $\partial f = 0$. These properties are sufficient to ensure that the structure is that of a symplectic Lefschetz pencil. For details, the reader is referred to [10].

The statement that the constructed pencils are canonical up to isotopy for $k \gg 0$ is to be interpreted as follows. Consider two sequences $(s_k^0)_{k \gg 0}$ and $(s_k^1)_{k \gg 0}$ of approximately holomorphic sections of $\mathbb{C}^2 \otimes L^{\otimes k}$ for increasing values of k. Assume that they satisfy the three above-described transversality and compatibility properties and hence define symplectic Lefschetz pencils. Then, for large enough k (how large exactly depends on the estimates on the given sections), there exists an interpolating family $(s_k^t)_{t \in [0,1]}$ of approximately holomorphic sections, depending continuously on the parameter t, such that for all values of t the sections s_k^t satisfy the transversality and compatibility properties. In particular, for large enough k the symplectic Lefschetz pencils defined by s_k^0 and s_k^1 are isotopic to each other. Moreover, the same result remains true if the almost-complex structures J_0 and J_1 with respect to which s_k^0 and s_k^1 are approximately holomorphic differ, so the topology of the constructed pencils depends only on the topology of the symplectic manifold X (and on k of course). However, because isotopy holds only for large values of k, this is only a weak (asymptotic) uniqueness result.

A convenient way to study the topology of a Lefschetz pencil is to blow up X along the submanifold Z. The resulting symplectic manifold $\hat{X}$ is the total space of a *symplectic Lefschetz fibration* $\hat{f} : \hat{X} \to \mathbb{CP}^1$. Although in the following description we work on the blown up manifold $\hat{X}$, it is actually preferrable to work directly on X ; verifying that the discussion applies to X itself is a simple task left to the reader.

The fibers of $\hat{f}$ can be identified with the submanifolds Σ_α, made mutually disjoint by the blow-up process. It is then possible to study the *monodromy* of the fibration $\hat{f}$ around its singular fibers.

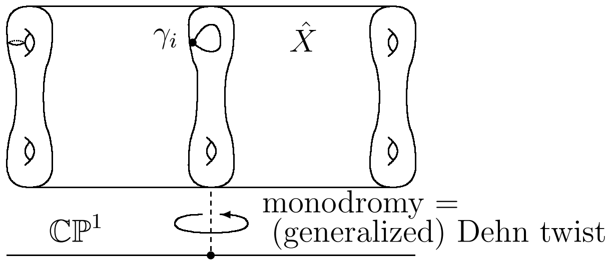
One easily checks that this monodromy consists of symplectic automorphisms of the fiber Σ_α. Moreover, the exceptional divisor obtained by blowing up the set of base points Z is a subfibration of $\hat{f}$, with fiber Z, which is unaffected by the monodromy ; after restricting to an affine slice, the normal bundle to the exceptional divisor can be trivialized, so that it becomes natural to consider that the monodromy of $\hat{f}$ takes values in the symplectic mapping class group $\mathrm{Map}^\omega(\Sigma, Z) = \pi_0(\{\phi \in \mathrm{Symp}(\Sigma, \omega), \phi_{|U(Z)} = \mathrm{Id}\})$, i.e. the set of isotopy classes of symplectomorphisms of the generic fiber Σ which coincide with the identity near Z.

In the four-dimensional case, Z consists of a finite number n of points, and Σ is a compact surface with a certain genus g (note that Σ is always connected because it satisfies a Lefschetz hyperplane type property) ; $\mathrm{Map}^\omega(\Sigma, Z)$ is then the classical mapping class group $\mathrm{Map}_{g,n}$ of a genus g surface with n boundary components.

In fact, the image of the monodromy map is contained in the subgroup of *exact* symplectomorphisms in $\mathrm{Map}^\omega(\Sigma, Z)$: the connection on $L^{\otimes k}$ induces over $\Sigma - Z$ a 1-form α such that $d\alpha = \omega$. This endows $\Sigma - Z$ with a structure of exact symplectic manifold. Monodromy transformations are then exact symplectomorphisms in the sense that they preserve not only ω but also the 1-form α : every monodromy transformation f satisfies $f^*\alpha - \alpha = dh$ for some function h vanishing near Z (see [17] for details).

It is well-known (see e.g. [16], [17]) that the singular fibers of a Lefschetz fibration are obtained from the generic fiber by collapsing a *vanishing cycle* to a point. The vanishing cycle is an embedded closed loop in Σ in the four-dimensional case ; more generally, it is an embedded Lagrangian sphere $S^{n-1} \subset \Sigma$. Then, the monodromy of $\hat{f}$ around one of its singular fibers consists in a *generalized Dehn twist* in the positive direction along the vanishing cycle.

The picture is the following :



Because the normal bundle to the exceptional divisor is not trivial, the monodromy map cannot be defined over all of $\mathbb{CP}^1$, and we need to restrict ourselves to the preimage of an affine subset $\mathbb{C}$ (the fiber at infinity can be assumed regular). The monodromy around the fiber at infinity of $\hat{f}$ is given by a mapping class group element δ_Z corresponding to a *twist around* Z. In the four-dimensional case Z consists of n points, and δ_Z is the product of positive Dehn twists along n loops each encircling one of the base points ; in the higher-dimensional case δ_Z is a positive Dehn twist along the unit sphere bundle in the normal bundle of Z in Σ (i.e. it restricts to each fiber of the normal bundle as a Dehn twist around the origin).

It follows from the above observations that the monodromy of the Lefschetz fibration $\hat{f}$ with critical levels $p_1, \dots, p_d$ is given by a group homomorphism

$$\psi : \pi_1(\mathbb{C} - \{p_1, \dots, p_d\}) \to \mathrm{Map}^\omega(\Sigma^{2n-2}, Z) \tag{1}$$

which maps the *geometric generators* of $\pi_1(\mathbb{C} - \{p_1, \dots, p_d\})$, i.e. loops going around one of the points p_i, to Dehn twists.

Alternately, choosing a system of generating loops in $\mathbb{C} - \{p_1, \dots, p_d\}$, we can express the monodromy by a *factorization* of δ_Z in the mapping class group :

$$\delta_Z = \prod_{i=1}^{d} \tau_{\gamma_i}, \tag{2}$$

where γ_i is the image in a chosen reference fiber of the vanishing cycle of the singular fiber above p_i and τ_{γ_i} is the corresponding positive Dehn twist. The identity (2) in $\mathrm{Map}^\omega(\Sigma, Z)$ expresses the fact that the monodromy of the fibration around the point at infinity in $\mathbb{CP}^1$ decomposes as the product of the elementary monodromies around each of the singular fibers.

The monodromy morphism (1), or equivalently the mapping class group factorization (2), completely characterizes the topology of the Lefschetz fibration $\hat{X}$.

However, they are not entirely canonical, because two choices have been implicitly made in order to define them.

First, a base point in $\mathbb{C} - \{p_1, \dots, p_d\}$ and an identification symplectomorphism between Σ and the chosen reference fiber of $\hat{f}$ are needed in order to view the monodromy transformations as elements in the mapping class group of Σ. The choice of a different identification affects the monodromy morphism ψ by conjugation by a certain element $g \in \mathrm{Map}^\omega(\Sigma, Z)$. The corresponding operation on the mapping class group factorization (2) is a *simultaneous conjugation* of all factors : each factor τ_{γ_i} is replaced by $\tau_{g(\gamma_i)} = g^{-1}\tau_{\gamma_i} g$.

Secondly, a system of generating loops has to be chosen in order to define a factorization of δ_Z. Different choices of generating systems differ by a sequence of *Hurwitz operations*, i.e. moves in which two consecutive generating loops are exchanged, one of them being conjugated by the other in order to preserve the counterclockwise ordering. On the level of the factorization, this amounts to replacing two consecutive factors τ_1 and τ_2 by respectively τ_2 and $\tau_2^{-1}\tau_1\tau_2$ (or, by the reverse operation, $\tau_1\tau_2\tau_1^{-1}$ and τ_1).

It is quite easy to see that any two factorizations of δ_Z describing the Lefschetz fibration $\hat{f}$ differ by a sequence of these two operations (simultaneous conjugation and Hurwitz moves). Therefore, Donaldson's uniqueness statement implies that, for large enough values of k, the mapping class group factorizations associated to the symplectic Lefschetz pencil structures obtained in Theorem 2.1 are, up to simultaneous conjugation and Hurwitz moves, *symplectic invariants* of the manifold (X, ω).

Conversely, given any factorization of δ_Z in $\mathrm{Map}^\omega(\Sigma, Z)$ as a product of positive Dehn twists, it is possible to construct a symplectic Lefschetz fibration with the given monodromy. It follows from a result of Gompf that the total space of such a fibration is always a symplectic manifold. In fact, because the monodromy preserves the symplectic submanifold $Z \subset \Sigma$, it is also possible to reconstruct the blown down manifold X. More precisely, the following result holds :

Theorem 2.2 (Gompf)**.** *Let (Σ, ω_Σ) be a compact symplectic manifold, and $Z \subset \Sigma$ a codimension 2 symplectic submanifold such that $[Z] = PD([\omega_\Sigma])$. Consider a factorization of δ_Z as a product of positive Dehn twists in $\mathrm{Map}^\omega(\Sigma, Z)$. In the case $\dim(\Sigma) = 2$, assume moreover that all the Dehn twists in the factorization are along loops that are not homologically trivial in $\Sigma - Z$.*

Then the total space X of the corresponding Lefschetz pencil carries a symplectic form ω_X such that, given a generic fiber Σ_0 of the pencil, $[\omega_X]$ is Poincaré dual to $[\Sigma_0]$, and $(\Sigma_0, \omega_{X|\Sigma_0})$ is symplectomorphic to (Σ, ω_Σ). This symplectic structure on X is canonical up to symplectic isotopy.

The strategy of proof is to first construct a symplectic structure in the correct cohomology class on a neighborhood of any fiber of the pencil, which is easily done as Σ already carries a symplectic structure and the monodromy lies in the exact symplectomorphism group. More precisely, the symplectic structure on $\Sigma - Z$ is exact, and Dehn twists along exact Lagrangian spheres are exact symplectomorphisms [17]. When $\dim \Sigma \geq 4$, the exactness condition is always trivially satisfied, while in the case $\dim \Sigma = 2$ it can be ensured by suitably choosing the vanishing loop in its homotopy class provided that it does not separate Σ into connected components without base points. With this understood, it is possible to define local symplectic structures over neighborhoods of the singular fibers, coinciding with a fixed standard symplectic form near Z, and to combine them into a globally defined symplectic form, singular near the base locus Z. Since the total monodromy is δ_Z, the structure of X near Z is completely standard, and so a non-singular symplectic form on X can be recovered (this process can also be viewed as a symplectic blow-down along the exceptional hypersurface $\mathbb{CP}^1 \times Z$ in the total space of the corresponding Lefschetz fibration). This operation changes the cohomology class of the symplectic form on X, but one easily checks that the resulting class is a nonzero multiple of the Poincaré dual to a fiber ; scaling the symplectic form by a suitable factor then yields ω_X. The proof that this process is canonical up to symplectic isotopy is a direct application of Moser's stability theorem. The reader is referred to [11] and references therein for details.

In conclusion, the study of the monodromy of symplectic Lefschetz pencils makes it possible to define invariants of compact symplectic manifolds, which in principle provide a complete description of the topology. However, the complexity of mapping class groups and the difficulties in computing the invariants in concrete situations greatly decrease their usefulness in practice. This motivates the introduction of other similar topological constructions which may lead to more usable invariants.

3. Branched covers of $\mathbb{CP}^2$ and invariants of symplectic 4-manifolds

Throughout §3, we assume that (X, ω) is a compact symplectic 4-manifold. In that case, three generic approximately holomorphic sections s_0, s_1 and s_2 of $L^{\otimes k}$ never vanish simultaneously, and so they define a projective map $f = (s_0 : s_1 : s_2) : X \to \mathbb{CP}^2$. It was shown in [3] that, if the sections are suitably chosen, this map is a *branched covering*, whose branch curve $R \subset X$ is a smooth connected symplectic submanifold in X.

There are two possible local models in approximately holomorphic coordinates for the map f near the branch curve. The first one, corresponding to a generic point of R, is the map $(x, y) \mapsto (x^2, y)$; locally, both the branch curve R and its image by f are smooth. The other local model corresponds to the isolated points where f does not restrict to R as an immersion. The model map is then $(x, y) \mapsto (x^3 - xy, y)$, and the image of the smooth branch curve $R : 3x^2 - y = 0$ has equation $f(R) : 27z_1^2 = 4z_2^3$ and presents a cusp singularity. These two local models are the same as in the complex algebraic setting.

It is easy to see by considering the two model maps that R is a smooth approximately holomorphic (and therefore symplectic) curve in X, and that $f(R)$ is an approximately holomorphic symplectic curve in $\mathbb{CP}^2$, immersed away from its cusps. After a generic perturbation, we can moreover require that the branch curve $D = f(R)$ satisfies a self-transversality property, i.e. that its only singular points besides the cusps are transverse double points ("nodes"). Even though D is approximately holomorphic, it is not immediately possible to require that all of its double points correspond to a positive intersection number with respect to the standard orientation of $\mathbb{CP}^2$; the presence of (necessarily badly transverse) negative double points is a priori possible.

It was also shown in [3] that the branched coverings obtained from sections of $L^{\otimes k}$ are, for large values of k, canonical up to isotopy (this weak uniqueness statement holds in the same sense as that of Theorem 2.1). Therefore, the topology of the branch curve $D = f(R)$ can be used to define symplectic invariants, provided that one takes into account the possibility of cancellations or creations of pairs of nodes with opposite orientations in isotopies of branched coverings.

Most of the results cited below were obtained in a joint work with L. Katzarkov [5].

3.1. **Quasiholomorphic maps to $\mathbb{CP}^2$.** In order to study the topology of the singular plane curve D, it is natural to try to adapt the braid group techniques previously used by Moishezon and Teicher in the algebraic case (see e.g. [13], [14], [18]). However, in order to apply this method it is necessary to ensure that the branch curve satisfies suitable transversality properties with respect to a generic projection map from $\mathbb{CP}^2$ to $\mathbb{CP}^1$. This leads naturally to the notion of *quasiholomorphic covering* introduced in [5], which we now describe carefully.

We slightly rephrase the conditions listed in [5] in such a way that they extend naturally to the higher dimensional case ; the same definitions will be used again in §4. It is important to be aware that these concepts only apply to sequences of objects obtained for increasing values of the degree k ; the general strategy is always to work simultaneously with a whole family of sections indexed by the parameter k, in order to ultimately ensure the desired properties for large values of k. We start with the following terminology :

Definition 3.1. A sequence of sections s_k of complex vector bundles E_k over X (endowed with Hermitian metrics and connections) is *asymptotically holomorphic* if there exist constants C_j independent of k such that $|\nabla^j s_k|_{g_k} \leq C_j$ and $|\nabla^{j-1}\bar\partial s_k|_{g_k} \leq C_j k^{-1/2}$ for all j, all norms being evaluated with respect to the rescaled metric $g_k = kg$ on X.

The sections s_k are *uniformly transverse to* 0 if there exists a constant $\gamma > 0$ such that, at every point $x \in X$ where $|s_k(x)| \leq \gamma$, the covariant derivative $\nabla s_k(x)$ is surjective and has a right inverse of norm less than γ^{-1} w.r.t. g_k (we then say that s_k is γ-transverse to 0).

In the case where the rank of the bundle E_k is greater than the dimension of X, the surjectivity condition imposed by transversality is never satisfied ; γ-transversality to 0 then means that the norm of the section is greater than γ at every point of X.

As mentioned in the introduction, it is easy to check that, if sections are asymptotically holomorphic and uniformly transverse to 0, then for large k their zero sets are smooth approximately holomorphic symplectic submanifolds. This principle, which plays a key role in Donaldson's construction of symplectic submanifolds [8], can also be applied to the Jacobian of the maps defined below and now implies the symplecticity of their branch curves.

Definition 3.2. A sequence of projective maps $f_k : X \to \mathbb{CP}^2$ determined by asymptotically holomorphic sections $s_k = (s_k^0, s_k^1, s_k^2)$ of $\mathbb{C}^3 \otimes L^{\otimes k}$ for $k \gg 0$ is *quasiholomorphic* if there exist constants C_j, γ, δ independent of k, almost-complex structures $\tilde J_k$ on X, and finite sets $\mathcal{C}_k, \mathcal{T}_k, \mathcal{I}_k \subset X$ such that the following properties hold (using $\tilde J_k$ to define the $\bar\partial$ operator) :

(0) $|\nabla^j(\tilde J_k - J)|_{g_k} \leq C_j k^{-1/2}$ for every $j \geq 0$; $\tilde J_k = J$ outside of the 2δ-neighborhood of $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$; $\tilde J_k$ is integrable in the δ-neighborhood of $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$;

(1) the section s_k of $\mathbb{C}^3 \otimes L^{\otimes k}$ is γ-transverse to 0 ;

(2) $|\nabla f_k(x)|_{g_k} \geq \gamma$ at every point $x \in X$;

(3) the (2, 0)-Jacobian $\mathrm{Jac}(f_k) = \bigwedge^2 \partial f_k$ is γ-transverse to 0 ; in particular it vanishes transversely along a smooth symplectic curve $R_k \subset X$ (the branch curve).

(3′) the restriction of $\bar\partial f_k$ to $\mathrm{Ker}\,\partial f_k$ vanishes at every point of R_k ;

(4) the quantity $\partial(f_{k|R_k})$, which can be seen as a section of a line bundle over R_k, is γ-transverse to 0 and vanishes at the finite set $\mathcal{C}_k$ (the cusp points of f_k) ; in particular $f_k(R_k) = D_k$ is an immersed symplectic curve away from the image of $\mathcal{C}_k$;

(4′) f_k is $\tilde J_k$-holomorphic over the δ-neighborhood of $\mathcal{C}_k$;

(5) the section (s_k^0, s_k^1) of $\mathbb{C}^2 \otimes L^{\otimes k}$ is γ-transverse to 0 ; as a consequence D_k remains away from the point $(0:0:1)$;

(6) let $\pi : \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$ be the map defined by $\pi(x:y:z) = (x:y)$, and let $\phi_k = \pi \circ f_k$. Then the quantity $\partial(\phi_{k|R_k})$ is γ-transverse to 0 over R_k, and it vanishes over the union of $\mathcal{C}_k$ with the finite set $\mathcal{T}_k$ (the tangency points of the branch curve D_k with respect to the projection π) ;

(6′) f_k is $\tilde J_k$-holomorphic over the δ-neighborhood of $\mathcal{T}_k$;

(7) the projection $f_k : R_k \to D_k$ is injective outside of the singular points of D_k, and the self-intersections of D_k are transverse double points. Moreover, all special points of D_k (cusps, nodes, tangencies) lie in different fibers of the projection π, and none of them lies in $\pi^{-1}(0:1)$;

(8) the section s_k^0 of $L^{\otimes k}$ is γ-transverse to 0 ;

(8′) R_k intersects the zero set of s_k^0 at the points of $\mathcal{I}_k$; f_k is $\tilde J_k$-holomorphic over the δ-neighborhood of $\mathcal{I}_k$.

Remark 3.1. Definition 3.2 is slightly stronger than the definition given in [5]. Most notably, property (8), which ensures that the fiber of $\pi \circ f_k$ above $(0:1)$ enjoys suitable genericity properties, has been added for our purposes. Similarly, condition (6′) is significantly stronger than in [5], where it was only required that $\bar\partial f_k$ vanish at the points of $\mathcal{T}_k$. These extra conditions only require minor modifications of the arguments, while allowing the inductive construction described in §5 to be largely simplified.

Observe that, because of property (0), the notions of asymptotic holomorphicity with respect to J or $\tilde J_k$ coincide. Moreover, even though $\tilde J_k$ is used implicitly thoughout the definition, the choice of J or $\tilde J_k$ is irrelevant as far as transversality properties are concerned since they differ by $O(k^{-1/2})$.

Property (1) means that s_k is everywhere bounded from below by γ ; this implies that the projective map f_k is well-defined, and that $|\nabla^j f_k|_{g_k} = O(1)$ and $|\nabla^{j-1}\bar\partial f_k|_{g_k} = O(k^{-1/2})$ for all j. The second property can be interpreted in terms of transversality to the codimension 4 submanifold in the space of 1-jets given by the equation $\partial f = 0$. Properties (3) and (3′) yield the correct structure near generic points of the branch curve : the transverse vanishing of $\mathrm{Jac}(f_k)$ implies that the branching order is 2, and the compatibility property (3′) ensures that $\bar\partial f_k$ remains much smaller than ∂f_k in all directions, which is needed to obtain the correct local model.

Properties (4) and (4′) determine the structure of the covering near the cusp points. More precisely, observe that along R_k the tangent plane field TR_k and the plane field $\mathrm{Ker}\,\partial f_k$ coincide exactly at the cusp points ; condition (4) expresses that these two plane fields are transverse to each other (in [3] and [5] this condition was formulated in terms of a more complicated quantity; the two formulations are easily seen to be equivalent). This implies that cusp points are isolated and non-degenerate. The compatibility condition (4′) then ensures that the expected local model indeed holds.

The remaining conditions are used to ensure the compatibility of the branch curve $D_k = f_k(R_k)$ with the projection π to $\mathbb{CP}^1$. In particular, the transversality condition (6) and the corresponding compatibility condition (6′) imply that the points where the branch curve D_k fails to be transverse to the fibers of π are isolated non-degenerate tangency points. Moreover, property (7) states that the curve D_k is transverse to itself. This implies that D_k is a *braided curve* in the following sense :

Definition 3.3. A real 2-dimensional singular submanifold $D \subset \mathbb{CP}^2$ is a *braided curve* if it satisfies the following properties : (1) the only singular points of D are cusps (with positive orientation) and transverse double points (with either orientation) ; (2) the point $(0:0:1)$ does not belong to D ; (3) the fibers of the projection $\pi : (x:y:z) \mapsto (x:y)$ are everywhere transverse to D, except at a finite set of nondegenerate tangency points where a local model for D in orientation-preserving coordinates is $z_2^2 = z_1$; (4) the cusps, nodes and tangency points are all distinct and lie in different fibers of π.

We will see in §3.2 that these properties are precisely those needed in order to apply the braid monodromy techniques of Moishezon-Teicher to the branch curve D_k.

The main result of [5] can be formulated as follows :

Theorem 3.1 ([3],[5]). *For $k \gg 0$, it is possible to find asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ such that the corresponding projective maps $f_k : X \to \mathbb{CP}^2$ are quasiholomorphic branched coverings. Moreover, for large k these coverings are canonical up to isotopy and up to cancellations of pairs of nodes in the branch curves D_k.*

The uniqueness statement is to be understood in the same weak sense as for Theorem 2.1 : given two sequences of quasiholomorphic branched coverings (possibly for different choices of almost-complex structures on X), for large k it is possible to find an interpolating one-parameter family of quasiholomorphic coverings, the only possible non-trivial phenomenon being the cancellation or creation of pairs of nodes in the branch curve for certain parameter values.

The proof of Theorem 3.1 follows a standard pattern : in order to construct quasiholomorphic coverings, one starts with any sequence of asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ and proceeds by successive perturbations in order to obtain all the required properties, starting with uniform transversality. Since transversality is an open condition, it is preserved by the subsequent perturbations.

So the first part of the proof consists in obtaining, by successive perturbation arguments, the transversality properties (1), (2), (3) and (4) of Definition 3.2 as in [3], (5) and (6) as in [5], and also (8) by a direct application of the result of [8]. The argument is notably more technical in the case of (4) and (6) because the transversality conditions involve derivatives along the branch curve, but these can actually all be thought of as immediate applications of the general transversality principle mentioned in the Introduction.

The second part of the proof, which is comparatively easier, deals with the compatibility conditions. The idea is to ensure these properties by perturbing the sections s_k by quantities bounded by $O(k^{-1/2})$, which clearly affects neither holomorphicity nor transversality properties. One first chooses suitable almost-complex structures $\tilde{J}_k$ differing from J by $O(k^{-1/2})$ and integrable near the finite set $\mathcal{C}_k \cup \mathcal{T}_k \cup \mathcal{I}_k$. It is then possible to perturb f_k near these points in order to obtain conditions (4′), (6′) and (8′), by the same argument as in §4.1 of [3]. Next, a generic small perturbation yields the self-transversality of D (property (7)). Finally, a suitable perturbation yields property (3′) along the branch curve without modifying R_k and D_k and without affecting the other compatibility properties.

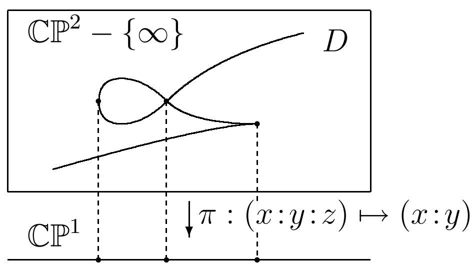
The uniqueness statement is obtained by showing that, provided that k is large enough, all the arguments extend verbatim to one-parameter families of sections. Therefore, given two sequences of quasiholomorphic coverings, one starts with a one-parameter family of sections interpolating between them in a trivial way and perturbs it in such a way that the required properties hold for all parameter values (with the exception of (7) when a node cancellation occurs). Since this construction can be performed in such a way that the two end points of the one-parameter family are not affected by the perturbation, the isotopy result follows immediately.

The reader is referred to [3] and [5] for more details (incorporating requirement (8) in the arguments is a trivial task).

3.2. **Braid monodromy invariants.** We now describe the monodromy invariants that naturally arise from the quasiholomorphic coverings described in the previous section. This is a relatively direct extension to the symplectic framework of the braid group techniques studied by Moishezon and Teicher in the algebraic case (see [13], [14], [18]).

Recall that the braid group on d strings is the fundamental group $B_d = \pi_1(\mathcal{X}_d)$ of the space $\mathcal{X}_d$ of unordered configurations of d distinct points in the plane $\mathbb{R}^2$. A braid can therefore be thought as a motion of d points in the plane. An alternate description involves compactly supported orientation-preserving diffeomorphisms of $\mathbb{R}^2$ which globally preserve a set of d given points : $B_d = \pi_0(\mathrm{Diff}^+_c(\mathbb{R}^2, \{q_1, \dots, q_d\}))$. The group B_d is generated by *half-twists*, i.e. braids in which two of the d points rotate around each other by 180 degrees while the other points are preserved. For more details see [6].

Consider a braided curve $D \subset \mathbb{CP}^2$ (see Definition 3.3) of fixed degree d, for example the branch curve of a quasiholomorphic covering as given by Theorem 3.1. Projecting to $\mathbb{CP}^1$ via the map π makes D a singular branched covering of $\mathbb{CP}^1$. The picture is the following :



Let $p_1, \dots, p_r$ be the images by π of the special points of D (nodes, cusps and tangencies). Observing that the fibers of π are complex lines (or equivalently real planes) which generically intersect D in d points, we easily get that the monodromy of the map $\pi_{|D}$ around the fibers above $p_1, \dots, p_r$ takes values in the braid group B_d.

The monodromy around one of the points $p_1, \ldots, p_r$ is as follows. In the case of a tangency point, a local model for the curve D is $y^2 = x$ (with projection to the x factor), so one easily checks that the monodromy is a half-twist exchanging two sheets of $\pi_{|D}$. Since all half-twists in B_d are conjugate, it is possible to write this monodromy in the form $Q^{-1}X_1Q$, where $Q \in B_d$ is any braid and X_1 is a fixed half-twist (aligning the points $q_1, \ldots, q_d$ in that order along the real axis, X_1 is the half-twist exchanging the points q_1 and q_2 along a straight line segment). In the case of a transverse double point with positive intersection, the local model $y^2 = x^2$ implies that the monodromy is the square of a half-twist, which can be written in the form $Q^{-1}X_1^2Q$. The monodromy around a double point with negative intersection is the mirror image of the previous case, and can therefore be written as $Q^{-1}X_1^{-2}Q$. Finally, the monodromy around a cusp (local model $y^2 = x^3$) is the cube of a half-twist and can be expressed as $Q^{-1}X_1^3Q$.

However, in order to describe the monodromy automorphisms as braids, one needs to identify up to compactly supported diffeomorphisms the fibers of π with a reference plane $\mathbb{R}^2$. This implicitly requires a trivialization of the fibration π, which is not available over all of $\mathbb{CP}^1$. Therefore, as in the case of Lefschetz pencils, it is necessary to restrict oneself to the preimage of an affine subset $\mathbb{C} \subset \mathbb{CP}^1$, by removing the fiber above the point at infinity (which may easily be assumed to be regular). So the monodromy map is only defined as a group homomorphism

$$\rho : \pi_1(\mathbb{C} - \{p_1, \ldots, p_r\}) \to B_d. \tag{3}$$

Since the fibration π defines a line bundle of degree 1 over $\mathbb{CP}^1$, the monodromy around the fiber at infinity is given by the *full twist* Δ^2, i.e. the braid which corresponds to a rotation of all points by 360 degrees (Δ^2 generates the center of B_d).

Therefore, choosing as in §2 a system of generating loops in $\mathbb{C} - \{p_1, \ldots, p_r\}$, we can express the monodromy by a factorization of Δ^2 in the braid group :

$$\Delta^2 = \prod_{j=1}^{r} Q_j^{-1} X_1^{r_j} Q_j, \tag{4}$$

where the elements $Q_j \in B_d$ are arbitrary braids and the degrees $r_j \in \{1, \pm 2, 3\}$ depend on the types of the special points lying above p_j.

As in the case of Lefschetz pencils, this braid factorization, which completely characterizes the braided curve D up to isotopy, is only well-defined up to two algebraic operations: simultaneous conjugation of all factors by a given braid in B_d, and Hurwitz moves. As previously, simultaneous conjugation reflects the different possible choices of an identification diffeomorphism between the fiber of π above the base point and the standard plane $(\mathbb{C}, \{q_1, \ldots, q_d\})$, while Hurwitz moves arise from changes in the choice of a generating system of loops in $\mathbb{C} - \{p_1, \ldots, p_r\}$.

Starting with any braid factorization of the form (4), it is possible to reconstruct a braided curve D in a canonical way up to isotopy (see [5]; similar statements were also obtained by Moishezon, Teicher and Catanese). Moreover, one easily checks that factorizations which differ only by global conjugations and Hurwitz moves lead to isotopic braided curves (each such operation amounts to a diffeomorphism isotopic to the identity, obtained in the case of a Hurwitz move by lifting by π a diffeomorphism of $\mathbb{CP}^1$, and in the case of a global conjugation by a diffeomorphism in each of the fibers of π).

Moreover, it is important to observe that every braided curve D can be made symplectic by a suitable isotopy. In fact, it is sufficient to perform a radial contraction in all the fibers of π, which brings the given curve into an arbitrarily small neighborhood of the zero section of π (the complex line $\{z = 0\}$ in $\mathbb{CP}^2$). The tangent space to D is then very close to that of the complex line (and therefore symplectic) everywhere except near the tangency points; verifying that the property also holds near tangencies by means of the local model, one obtains that D is symplectic.

We now briefly describe the structure of the fundamental group $\pi_1(\mathbb{CP}^2 - D)$. Consider a generic fiber of π, intersecting D in d points $q_1, \dots, q_d$. Then the inclusion map $i : \mathbb{C} - \{q_1, \dots, q_d\} \to \mathbb{CP}^2 - D$ induces a surjective homomorphism on fundamental groups. Therefore, a generating system of loops $\gamma_1, \dots, \gamma_d$ in $\mathbb{C} - \{q_1, \dots, q_d\}$ provides a set of generators for $\pi_1(\mathbb{CP}^2 - D)$ (*geometric generators*). Because the fiber of π can be compactified by adding the pole of the projection, an obvious relation is $\gamma_1 \dots \gamma_d = 1$. Moreover, each special point of the curve D, or equivalently every term in the braid factorization, determines a relation in $\pi_1(\mathbb{CP}^2 - D)$ in a very explicit way.

Namely, recall that there exists a natural right action of B_d on the free group $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$, that we shall denote by $*$, and consider a factor $Q_j^{-1} X_1^{r_j} Q_j$ in (4). Then, if $r_j = 1$, the tangency point above p_j yields the relation $\gamma_1 * Q_j = \gamma_2 * Q_j$ (the two elements $\gamma_1 * Q_j$ and $\gamma_2 * Q_j$ correspond to small loops going around the two sheets of $\pi_{|D}$ that merge at the tangency point). Similarly, in the case of a node ($r_j = \pm 2$), the relation is $[\gamma_1 * Q_j, \gamma_2 * Q_j] = 1$. Finally, in the case of a cusp ($r_j = 3$), the relation becomes $(\gamma_1\gamma_2\gamma_1) * Q_j = (\gamma_2\gamma_1\gamma_2) * Q_j$. It is a classical result that $\pi_1(\mathbb{CP}^2 - D)$ is exactly the quotient of $F_d = \langle \gamma_1, \dots, \gamma_d \rangle$ by the above-listed relations.

Given a branched covering map $f : X \to \mathbb{CP}^2$ with branch curve D, it is easy to see that the topology of X is determined by a group homomorphism from $\pi_1(\mathbb{CP}^2 - D)$ to the symmetric group S_n of order $n = \deg f$. Considering a generic fiber of π which intersects D in d points $q_1, \dots, q_d$, the restriction of f to its preimage Σ is a n-sheeted branched covering map from Σ to $\mathbb{C}$ with branch points $q_1, \dots, q_d$. This covering is naturally described by a monodromy representation

$$\theta : \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to S_n. \tag{5}$$

Because the branching index is 2 at a generic point of the branch curve of f, the group homomorphism θ maps geometric generators to transpositions. Also, θ necessarily factors through the surjective homomorphism $i_* : \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to \pi_1(\mathbb{CP}^2 - D)$, because the covering f is defined everywhere, and the resulting map from $\pi_1(\mathbb{CP}^2 - D)$ to S_n is exactly what is needed to recover the 4-manifold X from the branch curve D. The properties of θ are summarized in the following definition due to Moishezon :

Definition 3.4. A geometric monodromy representation associated to a braided curve $D \subset \mathbb{CP}^2$ is a surjective group homomorphism θ from the free group $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) = F_d$ to the symmetric group S_n of order n, mapping the geometric generators γ_i (and thus also the $\gamma_i * Q_j$) to transpositions, and such that

$\theta(\gamma_1 \dots \gamma_d) = 1$,
$\theta(\gamma_1 * Q_j) = \theta(\gamma_2 * Q_j)$ if $r_j = 1$,
$\theta(\gamma_1 * Q_j)$ and $\theta(\gamma_2 * Q_j)$ are distinct and commute if $r_j = \pm 2$,
$\theta(\gamma_1 * Q_j)$ and $\theta(\gamma_2 * Q_j)$ do not commute if $r_j = 3$.

Observe that, when the braid factorization defining D is affected by a Hurwitz move, θ remains unchanged and the compatibility conditions are preserved. On the contrary, when the braid factorization is modified by simultaneously conjugating all factors by a certain braid $Q \in B_d$, the system of geometric generators $\gamma_1, \dots, \gamma_d$ changes accordingly, and so the geometric monodromy representation θ should be replaced by $\theta \circ Q_*$, where Q_* is the automorphism of F_d induced by the braid Q.

One easily checks that, given a braided curve $D \subset \mathbb{CP}^2$ and a compatible monodromy representation $\theta : F_d \to S_n$, it is possible to recover a compact 4-manifold X and a branched covering map $f : X \to \mathbb{CP}^2$ in a canonical way. Moreover, as observed above we can assume that the curve D is symplectic; in that case, the branched covering map makes it possible to endow X with a symplectic structure, canonically up to symplectic isotopy (see [3],[5] ; a similar result has also been obtained by Catanese).

The above discussion leads naturally to the definition of symplectic invariants arising from the quasiholomorphic coverings constructed in Theorem 3.1. However, things are complicated by the fact that the branch curves of these coverings are only canonical up to cancellations of double points.

On the level of the braid factorization, a pair cancellation amounts to removing two consecutive factors which are the inverse of each other (necessarily one must have degree 2 and the other degree -2); the geometric monodromy representation is not affected. The opposite operation is the creation of a pair of nodes, in which two factors $(Q^{-1}X_1^{-2}Q).(Q^{-1}X_1^2Q)$ are added anywhere in the factorization ; it is allowed only if the new factorization remains compatible with the monodromy representation θ, i.e. if $\theta(\gamma_1 * Q)$ and $\theta(\gamma_2 * Q)$ are commuting disjoint transpositions.

Definition 3.5. Two braid factorizations (along with the corresponding geometric monodromy representations) are m-equivalent if there exists a sequence of operations which turns one into the other, each operation being either a global conjugation, a Hurwitz move, or a pair cancellation or creation.

In conclusion, we get the following result :

Theorem 3.2 ([5]). *The braid factorizations and geometric monodromy representations associated to the quasiholomorphic coverings obtained in Theorem 3.1 are, for $k \gg 0$, canonical up to m-equivalence, and define symplectic invariants of (X^4, ω).*

Conversely, the data consisting of a braid factorization and a geometric monodromy representation, or a m-equivalence class of such data, determines a symplectic 4-manifold in a canonical way up to symplectomorphism.

3.3. **The braid group and the mapping class group.** Let $f : X \to \mathbb{CP}^2$ be a branched covering map, and let $D \subset \mathbb{CP}^2$ be its branch curve. It is a simple observation that, if D is braided, then the map $\pi \circ f$ with values in $\mathbb{CP}^1$ obtained by forgetting one of the components of f topologically defines a Lefschetz pencil. This pencil is obtained by lifting via the covering f the pencil of lines on $\mathbb{CP}^2$ defined by π, and its base points are the preimages by f of the pole of the projection π.

Moreover, if one starts with the quasiholomorphic coverings given by Theorem 3.1, then the corresponding Lefschetz pencils coincide for $k \gg 0$ with those obtained by Donaldson in [10] and described in §2.

As a consequence, in the case of a 4-manifold, the invariants described in §3.2 (braid factorization and geometric monodromy representation) completely determine those described in §2 (factorizations in mapping class groups). It is therefore natural to look for a more explicit description of the relation between branched coverings and Lefschetz pencils. This description involves the group of *liftable braids*, which has been studied in a special case by Birman and Wajnryb in [7]. We recall the following construction from §5 of [5].

Let $\mathcal{C}_n(q_1, \dots, q_d)$ be the (finite) set of all surjective group homomorphisms $F_d \to S_n$ which map each of the geometric generators $\gamma_1, \dots, \gamma_d$ of F_d to a transposition and map their product $\gamma_1 \cdots \gamma_d$ to the identity element in S_n. Each element of $\mathcal{C}_n(q_1, \dots, q_d)$ determines a simple n-fold covering of $\mathbb{CP}^1$ branched at $q_1, \dots, q_d$.

Let $\mathcal{X}_d$ be the space of configurations of d distinct points in the plane. The set of all simple n-fold coverings of $\mathbb{CP}^1$ with d branch points and such that no branching occurs above the point at infinity can be thought of as a covering $\tilde{\mathcal{X}}_{d,n}$ above $\mathcal{X}_d$, in which the fiber above the configuration $\{q_1, \dots, q_d\}$ identifies with $\mathcal{C}_n(q_1, \dots, q_d)$. Therefore, the braid group $B_d = \pi_1(\mathcal{X}_d)$ acts on the fiber $\mathcal{C}_n(q_1, \dots, q_d)$ by deck transformations of the covering $\tilde{\mathcal{X}}_{d,n}$. In fact, the action of a braid $Q \in B_d$ on $\mathcal{C}_n(q_1, \dots, q_d)$ is given by $\theta \mapsto \theta \circ Q_*$, where $Q_* \in \mathrm{Aut}(F_d)$ is the automorphism induced by Q on the fundamental group of $\mathbb{C} - \{q_1, \dots, q_d\}$.

Fix a base point $\{q_1, \dots, q_d\}$ in $\mathcal{X}_d$, and consider an element θ of $\mathcal{C}_n(q_1, \dots, q_d)$ (i.e., a monodromy representation $\theta : F_d \to S_n$). Let p_θ be the corresponding point in $\tilde{\mathcal{X}}_{d,n}$.

Definition 3.6. The subgroup $B^0_d(\theta)$ of liftable braids is the set of all the loops in $\mathcal{X}_d$ whose lift at the point p_θ is a closed loop in $\tilde{\mathcal{X}}_{d,n}$. Equivalently, $B^0_d(\theta)$ is the set of all braids which act on $F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ in a manner compatible with the covering structure defined by θ.

In other words, $B^0_d(\theta)$ is the set of all braids Q such that $\theta \circ Q_* = \theta$, i.e. the stabilizer of θ with respect to the action of B_d on $\mathcal{C}_n(q_1, \dots, q_d)$.

There exists a natural bundle $\mathcal{Y}_{d,n}$ over $\tilde{\mathcal{X}}_{d,n}$ (the *universal curve*) whose fiber is a Riemann surface of genus $g = 1 - n + (d/2)$ with n marked points. Each of these Riemann surfaces naturally carries a structure of branched covering of $\mathbb{CP}^1$, and the marked points are the preimages of the point at infinity.

Given an element Q of $B^0_d(\theta) \subset B_d$, it can be lifted to $\tilde{\mathcal{X}}_{d,n}$ as a loop based at the point p_θ, and the monodromy of the fibration $\mathcal{Y}_{d,n}$ along this loop defines an element of $\mathrm{Map}_{g,n}$ (the mapping class group of a Riemann surface of genus g with n boundary components), which we call $\theta_*(Q)$. This defines a group homomorphism $\theta_* : B^0_d(\theta) \to \mathrm{Map}_{g,n}$.

More geometrically, viewing Q as a compactly supported diffeomorphism of the plane preserving $\{q_1, \dots, q_d\}$, the fact that Q belongs to $B^0_d(\theta)$ means that it can be lifted via the covering map $\Sigma_g \to \mathbb{CP}^1$ to a diffeomorphism of Σ_g ; the corresponding element in the mapping class group is $\theta_*(Q)$.

It is easy to check that, when the given monodromy representation θ is compatible with a braided curve $D \subset \mathbb{CP}^2$, the image of the braid monodromy homomorphism

$\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d$ describing D is entirely contained in $B^0_d(\theta)$: this is because the geometric monodromy representation θ factors through $\pi_1(\mathbb{CP}^2 - D)$, on which the braids in $\mathrm{Im}\,\rho$ act trivially. Therefore, we can take the image of the braid factorization describing D by θ_* and obtain a factorization in the mapping class group $\mathrm{Map}_{g,n}$. One easily checks that $\theta_*(\Delta^2)$ is, as expected, the twist δ_Z around the n marked points.

As observed in [5], all the factors of degree ± 2 or 3 in the braid factorization lie in the kernel of θ_* ; therefore, the only terms whose contribution to the mapping class group factorization is non-trivial are those arising from the tangency points of the branch curve D, and each of these is a Dehn twist. More precisely, the image in $\mathrm{Map}_{g,n}$ of a half-twist $Q \in B^0_d(\theta)$ can be constructed as follows. Call γ the path joining two of the branch points naturally associated to the half-twist Q (i.e. the path along which the twisting occurs). Among the n lifts of γ to Σ_g, only two hit the branch points of the covering ; these two lifts have common end points, and together they define a loop δ in Σ_g. Then the element $\theta_*(Q)$ in $\mathrm{Map}_{g,n}$ is the positive Dehn twist along the loop δ (see Proposition 4 of [5]).

In conclusion, the following result holds :

Proposition 3.3. *Let $f : X \to \mathbb{CP}^2$ be a branched covering, and assume that its branch curve D is braided. Let $\rho : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B^0_d(\theta)$ and $\theta : F_d \to S_n$ be the corresponding braid monodromy and geometric monodromy representation. Then the monodromy map $\psi : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to \mathrm{Map}_{g,n}$ of the Lefschetz pencil $\pi \circ f$ is given by the identity $\psi = \theta_* \circ \rho$.*

In particular, for $k \gg 0$ the symplectic invariants obtained from Theorem 2.1 are obtained in this manner from those given by Theorem 3.2.

Remark 3.2. It is a basic fact that for $n \geq 3$ the group homomorphism $\theta_* : B^0_d(\theta) \to \mathrm{Map}_{g,n}$ is surjective, and that for $n \geq 4$ every Dehn twist is the image by θ_* of a half-twist. This makes it natural to ask whether every factorization of δ_Z in $\mathrm{Map}_{g,n}$ as a product of Dehn twists is the image by θ_* of a factorization of Δ^2 in $B^0_d(\theta)$ compatible with θ. This can be reformulated in more geometric terms as the classical problem of determining whether every Lefschetz pencil is topologically a covering of $\mathbb{CP}^2$ branched along a curve with node and cusp singularities (a similar question replacing pencils by Lefschetz fibrations and $\mathbb{CP}^2$ by ruled surfaces also holds ; presently the answer is only known in the hyperelliptic case, thanks to the results of Fuller, Siebert and Tian).

A natural approach to these problems is to understand the kernel of θ_*. For example, if one can show that this kernel is generated by squares and cubes of half-twists (factors of degree 2 and 3 compatible with θ), then the solution naturally follows : given a decomposition of δ_Z as a product of Dehn twists in $\mathrm{Map}_{g,n}$, any lift of this word to $B^0_d(\theta)$ as a product of half-twists differs from Δ^2 by a product of factors of degree 2 and 3 and their inverses. Adding these factors as needed, one obtains a decomposition of Δ^2 into factors of degrees 1, ± 2 and ± 3 ; the branch curve constructed in this way may have nodes and cusps with reversed orientation, but it can still be made symplectic.

Even if the kernel of θ_* is not generated by factors of degree 2 and 3, it remains likely that the result still holds and can be obtained by starting from a suitable lift to $B^0_d(\theta)$ of the word in $\mathrm{Map}_{g,n}$. A better understanding of the structure of $\mathrm{Ker}\,\theta_*$ would be extremely useful for this purpose.

4. The higher dimensional case

In this section we extend the results of §3 to the case of higher dimensional symplectic manifolds. In §4.1 we prove the existence of quasiholomorphic maps $X \to \mathbb{CP}^2$ given by triples of sections of $L^{\otimes k}$ for $k \gg 0$. The topological invariants arising from these maps are studied in §4.2 and §4.3, and the relation with Lefschetz pencils is described in §4.4.

4.1. **Quasiholomorphic maps to $\mathbb{CP}^2$.** Let (X^{2n}, ω) be a compact symplectic manifold, endowed with a compatible almost-complex structure J. Let L be the same line bundle as previously (if $\frac{1}{2\pi}[\omega]$ is not integral one works with a perturbed symplectic form as explained in the introduction). Consider three approximately holomorphic sections of $L^{\otimes k}$, or equivalently a section of $\mathbb{C}^3 \otimes L^{\otimes k}$. Then the following result states that exactly the same transversality and compatibility properties can be expected as in the four-dimensional case :

Theorem 4.1. *For $k \gg 0$, it is possible to find asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ such that the corresponding $\mathbb{CP}^2$ valued projective maps f_k are quasiholomorphic (cf. Definition 3.2). Moreover, for large k these projective maps are canonical up to isotopy and up to cancellations of pairs of nodes in the critical curves D_k.*

Before sketching a proof of Theorem 4.1, we briefly describe the behavior of quasiholomorphic maps, which will clarify some of the requirements of Definition 3.2.

Condition (1) in Definition 3.2 implies that the set Z_k of points where the three sections s_k^0, s_k^1, s_k^2 vanish simultaneously is a smooth codimension 6 symplectic (approximately holomorphic) submanifold. The projective map $f_k = (s_k^0 : s_k^1 : s_k^2)$ with values in $\mathbb{CP}^2$ is only defined over the complement of Z_k. The behavior near the set of base points is similar to what happens for Lefschetz pencils : in suitable local approximately holomorphic coordinates, Z_k is given by the equation $z_1 = z_2 = z_3 = 0$, and f_k behaves like the model map $(z_1, \dots, z_n) \mapsto (z_1 : z_2 : z_3)$. In fact, a map defined everywhere can be obtained by blowing up X along the submanifold Z_k. The behavior near Z_k being completely specified by condition (1), it is implicit that all the other conditions on f_k are only to be imposed outside of a small neighborhood of Z_k.

The correct statement of condition (3) of Definition 3.2 in the case of a manifold of dimension greater than 4 is a bit tricky. Indeed, $\mathrm{Jac}(f_k) = \bigwedge^2 \partial f_k$ is a priori a section of the vector bundle $\Lambda^{2,0}T^*X \otimes f_k^*(\Lambda^{2,0}T\mathbb{CP}^2)$ of rank $n(n-1)/2$. However, transversality to 0 in this sense is impossible to obtain, as the expected complex codimension of R_k is $n-1$ instead of $n(n-1)/2$. Indeed, the section $\mathrm{Jac}(f_k)$ takes values in the non-linear subbundle $\mathrm{Im}(\bigwedge^2)$, whose fibers are of dimension $n-1$ at their smooth points (away from the origin). However, transversality to 0 does not have any natural definition in this subbundle, because it is singular along the zero section. The problem is very similar to what happens in the construction of determinantal submanifolds performed in [15].

In our case, a precise meaning can be given to condition (3) by the following observation. Near any point $x \in X$, property (2) implies that it is possible to find local approximately holomorphic coordinates on X and local complex coordinates on $\mathbb{CP}^2$ in which the differential at x of the first component of f_k can be written

$\partial f_k^1(x) = \lambda\, dz_1$, with $|\lambda| > \gamma/2$. This implies that, near x, the projection of $\bigwedge^2 \partial f_k$ to its components along $dz_1 \wedge dz_2, \dots, dz_1 \wedge dz_n$ is a quasi-isometric isomorphism. In other words, the transversality to 0 of $\mathrm{Jac}(f_k)$ is to be understood as the transversality to 0 of its orthogonal projection to the linear subbundle of rank $n-1$ generated by $dz_1 \wedge dz_2, \dots, dz_1 \wedge dz_n$.

Another equivalent approach is to consider the (non-linear) bundle $\mathcal{J}^1(X, \mathbb{CP}^2)$ of holomorphic 1-jets of maps from X to $\mathbb{CP}^2$. Inside this bundle, the 1-jets whose differential is not surjective define a subbundle Σ of codimension $n-1$, smooth away from the stratum $\{\partial f = 0\}$. Since this last stratum is avoided by the 1-jet of f_k (because of condition (2)), the transversality to 0 of $\mathrm{Jac}(f_k)$ can be naturally rephrased in terms of estimated transversality to Σ in the bundle of jets (this approach will be developed in [4]).

With this understood, conditions (3) and (3′) imply, as in the four-dimensional case, that the set R_k of points where the differential of f_k fails to be surjective is a smooth symplectic curve $R_k \subset X$, disjoint from Z_k, and that the differential of f_k has rank 2 at every point of R_k. Also, as before, conditions (4) and (4′) imply that $f_k(R_k) = D_k$ is a symplectic curve in $\mathbb{CP}^2$, immersed outside of the cusp points.

We now describe the proof of Theorem 4.1 ; most of the argument is identical to the 4-dimensional case, and the reader is referred to [3] and [5] for notations and details.

Proof of Theorem 4.1. The strategy of proof is the same as in the 4-dimensional case. One starts with an arbitrary sequence of asymptotically holomorphic sections of $\mathbb{C}^3 \otimes L^{\otimes k}$ over X, and perturbs it first to obtain the transversality properties. Provided that k is large enough, each transversality property can be obtained over a ball by a small localized perturbation, using the local transversality result of Donaldson (Theorem 12 in [10]). A globalization argument then makes it possible to combine these local perturbations into a global perturbation that ensures transversality everywhere (Proposition 3 of [3]). Since transversality properties are open, successive perturbations can be used to obtain all the required properties : once a transversality property is obtained, subsequent perturbations only affect it by at most decreasing the transversality estimate.

Step 1. One first obtains the transversality statements in parts (1), (5) and (8) of Definition 3.2 ; as in the 4-dimensional case, these properties are obtained e.g. simply by applying the main result of [2]. Observe that all required properties now hold near the base locus Z_k of s_k, so we can assume in the rest of the argument that the points of X being considered lie away from Z_k, and therefore that f_k is locally well-defined.

One next ensures condition (2), for which the argument is an immediate adaptation of that in §2.2 of [3], the only difference being the larger number of coordinate functions.

Step 2. The next property we want to get is condition (3). Here a significant generalization of the argument in §3.1 of [3] is needed. The problem reduces, as usual, to showing that the uniform transversality to 0 of $\mathrm{Jac}(f_k)$ can be ensured over a small ball centered at a given point $x \in X$ by a suitable localized perturbation. As in [3] one can assume that $s_k(x)$ is of the form $(s_k^0(x), 0, 0)$ and therefore locally trivialize $\mathbb{CP}^2$ via the quasi-isometric map $(x\!:\!y\!:\!z) \mapsto (y/x, z/x)$; this reduces the problem to the study of a $\mathbb{C}^2$-valued map h_k. Because $|\partial f_k|$ is bounded from below, we can assume (after a suitable rotation) that $|\partial h_k^1(x)|$ is greater than some fixed constant. Also, fixing suitable approximately holomorphic Darboux coordinates $z_k^1, \dots, z_k^n$ (using Lemma 3 of [3], which trivially extends to dimensions larger than 4), we can after a rotation assume that $\partial h_k^1(x)$ is of the form $\lambda\, dz_k^1$, where the complex number λ is bounded from below.

By Lemma 2 of [3], there exist asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of $L^{\otimes k}$ with exponential decay away from x. Define the asymptotically holomorphic 2-forms $\mu_k^j = \partial h_k^1 \wedge \partial(z_k^j s_{k,x}^{\mathrm{ref}}/s_k^0)$ for $2 \le j \le n$. At x, the 2-form μ_k^j is proportional to $dz_k^1 \wedge dz_k^j$; therefore, over a small neighborhood of x, the transversality to 0 of $\mathrm{Jac}(f_k)$ in the sense explained above is equivalent to the transversality to 0 of the projection of $\mathrm{Jac}(h_k)$ onto the subspace generated by $\mu_k^2, \dots, \mu_k^n$. In terms of 1-jets, the 2-forms μ_k^j define a local frame in the normal bundle to the stratum of non-regular maps at $\mathcal{J}^1(f_k)$. Now, express $\mathrm{Jac}(h_k)$ in the form $u_k^2\mu_k^2 + \dots + u_k^n\mu_k^n + \alpha_k$ over a neighborhood of x, where $u_k^2, \dots, u_k^n$ are complex-valued functions and α_k has no component along dz_k^1. Then, the transversality to 0 of $\mathrm{Jac}(f_k)$ is equivalent to that of the $\mathbb{C}^{n-1}$-valued function $u_k = (u_k^2, \dots, u_k^n)$.

Since the functions u_k are asymptotically holomorphic, using suitable Darboux coordinates at x we can use Theorem 12 of [10] to obtain, for large enough k, the existence of constants $w_k^2, \dots, w_k^n$ smaller than any given bound $\delta > 0$ and such that $(u_k^2 - w_k^2, \dots, u_k^n - w_k^n)$ is η-transverse to 0 over a small ball centered at x, where $\eta = \delta(\log \delta^{-1})^{-p}$ (p is a fixed constant). Letting $\tilde{s}_k = (s_k^0, s_k^1, s_k^2 - \sum w_k^j z_k^j s_{k,x}^{\mathrm{ref}})$ and calling $\tilde{f}_k$ and $\tilde{h}_k$ the projective map defined by $\tilde{s}_k$ and the corresponding local $\mathbb{C}^2$-valued map, we get that $\mathrm{Jac}(\tilde{h}_k) = \mathrm{Jac}(h_k) - \sum w_k^j \mu_k^j$, and therefore that $\mathrm{Jac}(\tilde{f}_k)$ is transverse to 0 near x. Since the perturbation of s_k has exponential decay away from x, we can apply the standard globalization argument to obtain property (3) everywhere.

Step 3. The next properties that we want to get are (4) and (6). It is possible to extend the arguments of [3] and [5] to the higher dimensional case ; however this yields a very technical and lengthy argument, so we outline here a more efficient strategy following the ideas of [4]. Thanks to the previously obtained transversality properties (1) and (5), both f_k and ϕ_k are well-defined over a neighborhood of R_k, so the statements of (4) and (6) are well-defined. Moreover, observe that property (6) implies property (4), because at any point where $\partial(f_{k|R_k})$ vanishes, $\partial(\phi_{k|R_k})$ necessarily vanishes as well, and if it does so transversely then the same is true for $\partial(f_{k|R_k})$ as well. So we only focus on (6).

This property can be rephrased in terms of transversality to the codimension n stratum $S : \{\partial(\phi_{|R}) = 0\}$ in the bundle $\mathcal{J}^2(X, \mathbb{CP}^2)$ of holomorphic 2-jets of maps from X to $\mathbb{CP}^2$. However this stratum is singular, even away from the substratum S_{nt} corresponding to the non-transverse vanishing of $\mathrm{Jac}(f)$; in fact it is reducible and comes as a union $S_1 \cup S_2$, where $S_1 : \{\mathrm{Jac}(f) = 0,\ \partial(f_{|R}) = 0\}$ is the stratum corresponding to non-immersed points of the branch curve, and $S_2 : \{\partial\phi = 0\}$ is the stratum corresponding to tangency points of the branch curve. Therefore, one first needs to ensure transversality with respect to $S_0 = S_1 \cap S_2 : \{\partial\phi = 0,\ \partial(f_{|R}) = 0\}$, which is a smooth codimension $n+1$ stratum ("vertical cusp points of the branch curve") away from S_{nt}.

Step 3a. We first show that a small perturbation can be used to make sure that the quantity $(\partial\phi_k, \partial(f_{k|R_k}))$ remains bounded from below, i.e. that given any point

$x \in X$, either $\partial\phi_k(x)$ is larger than a fixed constant, or x lies at more than a fixed distance from R_k, or x lies close to a point of R_k where $\partial(f_{k|R_k}))$ is larger than a fixed constant. Since this transversality property is local and open, we can obtain it by successive small localized perturbations, as for the previous properties.

Fix a point $x \in X$, and assume that $\partial\phi_k(x)$ is small (otherwise no perturbation is needed). By property (5), we know that necessarily (s_k^0, s_k^1) is bounded away from zero at x ; a rotation in the first two coordinates makes it possible to assume that $s_k^1(x) = 0$ and s_k^0 is bounded from below near x. As above, we replace f_k by the $\mathbb{C}^2$-valued map $h_k = (h_k^1, h_k^2)$, where $h_k^i = s_k^i/s_k^0$. By assumption, we get that $\partial h_k^1(x)$ is small. This implies in particular that $\mathrm{Jac}(f_k)$ is small at x, and therefore property (3) gives a lower bound on its covariant derivative. Moreover, by property (2) we also have a lower bound on $\partial h_k^2(x)$, which after a suitable rotation can be assumed equal to $\lambda\, dz_k^1$ for some $\lambda \neq 0$. So, as above we can express $\bigwedge^2 \partial f_k$ by looking at its components along $dz_k^1 \wedge dz_k^j$ for $2 \leq j \leq n$; we again define the 2-forms $\mu_k^j = \partial h_k^2 \wedge \partial(z_k^j s_{k,x}^{\mathrm{ref}}/s_k^0)$, and the functions $u_2, \dots, u_n$ are defined as previously. Define a $(n,0)$-form θ over a neighborhood of x by $\theta = \partial u_2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^2$: at points of R_k, the vanishing of θ is equivalent to that of $\partial h^2_{k|R_k}$, or equivalently to that of $\partial f_{k|R_k}$. So our aim is to show that the quantity $(\partial h_k^1, \theta)$, which is a section of a rank $n+1$ bundle $\mathcal{E}_0$ near x, can be made bounded from below by a small perturbation.

For this purpose, we first show the existence of complex-valued polynomials (P_j^1, P_j^2) and local sections ϵ_j of $\mathcal{E}_0$, $1 \leq j \leq n+1$, such that :

(a) for any coefficients $w_j \in \mathbb{C}$, replacing the given sections of $L^{\otimes k}$ by $(s_k^0, s_k^1 + \sum w_j P_j^1 s_{k,x}^{\mathrm{ref}}, s_k^2 + \sum w_j P_j^2 s_{k,x}^{\mathrm{ref}})$ affects $(\partial h_k^1, \theta)$ by the addition of $\sum w_j \epsilon_j + O(w_j^2)$;

(b) the sections ϵ_j define a local frame in $\mathcal{E}_0$, and $\epsilon_1 \wedge \dots \wedge \epsilon_{n+1}$ is bounded from below by a universal constant.

First observe that, by property (3), $\partial u_2 \wedge \dots \wedge \partial u_n$ is bounded from below near x, whereas we may assume that $\theta = \partial u_2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^2$ is small (otherwise no perturbation is needed). Therefore, ∂h_k^2 (which at x is colinear to dz_k^1) lies close to the span of the ∂u_j. In particular, after a suitable rotation in the $n-1$ last coordinates on X, we can assume that $\partial u_2 \wedge \partial h_k^2$ is small at x. On the other hand, we know that there exists $j_0 \neq 1$ such that $dz_k^{j_0}$ lies far from the span of the $\partial u_j(x)$. We then define $P_{n+1}^1 = z_k^2 z_k^{j_0}$ and $P_{n+1}^2 = 0$. Adding to s_k^1 a quantity of the form $w\, z_k^2 z_k^{j_0}\, s_{k,x}^{\mathrm{ref}}$ does not affect $\partial h_k(x)$, but affects $\partial u_2(x)$ by the addition of a non-trivial multiple of $dz_k^{j_0}$, and similarly affects $\partial u_{j_0}(x)$ by the addition of a non-trivial multiple of dz_k^2. The other $\partial u_j(x)$ are not affected. Therefore, $\theta(x)$ changes by an amount of

$$cw\, dz_k^{j_0} \wedge \partial u_3 \wedge \dots \wedge \partial u_n \wedge \partial h_k^2 + c' w\, \partial u_2 \wedge \dots \wedge dz_k^2 \wedge \dots \wedge \partial u_n \wedge \partial h_k^2 + O(w^2),$$

where the constants c and c' are bounded from above and below. The first term is bounded from below by construction, while the second term is only present if $j_0 \neq 2$ (this requires $n \geq 3$), and in that case it is small because $\partial u_2 \wedge \partial h_k^2$ is small. Therefore, the local section ϵ_{n+1} of $\mathcal{E}_0$ naturally corresponding to such a perturbation is of the form $(0, \epsilon'_{n+1})$ at x, where ϵ'_{n+1} is bounded from below.

Next, for $1 \leq j \leq n$ we define $P_j^1 = z_k^j$ and $P_j^2 = 0$, and observe that adding $w\, z_k^j\, s_{k,x}^{\mathrm{ref}}$ to s_k^1 affects $\partial h_k^1(x)$ by adding a nontrivial multiple of dz_k^j. Therefore, the local section of $\mathcal{E}_0$ corresponding to this perturbation is at x of the form $\epsilon_j(x) = (c'' dz_k^j, \epsilon'_j)$, where c'' is a constant bounded from below.

It follows from this argument that the chosen perturbations P_j^1 and P_j^2 for $1 \leq j \leq n+1$, and the corresponding local sections ϵ_j of $\mathcal{E}_0$, satisfy the conditions (a) and (b) expressed above. Observe that, because ϵ_j define a local frame at x and $\epsilon_1 \wedge \cdots \wedge \epsilon_{n+1}$ is bounded from below at x, the same properties remain true over a ball of fixed radius around x.

Now that a local approximately holomorphic frame in $\mathcal{E}_0$ is given, we can write $(\partial h_k^1, \theta)$ in the form $\sum \zeta_j \epsilon_j$ for some complex-valued functions ζ_j ; it is easy to check that these functions are asymptotically holomorphic. Therefore, we can again use Theorem 12 of [10] to obtain, if k is large enough, the existence of constants $w_1, \dots, w_{n+1}$ smaller than any given bound $\delta > 0$ and such that $(\zeta_1 - w_1, \dots, \zeta_{n+1} - w_{n+1})$ is bounded from below by $\eta = \delta(\log \delta^{-1})^{-p}$ (p is a fixed constant) over a small ball centered at x. Letting $\tilde{s}_k = (s_k^0, s_k^1 - \sum w_j P_j^1 s_{k,x}^{\mathrm{ref}}, s_k^2 - \sum w_j P_j^2 s_{k,x}^{\mathrm{ref}})$ and calling $\tilde{f}_k$, $\tilde{h}_k$ and $\tilde{\theta}$ the projective map defined by $\tilde{s}_k$ and the corresponding local maps, we get that $(\partial \tilde{h}_k^1, \tilde{\theta})$ is by construction bounded from below by $c_0 \eta$, for a fixed constant c_0 ; indeed, observe that the non-linear term $O(w^2)$ in the perturbation formula does not play any significant role, as it is at most of the order of $\delta^2 \ll \eta$. Since the perturbation of s_k has exponential decay away from x, we can apply the standard globalization argument to obtain uniform transversality to the stratum $S_0 \subset \mathcal{J}^2(X, \mathbb{CP}^2)$ everywhere.

Step 3b. We now obtain uniform transversality to the stratum $S : \{\mathrm{Jac}(f) = 0,\ \partial(\phi_{|R}) = 0\}$. The strategy and notations are the same as above. We again fix a point $x \in X$, and assume that x lies close to a point of R_k where $\partial(\phi_{k|R_k})$ is small (otherwise, no perturbation is needed). As above, we can assume that $s_k^0(x)$ is bounded from below and define a $\mathbb{C}^2$-valued map h_k. Two cases can occur : either $\partial h_k^1(x)$ is bounded away from zero, or it is small and in that case by Step 3a we know that $\partial(h_{k|R_k}^2)$ is bounded from below near x.

We start with the case where ∂h_k^1 is bounded from below; in other words, we are not dealing with tangency points but only with cusps. In that case, we can use an argument similar to Step 3a, except that the roles of the two components of h_k are reversed. Namely, after a rotation we assume that $\partial h_k^1(x) = \lambda dz_k^1$ for some nonzero constant λ, and we define components $u_2, \dots, u_n$ of $\mathrm{Jac}(f_k)$ as previously (using ∂h_k^1 rather than ∂h_k^2 to define the μ_k^j). Let $\theta = \partial u_2 \wedge \cdots \wedge \partial u_n \wedge \partial h_k^1$: along R_k, the ratio between θ and $\partial(h_{k|R_k}^1)$, or equivalently $\partial(\phi_{k|R_k})$, is bounded between two fixed constants, so the transverse vanishing of θ is what we are trying to obtain. More precisely, our aim is to show that the quantity $(u_2, \dots, u_n, \theta)$, which is a section of a rank n bundle $\mathcal{E}$ near x, can be made uniformly transverse to 0 by a small perturbation.

For this purpose, we first show the existence of complex-valued polynomials (P_j^1, P_j^2) and local sections ϵ_j of $\mathcal{E}$, $2 \leq j \leq n+1$, such that :

(a) for any coefficients $w_j \in \mathbb{C}$, replacing the given sections of $L^{\otimes k}$ by $(s_k^0, s_k^1 + \sum w_j P_j^1 s_{k,x}^{\mathrm{ref}}, s_k^2 + \sum w_j P_j^2 s_{k,x}^{\mathrm{ref}})$ affects $(u_2, \dots, u_n, \theta)$ by the addition of $\sum w_j \epsilon_j + O(w_j^2)$;

(b) the sections ϵ_j define a local frame in $\mathcal{E}$, and $\epsilon_2 \wedge \cdots \wedge \epsilon_{n+1}$ is bounded from below by a universal constant.

By the same argument as in Step 3a, we find after a suitable rotation an index $j_0 \neq 1$ such that, letting $P^1_{n+1} = 0$ and $P^2_{n+1} = z^2_k z^{j_0}_k$, the corresponding local section ϵ_{n+1} of $\mathcal{E}$ is, at x, of the form $(0, \dots, 0, \epsilon'_{n+1})$, with ϵ'_{n+1} bounded from below by a fixed constant.

Moreover, adding $w\, z^j_k s^{\mathrm{ref}}_{k,x}$ to s^2_k amounts to adding w to u_j and does not affect the other u_i's, by the argument in Step 2. So, letting $P^1_j = 0$ and $P^2_j = z^j_k$, we get that the corresponding local sections of $\mathcal{E}$ are of the form $\epsilon_j = (0, \dots, 1, \dots, 0, \epsilon'_j)$, where the coefficient 1 is in j-th position.

So it is easy to check that both conditions (a) and (b) are satisfied by these perturbations. The rest of the argument is as in Step 3a : expressing $(u_2, \dots, u_n, \theta)$ as a linear combination of $\epsilon_2, \dots, \epsilon_{n+1}$, one uses Theorem 12 of [10] to obtain transversality to 0 over a small ball centered at x.

We now consider the second possibility, namely the case where $\partial h^1_k(x)$ is small, which corresponds to tangency points. By property (2) we know that $\partial h^2_k(x)$ is bounded from below, and we can assume that it is colinear to dz^1_k. We then define components $u_2, \dots, u_n$ of $\mathrm{Jac}(f_k)$ as usual (as in Step 3a and unlike the previous case, the μ^j_k are defined using ∂h^2_k rather than ∂h^1_k). Letting $\theta = \partial u_2 \wedge \cdots \wedge \partial u_n \wedge \partial h^1_k$, we want as before to obtain the transversality to 0 of the quantity $(u_2, \dots, u_n, \theta)$, which is a local section of a rank n bundle $\mathcal{E}$ near x. For this purpose, as usual we look for polynomials P^1_j, P^2_j and local sections ϵ_j satisfying the same properties (a) and (b) as above.

In order to construct P^i_{n+1}, observe that, by the result of Step 3a, the quantity $\partial u_2 \wedge \cdots \wedge \partial u_n \wedge \partial h^2_k$ is bounded from below at x. So, adding to s^1_k a small multiple of s^2_k does not affect the u_j's, but it affects θ non-trivially. However, this perturbation is not localized, so it is not suitable for our purposes (we can't apply the globalization argument). Instead, let P^1_{n+1} be a polynomial of degree 2 in the coordinates z^j_k and their complex conjugates, such that $P^1_{n+1} s^{\mathrm{ref}}_{k,x}$ coincides with s^2_k up to order two at x. Note that the coefficients of P^1_{n+1} are bounded by uniform constants, and that its antiholomorphic part is at most of the order $O(k^{-1/2})$ (because s^2_k and $s^{\mathrm{ref}}_{k,x}$ are asymptotically holomorphic); therefore, $P^1_{n+1} s^{\mathrm{ref}}_{k,x}$ is an admissible localized asymptotically holomorphic perturbation. Also, define $P^2_{n+1} = 0$. Then one easily checks that the local section ϵ_{n+1} of $\mathcal{E}$ corresponding to P^1_{n+1} and P^2_{n+1} is, at x, of the form $(0, \dots, 0, \epsilon'_{n+1})$, where ϵ'_{n+1} is bounded from below.

Moreover, let $P^1_j = z^j_k$ and $P^2_j = 0$: as above, this perturbation affects u_j and not the other u_i's, and we get that the corresponding local sections of $\mathcal{E}$ are of the form $\epsilon_j = (0, \dots, 1, \dots, 0, \epsilon'_j)$, where the coefficient 1 is in j-th position.

Once again, these perturbations satisfy both conditions (a) and (b). Therefore, expressing $(u_2, \dots, u_n, \theta)$ as a linear combination of $\epsilon_2, \dots, \epsilon_{n+1}$, Theorem 12 of [10] yields transversality to 0 over a small ball centered at x by the usual argument. Now that both possible cases have been handled, we can apply the standard globalization argument to obtain uniform transversality to the stratum $S \subset \mathcal{J}^2(X, \mathbb{CP}^2)$. This gives properties (4) and (6) of Definition 3.2.

Step 4. Now that all required transversality properties have been obtained, we perform further perturbations in order to achieve the other conditions in Definition 3.2. These new perturbations are bounded by a fixed multiple of $k^{-1/2}$, so the transversality properties are not affected. The argument is almost the same as in the case of 4-manifolds (see §4 of [3] and §3.1 of [5]); the adaptation to the higher-dimensional case is very easy.

One first defines a suitable almost-complex structure $\tilde{J}_k$, by the same argument as in §4.1 of [3] (except that one also considers the points of $\mathcal{T}_k$ and $\mathcal{I}_k$ besides the cusps). As explained in §4.1 of [3], a suitable perturbation makes it possible to obtain the local holomorphicity of f_k near these points, which yields conditions (4′), (6′) and (8′) ; the argument is the same in all three cases. Next, a generically chosen small perturbation yields the self-transversality of D (property (7)). Finally, as described in §4.2 of [3], a suitable perturbation yields property (3′) along the branch curve without modifying R_k and D_k and without affecting the other compatibility properties. This completes the proof of the existence statement in Theorem 4.1.

Uniqueness. The uniqueness statement is obtained by showing that, provided that k is large enough, the whole argument extends to the case of families of sections depending continuously on a parameter $t \in [0, 1]$. Then, given two sequences of quasiholomorphic maps, one can start with a one-parameter family of sections interpolating between them in a trivial way and perturb it in such a way that the required properties hold for all parameter values (with the exception of (7) when a node cancellation occurs). If one moreover checks that the construction can be performed in such a way that the two end points of the one-parameter family are not affected by the perturbation, the isotopy result becomes an immediate corollary. Observe that, in the one-parameter construction, the almost-complex structure is allowed to depend on t.

Most of the above argument extends to 1-parameter families in a straightforward manner, exactly as in the four-dimensional case ; the key observation is that all the standard building blocks (existence of approximately holomorphic Darboux coordinates z_k^j and of localized approximately holomorphic sections $s_{k,x}^{\mathrm{ref}}$, local transversality result, globalization principle, ...) remain valid in the parametric case, even when the almost-complex structure depends on t. The only places where the argument differs from the case of 4-manifolds are properties (3), (4) and (6), obtained in Steps 2 and 3 above.

For property (3), one easily checks that it is still possible in the parametric case to assume, after composing with suitable rotations depending continuously on the parameter t, that $s_k^1(x) = s_k^2(x) = 0$ and that $\partial h_k^1(x)$ is bounded from below and directed along dz_k^1. This makes it possible to define μ_k^j and u_k^j as in the non-parametric case, and the parametric version of Theorem 12 of [10] yields a suitable perturbation depending continuously on t.

The argument of Step 3a also extends to the parametric case, using the following observation. Fix a point $x \in X$, and let $\rho_k(t) = |\partial\phi_{k,t}(x)|$. For all values of t such that $\rho_k(t)$ is small enough (smaller than a fixed constant $\alpha > 0$), we can perform the construction as in the non-parametric case, defining $u_{j,t}$ and θ_t. If $\rho_k'(t) = |\theta_t(x)|$ is small enough (smaller than α), then we can apply the same argument as in the non-parametric case to define polynomials $(P_{j,t}^1, P_{j,t}^2)$ and local sections $\epsilon_{j,t}$ of $\mathcal{E}_0$. However the definition of P_{n+1}^1 needs to be modified as follows. Although it is still possible after a suitable rotation depending continuously on t to assume that $\partial u_2 \wedge \partial h_k^2(x)$ is small, the choice of an index $j_0 \neq 1$ such that $dz_k^{j_0}$ lies far from the span of the $\partial u_j(x)$ may depend on t. Instead, we define $\nu_{k,t}$ as a unit vector in $\mathbb{C}^{n-1}$ depending continuously on t and such that $\sum_{j=2}^n \nu_{k,t}^j \, dz_k^j$ lies far from the span of $\partial u_j(x)$, and let $P^1_{n+1,t} = \sum_{j=2}^n \nu^j_{k,t} z_k^2 z_k^j$. Then the required properties are satisfied, and we can proceed with the argument. So, provided that $\rho_k(t)$ and $\rho'_k(t)$ are both smaller than α, we can use Theorem 12 of [10] to obtain a localized perturbation $\tau_{k,t}$ depending continuously on t and such that $s_{k,t} + \tau_{k,t}$ satisfies the desired transversality property near x.

In order to obtain a well-defined perturbation for all values of t, we introduce a continuous cut-off function $\beta : \mathbb{R}_+ \to [0,1]$ which equals 1 over $[0, \alpha/2]$ and vanishes outside of $[0, \alpha]$. Then, we set $\tilde{\tau}_{k,t} = \beta(\rho_k(t))\beta(\rho'_k(t))\tau_{k,t}$, which is well-defined for all t and depends continuously on t. Since $s_{k,t} + \tilde{\tau}_{k,t}$ coincides with $s_{k,t} + \tau_{k,t}$ when $\rho_k(t)$ and $\rho'_k(t)$ are smaller than $\alpha/2$, the required transversality holds for these values of t ; moreover, for the other values of t we know that the 2-jet of $s_{k,t}$ already lies at distance more than $\alpha/2$ from the stratum S_0, and we can safely assume that $\tilde{\tau}_{k,t}$ is much smaller than $\alpha/2$, so the perturbation does not affect transversality. Therefore we obtain a well-defined local perturbation for all $t \in [0, 1]$, and the one-parameter version of the result of Step 3a follows by the standard globalization argument.

The argument of Step 3b is extended to one-parameter families in the same way : given a point $x \in X$, the same ideas as for Step 3a yield, for all values of the parameter t such that the 2-jet of $s_{k,t}$ at x lies close to the stratum S, small localized perturbations $\tau_{k,t}$ depending continuously on t and such that $s_{k,t} + \tau_{k,t}$ satisfies the desired property over a small ball centered at x. As seen above, two different types of formulas for $\tau_{k,t}$ arise depending on which component of the stratum S is being hit; however, the result of Step 3a implies that, in any interval of parameter values such that the jet of $s_{k,t}$ remains close to S, only one of the two components of S has to be considered, so $\tau_{k,t}$ indeed depends continuously on t. The same type of cut-off argument as for Step 3a then makes it possible to extend the definition of $\tau_{k,t}$ to all parameter values and complete the proof. □

4.2. **The topology of quasiholomorphic maps.** We now describe the topological features of quasiholomorphic maps and the local models which characterize them near the critical points.

Proposition 4.2. *Let $f_k : X - Z_k \to \mathbb{CP}^2$ be a sequence of quasiholomorphic maps. Then the fibers of f_k are codimension 4 symplectic submanifolds, intersecting at the set of base points Z_k, and smooth away from the critical curve $R_k \subset X$. The submanifolds R_k and Z_k of X are smooth and symplectic, and the image $f_k(R_k) = D_k$ is a symplectic braided curve in $\mathbb{CP}^2$.*

Moreover, given any point $x \in R_k$, there exist local approximately holomorphic coordinates on X near x and on $\mathbb{CP}^2$ near $f_k(x)$ in which f_k is topologically conjugate to one of the two following models :

(i) $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$ *(points where $f_{k|R_k}$ is an immersion) ;*

(ii) $(z_1, \dots, z_n) \mapsto (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$ *(near the cusp points).*

Proof. The smoothness and symplecticity properties of the various submanifolds appearing in the statement follow from the observation made by Donaldson in [8] that the zero sets of approximately holomorphic sections satisfying a uniform transversality property are smooth and approximately J-holomorphic, and therefore symplectic. In particular, the smoothness and symplecticity of the fibers of f_k away from R_k follow immediately from Definition 3.2 : since $\mathrm{Jac}(f_k)$ is bounded from below away from R_k (because it satisfies a uniform transversality property), and since the sections s_k are asymptotically holomorphic, it is easy to check that the level sets of f_k are, away from R_k, smooth symplectic submanifolds. Symplecticity near the singular points is an immediate consequence of the local models (i) and (ii) that we will obtain later in the proof.

The corresponding properties of Z_k and R_k are obtained by the same argument : Z_k and R_k are the zero sets of asymptotically holomorphic sections, both satisfying a uniform transversality property (by conditions (1) and (3) of Definition 3.2, respectively), so they are smooth and symplectic.

We now study the local models at critical points of f_k. We start with the case of a cusp point $x \in X$. By property (2) of Definition 3.2, ∂f_k has complex rank 1 at x, so we can find local complex coordinates (Z_1, Z_2) on $\mathbb{CP}^2$ near $f_k(x)$ such that $\operatorname{Im} \partial f_k(x)$ is the Z_2 axis. Pulling back Z_2 via the map f_k, we obtain, using property $(4')$, a $\tilde{J}_k$-holomorphic function whose differential does not vanish near x ; therefore, we can find a $\tilde{J}_k$-holomorphic coordinate chart $(z_1, \dots, z_n)$ on X at x such that $z_n = Z_2 \circ f_k$. In the chosen coordinates, we get $f_k(z_1, \dots, z_n) = (g(z_1, \dots, z_n), z_n)$, where g is holomorphic and $\partial g(0) = 0$.

Since x is by assumption a cusp point, the tangent direction to R_k at x lies in the kernel of $\partial f_k(0)$, i.e. in the span of the $n-1$ first coordinate axes ; after a suitable rotation we may assume that $T_x R_k$ is the z_1 axis. Near the origin, $\operatorname{Jac}(f_k)$ is characterized by its $n-1$ components $(\partial g/\partial z_1, \dots, \partial g/\partial z_{n-1})$, and the critical curve R_k is the set of points where these quantities vanish. Therefore, at the origin, $\partial^2 g/\partial z_1^2 = \partial^2 g/\partial z_1 \partial z_2 = \dots = \partial^2 g/\partial z_1 \partial z_{n-1} = 0$. Nevertheless, $\operatorname{Jac}(f_k)$ vanishes transversely to 0 at the origin, so the matrix of second derivatives $M = (\partial^2 g/\partial z_i \partial z_j(0))$, $2 \le i \le n$, $1 \le j \le n-1$, is non-degenerate (invertible) at the origin. In particular, the first column of M (corresponding to $j = 1$) is non-zero, and therefore $\partial^2 g/\partial z_1 \partial z_n(0)$ is necessarily non-zero ; after a suitable rescaling of the coordinates we may assume that this coefficient is equal to 1. Moreover, the invertibility of M implies that the submatrix $M' = (\partial^2 g/\partial z_i \partial z_j(0))$, $2 \le i, j \le n-1$ is also invertible, i.e. it represents a non-degenerate quadratic form.

Diagonalizing this quadratic form, we can assume after a suitable linear change of coordinates that the diagonal coefficients of M' are equal to 2 and the others are zero. Therefore g is of the form $g(z_1, \dots, z_n) = z_1 z_n + \sum_{j=2}^{n-1} z_j^2 + \sum_{j=2}^{n-1} \alpha_j z_j z_n + O(z^3)$. Changing coordinates on X to replace z_j by $z_j + \frac{1}{2}\alpha_j z_n$ for all $2 \le j \le n-1$, and on $\mathbb{CP}^2$ to replace Z_1 by $Z_1 + \frac{1}{4}\sum \alpha_j^2 Z_2^2$, we can ensure that $g(z_1, \dots, z_n) = z_1 z_n + \sum_{j=2}^{n-1} z_j^2 + O(z^3)$.

Observe that R_k can be described near the origin by expressing the coordinates $z_2, \dots, z_n$ as functions of z_1. By assumption the expressions of $z_2, \dots, z_n$ are all of the form $O(z_1^2)$. Substituting into the formula for $\operatorname{Jac}(f_k)$, and letting $g_{ijk} = \partial^3 g/\partial z_i \partial z_j \partial z_k(0)$, we get that local equations of R_k near the origin are $z_j = -\frac{3}{2} g_{j11} z_1^2 + O(z_1^3)$ for $2 \le j \le n-1$, and $z_n = -3 g_{111} z_1^2 + O(z_1^3)$. It follows that $f_{k|R_k}$ is locally given in terms of z_1 by the map $z_1 \mapsto (-2 g_{111} z_1^3 + O(z_1^4), -3 g_{111} z_1^2 + O(z_1^3))$. Therefore, the transverse vanishing of $\partial(f_{k|R_k})$ at the origin implies that $g_{111} \neq 0$, so after a suitable rescaling we may assume that the coefficient of z_1^3 in the power series expansion of g is equal to one.

On the other hand, suitable coordinate changes can be used to kill all other degree 3 terms in the expansion of g : if $2 \le i \le n-1$ the coefficient of $z_i z_j z_k$ can be made zero by replacing z_i by $z_i + \frac{c}{2} z_j z_k$; similarly for z_n^3 (replace Z_1 by $Z_1 + cZ_2^3$), $z_1 z_n^2$ and $z_1^2 z_n$ (replace z_1 by $z_1 + cz_n^2 + c' z_1 z_n$). So we get that $f_k(z_1, \dots, z_n) = (z_1^3 + z_1 z_n + z_2^2 + \cdots + z_{n-1}^2 + O(z^4), z_n)$. It is then a standard result of singularity theory that the higher order terms can be absorbed by suitable coordinate changes (see e.g. [1]).

We now turn to the case of where x is a point of R_k which does not lie close to any of the cusp points. Conditions (2) and (3′) imply that the differential of f_k at x has real rank 2 and that its image lies close to a complex line in the tangent plane to $\mathbb{CP}^2$ at $f_k(x)$. Therefore, there exist local approximately holomorphic coordinates (Z_1, Z_2) on $\mathbb{CP}^2$ such that $\mathrm{Im}\, \nabla f_k(x)$ is the Z_2 axis. Moreover, because $Z_2 \circ f_k$ is an approximately holomorphic function whose derivative at x satisfies a uniform lower bound, it remains possible to find local approximately holomorphic coordinates $z_1, \dots, z_n$ on X such that $z_n = Z_2 \circ f_k$. As before, we can write $f_k(z_1, \dots, z_n) = (g(z_1, \dots, z_n), z_n)$, where g is an approximately holomorphic function such that $\nabla g(0) = 0$.

By assumption f_k restricts to R_k as an immersion at x, so the projection to the z_n axis of $T_x R_k$ is non-trivial. In fact, property (4) implies that, if $\partial(f_{k|R_k})$ is very small at x, then a cusp point lies nearby ; so we can assume that the z_n component of $T_x R_k$ is larger than some fixed constant. As a consequence, one can show that R_k is locally given by equations of the form $z_j = h_j(z_n)$, where the functions h_j are approximately holomorphic and have bounded derivatives. Therefore, a suitable change of coordinates on X makes it possible to assume that R_k is locally given by the equations $z_1 = \cdots = z_{n-1} = 0$. Similarly, a suitable approximately holomorphic change of coordinates on $\mathbb{CP}^2$ makes it possible to assume that $f_k(R_k)$ is locally given by the equation $Z_1 = 0$.

As a consequence, we have that $g_{|R_k} = 0$ and, since the image of ∇f_k at a point of R_k coincides with the tangent space to $f_k(R_k)$, ∇g vanishes at all points of R_k. In particular this implies that $\partial^2 g / \partial z_j \partial z_n(0) = 0$ for all $1 \le j \le n$. Moreover, property (3) implies that $\mathrm{Jac}(f_k)$ vanishes transversely at the origin, and therefore that the matrix $(\partial^2 g / \partial z_i \partial z_j(0))$, $1 \le i, j \le n-1$ is invertible, i.e. it represents a non-degenerate quadratic form. This quadratic form can be diagonalized by a suitable change of coordinates ; because the transversality property (3) is uniform, the coefficients are bounded between fixed constants. After a suitable rescaling, we can therefore assume that $\partial^2 g / \partial z_i \partial z_j(0)$ is equal to 2 if $i = j$ and 0 otherwise.

In conclusion, we get that $g(z_1, \dots, z_n) = z_1^2 + \cdots + z_{n-1}^2 + h(z_1, \dots, z_n)$, where h is the sum of a holomorphic function which vanishes up to order 3 at the origin and of a non-holomorphic function which vanishes up to order 2 at the origin and has derivatives bounded by $O(k^{-1/2})$.

Let z be the column vector $(z_1, \dots, z_{n-1})$, and denote by $\mathbf{z}$ the vector $(z_1, \dots, z_n)$. Using the fact that g vanishes up to order 2 along R_k, we conclude that there exist matrix-valued functions α, β and γ with the following properties :

(*a*) $g(\mathbf{z}) = {}^t z \alpha(\mathbf{z}) z + {}^t \bar{z} \beta(\mathbf{z}) z + {}^t \bar{z} \gamma(\mathbf{z}) \bar{z}$; (α and γ are symmetric) ;

(*b*) α is approximately holomorphic and has uniformly bounded derivatives ; $\alpha(0) = I$;

(*c*) β and γ and their derivatives are bounded by fixed multiples of $k^{-1/2}$.

The implicit function theorem then makes it possible to construct a C^∞ approximately holomorphic change of coordinates of the form $z \mapsto \lambda(\mathbf{z})z+\mu(\mathbf{z})\bar{z}$ (with $\lambda(0)$ orthogonal, λ approximately holomorphic, $\mu = O(k^{-1/2})$), such that g becomes of the form $g(\mathbf{z}) = {}^t zz + {}^t\bar{z}\tilde{\gamma}(\mathbf{z})\bar{z}$.

Unfortunately, smooth coordinate changes are not sufficient to further simplify this expression; instead, in order to obtain the desired local model one must use as coordinate change an "approximately holomorphic homeomorphism", which is smooth away from R_k but admits only directional derivatives at the points of R_k. More precisely, starting from $g = {}^t zz + h$ and using that $h/|z|^2$ is bounded by $O(k^{-1/2}) + O(\mathbf{z})$, we can write

$$g(\mathbf{z}) = \sum_{j=1}^{n-1} \tilde{z}_j^2, \qquad \tilde{z}_j = z_j\Big(1 + \frac{\bar{z}_j}{z_j}\frac{h(\mathbf{z})}{|z|^2}\Big)^{1/2}.$$

This gives the desired local model and ends the proof. □

Remark 4.1. The local model at points of R_k only holds topologically (up to an approximately holomorphic homeomorphism), which is not fully satisfactory. However, by replacing (3′) by a stronger condition, it is possible to obtain the same result in smooth approximately holomorphic coordinates. This new condition can be formulated as follows. Away from the cusp points, the complex lines $(\operatorname{Im}\partial f_k)^\perp$ define a line bundle $V \subset T\mathbb{CP}^2_{|D_k}$, everywhere transverse to TD_k. A neighborhood of the zero section in V can be sent via the exponential map of the Fubini-Study metric onto a neighborhood of D_k (away from the cusps), in such a way that each fiber V_x is mapped holomorphically to a subset $\mathcal{V}_x$ contained in a complex line in $\mathbb{CP}^2$.

Lifting back to a neighborhood of R_k in X, we can define slices $\mathcal{W}_x = f_k^{-1}(\mathcal{V}_{f_k(x)})$ for all $x \in R_k$ lying away from $\mathcal{C}_k$. It is then possible to identify a neighborhood of R_k (away from $\mathcal{C}_k$) with a neighborhood of the zero section in the vector bundle W whose fiber at $x \in R_k$ is $\operatorname{Ker}\partial f_k(x)$, in such a way that each fiber W_x gets mapped to $\mathcal{W}_x$. Observe moreover that, since W_x is a complex subspace in $(T_xX, \tilde{J}_k)$, W is endowed with a natural complex structure induced by $\tilde{J}_k$. It is then possible to ensure that the "exponential map" from W_x to $\mathcal{W}_x$ is approximately $\tilde{J}_k$-holomorphic for every x, and, using condition (4′), holomorphic when x lies at distance less than $\delta/2$ from a cusp point.

With this setup understood, and composing on both sides with the exponential maps, f_k induces a fiber-preserving map ψ_k between the bundles W and V ; this map is approximately holomorphic everywhere, and holomorphic at distance less than $\delta/2$ from $\mathcal{C}_k$. The condition which we impose as a replacement of (3′) is that ψ_k should be fiberwise holomorphic over a neighborhood of the zero section in W.

The proof of existence of quasiholomorphic maps satisfying this strengthened condition follows a standard argument : trivializing locally V and W for each value of k, and given asymptotically holomorphic maps ψ_k, Lemma 8 of [3] (see also [8]) implies the existence of a fiberwise holomorphic map $\tilde{\psi}_k$ differing from ψ_k by $O(k^{-1/2})$ over a neighborhood of the zero section. It is moreover easy to check that $\tilde{\psi}_k = \psi_k$ near the cusp points. So, in order to obtained the desired property, we introduce a smooth cut-off function and define a map $\hat{\psi}_k$ which equals $\tilde{\psi}_k$ near the zero section and coincides with ψ_k beyond a certain distance. Going back through the exponential maps, we obtain a map $\hat{f}_k$ which differs from f_k by $O(k^{-1/2})$ and coincides with f_k outside a small neighborhood of R_k and near the cusp points. The corresponding perturbations of the asymptotically holomorphic sections $s_k \in \Gamma(\mathbb{C}^3 \otimes L^{\otimes k})$ are easy to construct. Moreover, we can always assume that $\tilde{\psi}_k$ and ψ_k coincide at order 1 along the zero section, i.e. that $\hat{f}_k$ and f_k coincide up to order 1 along the branch curve ; therefore, the branch curve of $\hat{f}_k$ and its image are the same as for f_k, and so all properties of Definition 3.2 hold for $\hat{f}_k$.

Once this condition is satisfied, getting the correct local model at a point $x \in R_k$ in smooth approximately holomorphic coordinates is an easy task. Namely, we can define, near $f_k(x)$, local approximately holomorphic coordinates Z_2 on D_k and Z_1 on the fibers of V (Z_1 is a complex linear function on each fiber, depending approximately holomorphically on Z_2). Using the exponential map, we can use (Z_1, Z_2) as local coordinates on $\mathbb{CP}^2$. Lifting Z_2 via $\hat{f}_k$ yields a local coordinate z_n on R_k near x. Moreover, we can locally define complex linear coordinates $z_1, \dots, z_{n-1}$ in the fibers of W, depending approximately holomorphically on z_n. Using again the exponential map, $(z_1, \dots, z_n)$ define local approximately holomorphic coordinates on X. Then, by construction, local equations are $z_1 = \dots = z_{n-1} = 0$ for R_k and $Z_1 = 0$ for D_k, and f_k is given by $f_k(z_1, \dots, z_n) = (\psi_k(z_1, \dots, z_n), z_n)$. Moreover, we know that ψ_k is, for each value of z_n, a holomorphic function of $z_1, \dots, z_{n-1}$, vanishing up to order 2 at the origin. We can then use the argument in the proof of Proposition 4.2 to obtain the expected local model in smooth approximately holomorphic coordinates.

4.3. **Monodromy invariants of quasiholomorphic maps.** We now look at the monodromy invariants naturally arising from quasiholomorphic maps to $\mathbb{CP}^2$. Let $f : X - Z \to \mathbb{CP}^2$ be one of the maps constructed in Theorem 3.1 for large enough k. The fibers of f are singular along the smooth symplectic curve $R \subset X$, whose image in $\mathbb{CP}^2$ is a symplectic braided curve. Therefore, we obtain a first interesting invariant by considering the critical curve $D \subset \mathbb{CP}^2$.

As in the four-dimensional case, using the projection $\pi : \mathbb{CP}^2 - \{(0\!:\!0\!:\!1)\} \to \mathbb{CP}^1$ we can describe the topology of D by a braid monodromy map

$$\rho_n : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d, \tag{6}$$

where $p_1, \dots, p_r$ are the images by π of the cusps, nodes and tangency points of D, and $d = \deg D$. Alternately, we can also express this monodromy as a braid group factorization

$$\Delta^2 = \prod_{j=1}^{r} Q_j^{-1} X_1^{r_j} Q_j. \tag{7}$$

Like in the four-dimensional case, this braid factorization completely characterizes the curve D up to isotopy, but it is only well-defined up to simultaneous conjugation and Hurwitz equivalence.

We now turn to the second part of the problem, namely describing the topology of the map $f : X - Z \to \mathbb{CP}^2$ itself. As in the case of Lefschetz pencils, we blow up X along Z in order to obtain a well-defined map $\hat{f} : \hat{X} \to \mathbb{CP}^2$. The fibers of $\hat{f}$ are naturally identified with those of f, made mutually disjoint by the blow-up process.

Denote by Σ^{2n-4} the generic fiber, i.e. the fiber above a point of $\mathbb{CP}^2 - D$. The structure of the singular fibers of $\hat{f}$ can be easily understood by looking at the local models obtained in Proposition 4.2. The easiest case is that of the fiber above a smooth point of D. This fiber intersects R transversely in one point, where the local model is $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$, which can be thought of as a one-parameter version of the model map for the singularities of a Lefschetz pencil in dimension $2n-2$. Therefore, as in that case, the singular fiber is obtained by collapsing a vanishing cycle, namely a Lagrangian sphere S^{n-2}, in the generic fiber Σ, and the monodromy of $\hat{f}$ maps a small loop around D to a positive Dehn twist along the vanishing cycle.

The fiber of $\hat{f}$ above a nodal point of D intersects R transversely in two points, and is similarly obtained from Σ by collapsing two disjoint Lagrangian spheres. In fact, the nodal point does not give rise to any specific local model in X, as it simply corresponds to the situation where two points of R happen to lie in the same fiber.

Finally, in the case of a cusp point of D, the local model $(z_1, \dots, z_n) \mapsto (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$ can be used to show that the singular fiber is a "fishtail" fiber, obtained by collapsing two Lagrangian spheres which intersect transversely in one point.

With this understood, the topology of $\hat{f}$ is described by its monodromy around the singular fibers. As in the case of Lefschetz fibrations, the monodromy consists of symplectic automorphisms of Σ preserving the submanifold Z. However, as in §2, defining a monodromy map with values in $\mathrm{Map}^\omega(\Sigma, Z)$ requires a trivialization of the normal bundle of Z, which is only possible over an affine subset $\mathbb{C}^2 \subset \mathbb{CP}^2$. So, the monodromy of $\hat{f}$ is described by a group homomorphism

$$\psi_n : \pi_1(\mathbb{C}^2 - D) \to \mathrm{Map}^\omega(\Sigma, Z). \tag{8}$$

A simpler description can be obtained by restricting oneself to a generic line $L \subset \mathbb{CP}^2$ which intersects D transversely in d points $q_1, \dots, q_d$. In fact, Definition 3.2 implies that we can use the fiber of π above $(0\!:\!1)$ for this purpose. As in §3.2, the inclusion $i : \mathbb{C} - \{q_1, \dots, q_d\} \to \mathbb{C}^2 - D$ induces a surjective homomorphism on fundamental groups. The relations between the geometric generators $\gamma_1, \dots, \gamma_d$ of $\pi_1(\mathbb{C}^2 - D)$ are again given by the braid factorization (one relation for each factor) in the same manner as in §3.2. Note that the relation $\gamma_1 \dots \gamma_d = 1$ only holds in $\pi_1(\mathbb{CP}^2 - D)$, not in $\pi_1(\mathbb{C}^2 - D)$.

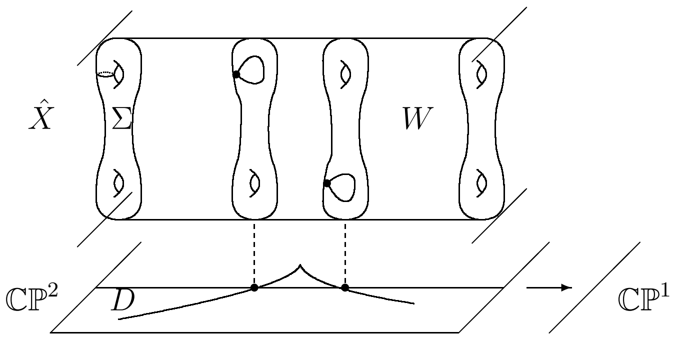
It follows from these observations that the monodromy of $\hat{f}$ can be described by the monodromy morphism

$$\theta_{n-1} : \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to \mathrm{Map}^\omega(\Sigma, Z) \tag{9}$$

defined by $\theta_{n-1} = \psi_n \circ i_*$. We know from the above discussion on the structure of $\hat{f}$ near its critical points that θ_{n-1} maps the geometric generators of $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$ to positive Dehn twists. Moreover, by considering the normal bundle to the exceptional divisor in $\hat{X}$ one easily checks that the monodromy around infinity is again a twist along Z in Σ, i.e. $\theta_{n-1}(\gamma_1 \dots \gamma_d) = \delta_Z$.

These properties of θ_{n-1} are strikingly similar to those of the monodromy of a symplectic Lefschetz pencil. In fact, let $W = f^{-1}(L)$ be the preimage of a complex line $L = \mathbb{CP}^1 \subset \mathbb{CP}^2$ intersecting D transversely. Then the restriction of f to the smooth symplectic hypersurface $W \subset X$ endows it with a structure of symplectic Lefschetz pencil with generic fiber Σ and base set Z ; for example, if one chooses $L = \pi^{-1}(0:1)$, then W is the zero set of s_k^0 and the restricted pencil $f_{|W} : W - Z \to \mathbb{CP}^1$ is defined by the two sections s_k^1 and s_k^2. The monodromy of the restricted pencil is, by construction, given by the map θ_{n-1}.

The situation is summarized in the following picture :



Remark 4.2. If a cusp point of D happens to lie close to the chosen line L, then two singular points of the restricted pencil $f_{|W}$ lie close to each other. This is not a problem here, but in general if we want to avoid this situation we need to impose one additional transversality condition on f. Namely, we must require the uniform transversality to 0 of $\partial(f_{|W})$, which is easily obtained by imitating Donaldson's argument from [10]. Another situation in which this property naturally becomes satisfied is the one described in §5.

Given a braided curve $D \subset \mathbb{CP}^2$ of degree d described by a braid factorization as in (7), and given a monodromy map θ_{n-1} as in (9), certain compatibility conditions need to hold between them in order to ensure the existence of a $\mathbb{CP}^2$-valued map with critical curve D and monodromy θ_{n-1}. Namely, θ_{n-1} must factor through $\pi_1(\mathbb{C}^2 - D)$, and the fibration must behave in accordance with the expected models near the special points of D. We introduce the following definition summarizing these compatibility properties :

Definition 4.1. A geometric $(n-1)$-dimensional monodromy representation associated to a braided curve $D \subset \mathbb{CP}^2$ is a surjective group homomorphism θ_{n-1} from the free group $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) = F_d$ to a symplectic mapping class group $\mathrm{Map}^\omega(\Sigma^{2n-4}, Z^{2n-6})$, mapping the geometric generators γ_i (and thus also the $\gamma_i * Q_j$) to positive Dehn twists and such that

$\theta_{n-1}(\gamma_1 \dots \gamma_d) = \delta_Z$,

$\theta_{n-1}(\gamma_1 * Q_j) = \theta_{n-1}(\gamma_2 * Q_j)$ if $r_j = 1$,

$\theta_{n-1}(\gamma_1 * Q_j)$ and $\theta_{n-1}(\gamma_2 * Q_j)$ are twists along disjoint Lagrangian spheres if $r_j = \pm 2$,

$\theta_{n-1}(\gamma_1 * Q_j)$ and $\theta_{n-1}(\gamma_2 * Q_j)$ are twists along Lagrangian spheres transversely intersecting in one point if $r_j = 3$.

As in the four-dimensional case, θ_{n-1} remains unchanged and the compatibility conditions are preserved when the braid factorization defining D is affected by a Hurwitz move. However, when all factors in the braid factorization are simultaneously conjugated by a certain braid $Q \in B_d$, the system of geometric generators $\gamma_1, \dots, \gamma_d$ changes accordingly, and so the geometric monodromy representation θ_{n-1} should be replaced by $\theta_{n-1} \circ Q_*$, where Q_* is the automorphism of F_d induced by the braid Q. For example, conjugating the braid factorization by one of the generating half-twists in B_d affects the monodromy θ_{n-1} of the restricted pencil by a Hurwitz move.

One easily checks that, given a symplectic braided curve $D \subset \mathbb{CP}^2$ and a compatible monodromy representation $\theta_{n-1} : F_d \to \mathrm{Map}^\omega(\Sigma, Z)$, it is possible to recover a compact $2n$-manifold X and a map $f : X - Z \to \mathbb{CP}^2$ in a canonical way up to smooth isotopy. Moreover, it is actually possible to endow X with a symplectic structure, canonically up to symplectic isotopy. Indeed, by first applying Theorem 2.2 to the monodromy map θ_{n-1} we can recover a canonical symplectic structure on the total space W of the restricted Lefschetz pencil ; furthermore, as will be shown in §4.4 below, the braid monodromy of D and the compatible monodromy representation θ_{n-1} determine on X a structure of Lefschetz pencil with generic fiber W and base set Σ, which implies by a second application of Theorem 2.2 that X carries a canonical symplectic structure. The same result can also be obtained more directly, by adapting the statement and proof of Theorem 2.2 to the case of $\mathbb{CP}^2$-valued maps.

As in the four-dimensional case, we can naturally define symplectic invariants arising from the quasiholomorphic maps constructed in Theorem 4.1. However, we again need to take into account the possible presence of negative self-intersections in the critical curves of these maps. Therefore, the braid factorizations we obtain are only canonical up to global conjugation, Hurwitz equivalence, and pair cancellations or creations. As in the four-dimensional case, a pair creation operation (inserting two mutually inverse factors anywhere in the braid factorization) is only allowed if the new factorization remains compatible with the monodromy representation θ_{n-1}, i.e. if θ_{n-1} maps the two corresponding geometric generators to Dehn twists along disjoint Lagrangian spheres.

With this understood, we can introduce a notion of m-equivalence as in Definition 3.5. The following result then holds :

Theorem 4.3. *The braid factorizations and geometric monodromy representations associated to the quasiholomorphic maps to $\mathbb{CP}^2$ obtained in Theorem 4.1 are, for $k \gg 0$, canonical up to m-equivalence (up to a choice of line bundle L when the cohomology class $[\omega]$ is not integral), and define symplectic invariants of (X^{2n}, ω).*

Conversely, the data consisting of a braid factorization and a geometric $(n-1)$-dimensional monodromy representation, or a m-equivalence class of such data, determines a symplectic $2n$-manifold in a canonical way up to symplectomorphism.

Remark 4.3. The invariants studied in this section are a very natural generalization of those defined in §3.2 for 4-manifolds. Namely, when $\dim X = 4$, we naturally get that $Z = \emptyset$ and $\dim \Sigma = 0$, i.e. the generic fiber Σ consists of a finite number of points, as expected for a branched covering map. In particular, the mapping class group $\mathrm{Map}(\Sigma)$ of the 0-manifold Σ is in fact the symmetric group of order $\mathrm{card}(\Sigma)$. Finally, a Lagrangian 0-sphere in Σ is just a pair of points of Σ, and the associated Dehn twist is simply the corresponding transposition. With this correspondence, the results of §3 are the exact four-dimensional counterparts of those described here.

4.4. **Quasiholomorphic maps and symplectic Lefschetz pencils.** Consider again a symplectic manifold (X^{2n}, ω) and let $f : X - Z \to \mathbb{CP}^2$ be a map with the same topological properties as those obtained by Theorem 4.1 from sections of $L^{\otimes k}$ for k large enough. As in the four-dimensional case, the $\mathbb{CP}^1$-valued map $\pi \circ f$

defines a Lefschetz pencil structure on X, obtained by lifting via f a pencil of lines on $\mathbb{CP}^2$. The base set of this pencil is the fiber of f above the pole $(0:0:1)$ of the projection π.

In fact, starting from the quasiholomorphic maps f_k given by Theorem 4.1, the symplectic Lefschetz pencils $\pi \circ f_k$ coincide for $k \gg 0$ with those obtained by Donaldson in [10] and described in §2 ; calling s_k^0, s_k^1, s_k^2 the sections of $L^{\otimes k}$ defining f_k, the Lefschetz pencil $\pi \circ f_k$ is the one induced by the sections s_k^0 and s_k^1.

Therefore, as in the case of a 4-manifold, the invariants described in §4.3 (braid factorization and $(n-1)$-dimensional geometric monodromy representation) completely determine those discussed in §2 (factorizations in mapping class groups). Once again, the topological description of the relation between quasiholomorphic maps and Lefschetz pencils involves a subgroup of θ_{n-1}-*liftable braids* in the braid group, and a group homomorphism from this subgroup to a mapping class group.

Consider a symplectic braided curve $D \subset \mathbb{CP}^2$, described by its braid monodromy $\rho_n : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B_d$, and a compatible $(n-1)$-dimensional monodromy representation $\theta_{n-1} : F_d = \pi_1(\mathbb{C} - \{q_1, \dots, q_d\}) \to \mathrm{Map}^\omega(\Sigma^{2n-4}, Z^{2n-6})$. Then we can make the following definition :

Definition 4.2. The subgroup $B_d^0(\theta_{n-1})$ of liftable braids is the set of all braids $Q \in B_d$ such that $\theta_{n-1} \circ Q_* = \theta_{n-1}$, where $Q_* \in \mathrm{Aut}(F_d)$ is the automorphism induced by the braid Q on $\pi_1(\mathbb{C} - \{q_1, \dots, q_d\})$.

A topological definition of $B_d^0(\theta_{n-1})$ can also be given in terms of universal fibrations and coverings of configuration spaces, similarly to the description in §3.3.

More importantly, denote by W the total space of the symplectic Lefschetz pencil $LP(\theta_{n-1})$ with generic fiber Σ and monodromy θ_{n-1}. For example, if ρ_n and θ_{n-1} are the monodromy morphisms associated to a quasiholomorphic map given by sections s_k^0, s_k^1, s_k^2 of $L^{\otimes k}$ over X, then W is the smooth symplectic hypersurface in X given by the equation $s_k^0 = 0$; indeed, as seen in §4.3, this hypersurface carries a Lefschetz pencil structure with generic fiber Σ, induced by s_k^1 and s_k^2, and the monodromy of this restricted pencil is precisely θ_{n-1}. A braid $Q \in B_d$ can be viewed as a motion of the critical set $\{q_1, \dots, q_d\}$ of the Lefschetz pencil $LP(\theta_{n-1})$; after this motion we obtain a new Lefschetz pencil with monodromy $\theta_{n-1} \circ Q_*$. So the subgroup $B_d^0(\theta_{n-1})$ precisely consists of those braids which preserve the monodromy of the Lefschetz pencil $LP(\theta_{n-1})$.

Viewing braids as compactly supported symplectomorphisms of the plane preserving $\{q_1, \dots, q_d\}$, the fact that Q belongs to $B_d^0(\theta_{n-1})$ means that it can be lifted via the Lefschetz pencil map $W - Z \to \mathbb{CP}^1$ to a symplectomorphism of W. Since the monodromy of the pencil $LP(\theta_{n-1})$ preserves a neighborhood of the base set Z, the lift to W of the braid Q coincides with the identity over a neighborhood of Z. Even better, because Q is compactly supported, its lift to W coincides with Id near the fiber above the point at infinity in $\mathbb{CP}^1$, which can be identified with Σ. Therefore, the lift of Q to W is a well-defined element of the mapping class group $\mathrm{Map}^\omega(W, \Sigma)$, which we call $(\theta_{n-1})_*(Q)$. This construction defines a group homomorphism

$$(\theta_{n-1})_* : B_d^0(\theta_{n-1}) \to \mathrm{Map}^\omega(W^{2n-2}, \Sigma^{2n-4}).$$

Since the geometric monodromy representation θ_{n-1} is compatible with the braided curve $D \subset \mathbb{CP}^2$, the image of the braid monodromy homomorphism $\rho_n : \pi_1(\mathbb{C} -$

$\{p_1, \dots, p_r\}) \to B_d$ describing D is entirely contained in $B^0_d(\theta_{n-1})$. Indeed, it follows from Definition 4.1 that θ_{n-1} factors through $\pi_1(\mathbb{C}^2 - D)$, on which the braids of $\operatorname{Im}\rho_n$ act trivially. As a consequence, we can use the group homomorphism $(\theta_{n-1})_*$ in order to obtain, from the braid monodromy ρ_n, a group homomorphism

$$\theta_n = (\theta_{n-1})_* \circ \rho_n : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to \mathrm{Map}^\omega(W, \Sigma).$$

If ρ_n and θ_{n-1} describe the monodromy of a $\mathbb{CP}^2$-valued map f, then θ_n is by construction the monodromy of the corresponding Lefschetz pencil $\pi \circ f$. Therefore, the following result holds :

Proposition 4.4. *Let $f : X - Z \to \mathbb{CP}^2$ be one of the quasiholomorphic maps of Theorem 4.1. Let $D \subset \mathbb{CP}^2$ be its critical curve, and denote by $\rho_n : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to B^0_d(\theta_{n-1})$ and $\theta : F_d \to \mathrm{Map}^\omega(\Sigma, Z)$ be the corresponding monodromies. Then the monodromy map $\theta_n : \pi_1(\mathbb{C} - \{p_1, \dots, p_r\}) \to \mathrm{Map}^\omega(W, \Sigma)$ of the Lefschetz pencil $\pi \circ f$ is given by the identity $\theta_n = (\theta_{n-1})_* \circ \rho_n$.*

In particular, for $k \gg 0$ the symplectic invariants given by Theorem 2.1 are obtained in this manner from those defined in Theorem 4.3.

As in the four-dimensional case, all the factors of degree ± 2 or 3 in the braid monodromy (corresponding to the cusps and nodes of D) lie in the kernel of $(\theta_{n-1})_*$; the only terms which contribute non-trivially to the pencil monodromy θ_n are those arising from the tangency points of the branch curve D, and each of these contributions is a Dehn twist.

More precisely, the image in $\mathrm{Map}^\omega(W, \Sigma)$ of a half-twist $Q \in B^0_d(\theta_{n-1})$ arising as the braid monodromy around a tangency point of D can be constructed as follows. Consider the Lefschetz pencil $LP(\theta_{n-1})$ with total space W, generic fiber Σ, critical levels $q_1, \dots, q_d$ and monodromy θ_{n-1}. Call γ the path joining two of the points $q_1, \dots, q_d$ (e.g., q_{i_1} and q_{i_2}) and naturally associated to the half-twist Q (the path along which the twisting occurs). By Definition 4.1, the monodromies of $LP(\theta_{n-1})$ around the two end points q_{i_1} and q_{i_2} are the same Dehn twists (using γ to identify the two singular fibers). Even better, in this context one easily shows that the vanishing cycles at the two end points of γ are isotopic Lagrangian spheres in Σ. Then it follows from the work of Donaldson and Seidel that, above the path γ, one can find a Lagrangian sphere $L = S^{n-1} \subset W$, joining the singular points of the fibers above q_{i_1} and q_{i_2}, and intersecting each fiber inbetween in a Lagrangian sphere S^{n-2} (there is in fact a hidden subtlety in the argument, but working on pencils rather than fibrations it can be seen that the isotopy of the two vanishing cycles is sufficient). The element $(\theta_{n-1})_*(Q)$ in $\mathrm{Map}^\omega(W, \Sigma)$ is the positive Dehn twist along the Lagrangian sphere L.

Remark 4.4. Let (X^{2n}, ω) be a compact symplectic manifold, and consider the symplectic Lefschetz pencils given by Donaldson's result (Theorem 2.1) from pairs of sections of $L^{\otimes k}$ for $k \gg 0$; the monodromy of these Lefschetz pencils consists of generalized Dehn twists around Lagrangian $(n-1)$-spheres in the generic fiber W_k. It follows from Proposition 4.4 that these Lagrangian spheres are not arbitrary. Indeed, they can all be obtained by endowing W_k with a structure of symplectic Lefschetz pencil induced by two sections of $L^{\otimes k}$ (the existence of such a structure follows from the results of this section), and by looking for Lagrangian $(n-1)$-spheres which join two mutually isotopic vanishing cycles of this pencil above a path in the base.

As observed by Seidel, this remarkable structure of vanishing cycles makes it possible to hope for a purely combinatorial description of Lagrangian Floer homology, at least for Lagrangian spheres : one can try to use the structure of vanishing cycles in a $2n$-dimensional Lefschetz pencil to reduce things first to the $2n-2$-dimensional case, and then by induction eventually to the case of 0-manifolds, in which the calculations are purely combinatorial.

5. Complete linear systems and dimensional induction

We now show how the results of §4 can be used in order to reduce in principle the classification of compact symplectic manifolds to a purely combinatorial problem.

The idea behind this approach is to consider a linear system of rank greater than 3, using partial monodromy data to define invariants which allow a dimensional reduction process. This strategy is somewhat complementary to the result obtained by Gompf in [11], showing that the total space of a "hyperpencil" (a rank $n-1$ linear system) carries a canonical symplectic structure.

Definition 5.1. Let (X^{2n},ω) be a compact symplectic manifold. We say that asymptotically holomorphic $(n+1)$-tuples of sections of $L^{\otimes k}$ define *braiding complete linear systems* on X if, for large values of k, these sections $s_0,\dots,s_n \in \Gamma(L^{\otimes k})$ satisfy the following properties :

(a) for $0 \le r \le n-1$, the section $(s_{r+1},\dots,s_n)$ of $\mathbb{C}^{n-r}\otimes L^{\otimes k}$ satisfies a uniform transverslity property, and its zero set $\Sigma_r = \{s_{r+1}=\cdots=s_n=0\}$ is a smooth symplectic submanifold of dimension $2r$ in X. We also define $\Sigma_n = X$ and $\Sigma_{-1}=\emptyset$;

(b) for $1\le r\le n$, the pair of sections $(s_r,s_{r-1})\in\Gamma(\mathbb{C}^2\otimes L^{\otimes k})$ defines a structure of symplectic Lefschetz pencil on Σ_r, with generic fiber Σ_{r-1} and base set Σ_{r-2} ;

(c) for $2\le r\le n$, the triple of sections $(s_r,s_{r-1},s_{r-2})\in\Gamma(\mathbb{C}^3\otimes L^{\otimes k})$ defines a quasiholomorphic map from Σ_r to $\mathbb{CP}^2$, with generic fiber Σ_{r-2} and base set Σ_{r-3}.

One can think of a braiding complete linear system in the following way. First, the two sections s_n and s_{n-1} define a Lefschetz pencil structure on X. By adding the section s_{n-2}, this structure is refined into a quasiholomorphic map to $\mathbb{CP}^2$. As observed in §4, by restricting to the hypersurface Σ_{n-1} we get a symplectic Lefschetz pencil defined by s_{n-1} and s_{n-2}. This structure is in turn refined into a quasiholomorphic map by adding the section s_{n-3} ; and so on.

Note that, except for the case $r=1$, part (b) of Definition 5.1 is actually an immediate consequence of part (c), because by composing $\mathbb{CP}^2$-valued quasiholomorphic maps with the projection $\pi:\mathbb{CP}^2-\{(0\!:\!0\!:\!1)\}\to\mathbb{CP}^1$ one always obtains Lefschetz pencils. Also note that, in order to make sense out of these properties, one implicitly needs to endow the submanifolds Σ_r with ω-compatible almost-complex structures ; these restricted almost-complex structures can be chosen to differ from the almost-complex structure J on X by $O(k^{-1/2})$, so that asymptotic holomorphicity and transversality properties are not affected by this choice.

Theorem 5.1. *Let (X^{2n},ω) be a compact symplectic manifold. Then for all large enough values of k it is possible to find asymptotically holomorphic sections of $\mathbb{C}^{n+1}\otimes L^{\otimes k}$ determining braiding complete linear systems on X. Moreover, for large k these structures are canonical up to isotopy and up to cancellations of pairs of nodes in the critical curves of the quasiholomorphic $\mathbb{CP}^2$-valued maps.*

Proof. We only give a sketch of the proof of Theorem 5.1. As usual, we need to obtain two types of properties : uniform transversality conditions, which we ensure in the first part of the argument, and compatibility conditions, which are obtained by a subsequent perturbation. As in previous arguments, the various uniform transversality properties are obtained successively, using the fact that, because transversality is an open condition, it is preserved by any sufficiently small subsequent perturbations.

The first transversality properties to be obtained are those appearing in part (a) of Definition 5.1, i.e. the transversality to 0 of $(s_{r+1},\dots,s_n)$ for all $0 \le r \le n-1$; this easy case is e.g. covered by the main result of [2].

One next turns to the transversality conditions arising from the requirement that the three sections (s_n, s_{n-1}, s_{n-2}) define quasiholomorphic maps from X to $\mathbb{CP}^2$: it follows immediately from the proof of Theorem 4.1 that these properties can be obtained by suitable small perturbations.

Next, we try to modify s_{n-1}, s_{n-2} and s_{n-3} in order to ensure that the restrictions to $\Sigma_{n-1} = s_n^{-1}(0)$ of these three sections satisfy the transversality properties of Definition 3.2. A general strategy to handle this kind of situation is to use the following remark (Lemma 6 of [3]) : if ϕ is a section of a vector bundle $\mathcal{F}$ over X, satisfying a uniform transversality property, and if $W = \phi^{-1}(0)$, then the uniform transversality to 0 over W of a section ξ of a vector bundle $\mathcal{E}$ is equivalent to the uniform transversality to 0 over X of the section $\xi \oplus \phi$ of $\mathcal{E} \oplus \mathcal{F}$, up to a change in transversality estimates. This makes it possible to replace all transversality properties to be satisfied over submanifolds of X by transversality properties to be satisfied over X itself ; each property can then be ensured by the standard type of argument, using the globalization principle to combine suitably chosen local perturbations (see [4] for more details).

However, in our case the situation is significantly simplified by the fact that, no matter how we perturb the sections s_{n-1}, s_{n-2} and s_{n-3}, the submanifold Σ_{n-1} itself is not affected. Moreover, the geometry of Σ_{n-1} is controlled by the transversality properties obtained on s_n ; for example, a suitable choice of the constant $\rho > 0$ (independent of k) ensures that the intersection of Σ_{n-1} with any ball of g_k-radius ρ centered at one of its points is topologically a ball (see e.g. Lemma 4 of [2]). Therefore, we can actually imitate all steps of the argument used to prove Theorem 4.1, working with sections of $L^{\otimes k}$ over Σ_{n-1}. The localized reference sections of $L^{\otimes k}$ over Σ_{n-1} that we use in the arguments are now chosen to be the restrictions to Σ_{n-1} of the localized sections $s^{\mathrm{ref}}_{k,x}$ of $L^{\otimes k}$ over X ; similarly, the approximately holomorphic local coordinates over Σ_{n-1} in which we work are obtained as the restrictions to Σ_{n-1} of local coordinate functions on X. With these two differences understood, we can still construct localized perturbations by the same algorithms as in §4.1 and, using the standard globalization argument, achieve the desired transversality properties over Σ_{n-1}. Moreover, all these local perturbations are obtained as products of the localized reference sections by polynomial functions of the local coordinates. Therefore, they naturally arise as restrictions to Σ_{n-1} of localized sections of $L^{\otimes k}$ over X, and so we actually obtain well-defined perturbations of the sections s_{n-1}, s_{n-2} and s_{n-3} over X which yield the desired transversality properties over Σ_{n-1}.

We can continue similarly by induction on the dimension, until we obtain the transversality properties required of s_2, s_1 and s_0 over Σ_2, and finally the transversality properties required of s_1 and s_0 over Σ_1. Observe that, even though the perturbations performed over each Σ_r result in modifications of the submanifolds Σ_j $(j < r)$ lying inside them, these perturbations preserve the transversality properties of $(s_{j+1}, \dots, s_n)$, and so the submanifolds Σ_j retain their smoothness and symplecticity properties.

We now turn to the second part of the argument, i.e. obtaining the desired compatibility conditions. First observe that the proof of Theorem 4.1 shows how, by a perturbation of s_n, s_{n-1} and s_{n-2} smaller than $O(k^{-1/2})$, we can ensure that the various compatibility properties of Definition 3.2 are satisfied by the $\mathbb{CP}^2$-valued map f_n defined by these three sections.

Next, we proceed to perturb $f_{n-1} = (s_{n-1} : s_{n-2} : s_{n-3})$ over a neighborhood of its ramification curve $R_{n-1} \subset \Sigma_{n-1}$, in order to obtain the required compatibility properties for f_{n-1}, but without losing those previously achieved for f_n near its ramification curve $R_n \subset X$. For this purpose, we first show that the curve R_n satisfies a uniform transversality property with respect to the hypersurface Σ_{n-1} in X.

The only way in which R_n can fail to be uniformly transverse to Σ_{n-1} is if $\partial(\pi \circ f_{n|R_n})$ becomes small at a point of R_n near Σ_{n-1}. Because f_n satisfies property (6) in Definition 3.2, this can only happen if a cusp point or a tangency point of f_n lies close to Σ_{n-1}. However, property (7) of Definition 3.2 implies that this point cannot belong to Σ_{n-1}. Therefore, two of the intersection points of R_n with Σ_{n-1} must lie close to each other. Observe that the points of $R_n \cap \Sigma_{n-1}$ are precisely the critical points of the Lefschetz pencil induced on Σ_{n-1} by s_{n-1} and s_{n-2}, i.e. the tangency points of the map f_{n-1}. The transversality properties already obtained for f_{n-1} imply that two tangency points cannot lie close to each other ; we get a contradiction, so the cusps and tangencies of f_n must lie far away from Σ_{n-1}, and R_n and Σ_{n-1} are mutually transverse.

This implies in particular that a small perturbation of s_{n-1}, s_{n-2} and s_{n-3} localized near Σ_{n-1} cannot affect properties (4′) and (6′) for f_n, and also that the only place where perturbing f_{n-1} might affect f_n is near the tangency points of f_{n-1}.

We now consider the set $\mathcal{C}_{n-1} \cup \mathcal{T}_{n-1} \cup \mathcal{I}_{n-1}$ of points where we need to ensure properties (4′), (6′) and (8′) for f_{n-1}. The first step is as usual to perturb J into an almost-complex structure which is integrable near these points ; once this is done, we perturb f_{n-1} to make it locally holomorphic with respect to this almost-complex structure.

We start by considering a point $x \in \mathcal{C}_{n-1} \cup \mathcal{I}_{n-1}$, where the issue of preserving properties of f_n does not arise. We follow the argument in §4.1 of [3]. First, it is possible to perturb the almost-complex structure J over a neighborhood of x in X in order to obtain an almost-complex structure $\tilde{J}$ which differs from J by $O(k^{-1/2})$ and is integrable over a small ball centered at x. Recall from [3] that $\tilde{J}$ is obtained by choosing approximately holomorphic coordinates on X and using them to pull back the standard complex structure of $\mathbb{C}^n$; a cut-off function is used to splice J with this locally defined integrable structure. Since we can choose the local coordinates in such a way that a local equation of Σ_{n-1} is $z_n = 0$, we can easily ensure that Σ_{n-1} is, over a small neighborhood of x, a $\tilde{J}$-holomorphic submanifold of X. Next, we can perturb the sections $s_{n-1}, s_{n-2}, s_{n-3}$ of $L^{\otimes k}$ by $O(k^{-1/2})$ in order to make the projective map defined by them $\tilde{J}$-holomorphic over a neighborhood of x in X (see [3]). This holomorphicity property remains true for the restrictions to the locally $\tilde{J}$-holomorphic submanifold Σ_{n-1}. So, we have obtained the desired compatibility property near x.

We now consider the case of a point $x \in \mathcal{T}_{n-1}$, where we need to obtain property (6′) for f_{n-1} while preserving property (8′) for f_n. We first observe that, by the construction of the previous step (getting property (8′) for f_n at x), we have a readily available almost-complex structure $\tilde{J}$ integrable over a neighborhood of x in X. In particular, by construction f_n is locally $\tilde{J}$-holomorphic and Σ_{n-1} is locally a $\tilde{J}$-holomorphic submanifold of X. We next try to make the projective map f_{n-1} holomorphic over a neighborhood of x, using once again the argument of [3]. The key observation here is that, because one of the sections s_{n-1} and s_{n-2} is bounded from below at x, we can reduce to a $\mathbb{C}^2$-valued map whose first component is already holomorphic. Therefore, the perturbation process described in [3] only affects s_{n-3}, while the two other sections are preserved. This means that we can ensure the local $\tilde{J}$-holomorphicity of f_{n-1} without affecting f_n.

It is easy to combine the various localized perturbations performed near each point of $\mathcal{C}_{n-1} \cup \mathcal{T}_{n-1} \cup \mathcal{I}_{n-1}$; this yields properties (4′), (6′) and (8′) of Definition 3.2 for f_{n-1}.

We now use a generically chosen small perturbation of s_{n-1}, s_{n-2} and s_{n-3} in order to ensure property (7), i.e. the self-transversality of the critical curve of f_{n-1}. It is important to observe that, because f_n satisfies property (7), the images by the projective map $(s_{n-1} : s_{n-2})$ of the points of $R_n \cap \Sigma_{n-1} = \mathcal{I}_n = \mathcal{T}_{n-1}$ are all distinct from each other, and because f_n satisfies property (5) they are also distinct from $(0 : 1)$. Therefore, we can choose a perturbation which vanishes identically over a neighborhood of $\mathcal{T}_{n-1}$; this makes it possible to obtain property (7) for f_{n-1} without losing any property of f_n.

Finally, by the process described in §4.2 of [3] we construct a perturbation yielding property (3′) along the critical curve of f_{n-1} ; this perturbation is originally defined only for the restrictions to Σ_{n-1} but it can easily be extended outside of Σ_{n-1} by using a cut-off function. The two important properties of this perturbation are the following : first, it vanishes identically near the points where f_{n-1} has already been made $\tilde{J}$-holomorphic, and in particular near the points of $\mathcal{T}_{n-1}$; therefore, none of the properties of f_n are affected, and properties (4′), (6′) and (8′) of f_{n-1} are not affected either. Secondly, this perturbation does not modify the critical curve of f_{n-1} nor its image, so property (7) is preserved. We have therefore obtained all desired properties for f_{n-1}.

We can continue similarly by induction on the dimension, until all required compatibility properties are satisfied. Observe that, because the ramification curve of f_r remains away from its fiber at infinity Σ_{r-2}, we do not need to worry about the possible effects on f_r of perturbations of f_{r-2}. Therefore, the argument remains the same at each step, and we can complete the proof of the existence statement in Theorem 5.1 in this way.

The proof of the uniqueness statement relies, as usual, on the extension of the whole construction to one-parameter families ; this is easily done by following the same ideas as in previous arguments. □

The structures of braiding complete linear systems given by Theorem 5.1 are extremely rich, and lead to interesting invariants of compact symplectic manifolds. Indeed, recall from Definition 5.1 that, for $1 \leq r \leq n$, the sections s_r and s_{r-1}

define a symplectic Lefschetz pencil structure on Σ_r, with generic fiber Σ_{r-1} and base set Σ_{r-2}. The monodromy of this pencil is given by a group homomorphism

$$\theta_r : \pi_1(\mathbb{C} - \{p_1, \dots, p_{d_r}\}) \to \mathrm{Map}^\omega(\Sigma_{r-1}, \Sigma_{r-2}). \tag{10}$$

Moreover, for $2 \le r \le n$, the sections s_r, s_{r-1} and s_{r-2} define a quasiholomorphic map from $\Sigma_r - \Sigma_{r-3}$ to $\mathbb{CP}^2$, with generic fiber Σ_{r-2}. Denote by $D_r \subset \mathbb{CP}^2$ the critical curve of this map, and let $d_{r-1} = \deg D_r$. As shown in §4.3, we obtain two monodromy morphisms : on one hand, the braid monodromy homomorphism characterizing D_r,

$$\rho_r : \pi_1(\mathbb{C} - \{p_1, \dots, p_{s_r}\}) \to B_{d_{r-1}}, \tag{11}$$

and on the other hand, a compatible $(r-1)$-dimensional monodromy representation, which was shown in §4.3 to be none other than

$$\theta_{r-1} : \pi_1(\mathbb{C} - \{p_1, \dots, p_{d_{r-1}}\}) \to \mathrm{Map}^\omega(\Sigma_{r-2}, \Sigma_{r-3}).$$

Finally, it was shown in §4.4 that $\mathrm{Im}(\rho_r) \subseteq B^0_{d_{r-1}}(\theta_{r-1})$, and that the various monodromies are related to each other by the identity

$$\theta_r = (\theta_{r-1})_* \circ \rho_r. \tag{12}$$

In particular, the manifold X is completely characterized by the braid monodromies $\rho_2, \dots, \rho_n$ and by the map θ_1 with values in $\mathrm{Map}^\omega(\Sigma_0, \emptyset)$, which is a symmetric group ; this data is sufficient to successively reconstruct all morphisms θ_r and all submanifolds Σ_r by inductively using equation (12).

In other words, a symplectic $2n$-manifold is characterized by $n-2$ braid factorizations and a word in a symmetric group ; or, stopping at θ_2, we can also consider $n-3$ braid factorizations and a word in the mapping class group of a Riemann surface.

These results can be summarized by the following theorem :

Theorem 5.2. *The braid monodromies $\rho_2, \dots, \rho_n$ and the symmetric group representation θ_1 associated to the braiding complete linear systems obtained in Theorem 5.1 are, for $k \gg 0$, canonical up to m-equivalence, and define symplectic invariants of (X^{2n}, ω).*

Conversely, the data consisting of several braid factorizations and a symmetric group representation satisfying suitable compatibility conditions, or a m-equivalence class of such data, determines a symplectic $2n$-manifold in a canonical way up to symplectomorphism.

In principle, this result reduces the study of compact symplectic manifolds to purely combinatorial questions about braid groups and symmetric groups ; however, the invariants it introduces are probably quite difficult to compute as soon as one considers examples which are not complex algebraic. Nevertheless, it seems that this construction should be very helpful in improving our understanding of the topology of Lefschetz pencils in dimensions greater than 4.

References

[1] V. I. Arnold, S. M. Gusein-Zadeh, A. N. Varchenko, *Singularities of smooth mappings*, Monographs in Mathematics, Birkhäuser, Boston (1982).

[2] D. Auroux, *Asymptotically Holomorphic Families of Symplectic Submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[3] D. Auroux, *Symplectic 4-manifolds as Branched Coverings of* $\mathbb{CP}^2$, Invent. Math. **139** (2000), 551–602.
[4] D. Auroux, *Estimated transversality in symplectic geometry and projective maps*, to appear in Proc. KIAS International Conference on Symplectic Geometry, Seoul (2000), World Scientific.
[5] D. Auroux, L. Katzarkov, *Branched coverings of* $\mathbb{CP}^2$ *and invariants of symplectic 4-manifolds*, to appear in Invent. Math.
[6] J. Birman, *Braids, Links and Mapping class groups*, Annals of Math. Studies **82**, Princeton Univ. Press, Princeton, 1974.
[7] J. Birman, B. Wajnryb, *3-fold branched coverings and the mapping class group of a surface*, Geometry and Topology (College Park, 1983/84), Lecture Notes in Math. **1167**, Springer, 1985, 24–46.
[8] S.K. Donaldson, *Symplectic Submanifolds and Almost-complex Geometry*, J. Differential Geom. **44** (1996), 666–705.
[9] S.K. Donaldson, *Lefschetz Fibrations in Symplectic Geometry*, Documenta Math., Extra Volume ICM 1998, II, 309–314.
[10] S.K. Donaldson, *Lefschetz Pencils on Symplectic Manifolds*, preprint (1999).
[11] R.E. Gompf, *The Topology of Symplectic Manifolds*, Proc. 7th Gökova Geometry-Topology Conference (2000), International Press.
[12] R.E. Gompf, A.I. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Math. **20**, Amer. Math. Soc., Providence, 1999.
[13] B. Moishezon, *Stable Branch Curves and Braid Monodromies*, Algebraic Geometry (Chicago, 1980), Lecture Notes in Math. **862**, Springer, 1981, 107–192.
[14] B. Moishezon, *The Arithmetic of Braids and a Statement of Chisini*, Geometric Topology (Haifa, 1992), Contemp. Math. **164**, Amer. Math. Soc., Providence, 1994, 151–175.
[15] V. Muñoz, F. Presas, I. Sols, *Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds*, preprint (1999).
[16] P. Seidel, Ph.D thesis, Oxford University.
[17] P. Seidel, *Vanishing cycles and mutation*, preprint, `math.SG/0007115`.
[18] M. Teicher, *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math., **62** (part 1), Amer. Math. Soc., 1997, 127–150.

Centre de Mathématiques, Ecole Polytechnique, F-91128 Palaiseau, France
E-mail address: `auroux@math.polytechnique.fr`

ESTIMATED TRANSVERSALITY IN SYMPLECTIC GEOMETRY AND PROJECTIVE MAPS

DENIS AUROUX

1. Introduction

Since Donaldson's original work [7], approximately holomorphic techniques have proven themselves most useful in symplectic geometry and topology, and various classical constructions from algebraic geometry have been extended to the case of symplectic manifolds [3, 4, 8, 11]. All these results rely on an estimated transversality statement for approximately holomorphic sections of very positive bundles, obtained by Donaldson [7, 8]. However, the arguments require transversality not only for sections but also for their covariant derivatives, which makes it necessary to painstakingly imitate the arguments underlying Thom's classical *strong transversality theorem* for jets.

It is our aim in this paper to formulate and prove a general result of estimated transversality with respect to finite stratifications in jet bundles. The transversality properties obtained in the various above-mentioned papers then follow as direct corollaries of this result, thus allowing some of the arguments to be greatly simplified. The result can be formulated as follows (see §2 and §3 for definitions) :

Theorem 1.1. *Let $(E_k)_{k\gg 0}$ be an asymptotically very ample sequence of locally splittable complex vector bundles over a compact almost-complex manifold (X, J). Let $\mathcal{S}_k$ be asymptotically holomorphic finite Whitney quasi-stratifications of the holomorphic jet bundles $\mathcal{J}^r E_k$. Finally, let $\delta > 0$ be a fixed constant. Then there exist constants K and η such that, given any asymptotically holomorphic sections s_k of E_k over X, there exist asymptotically holomorphic sections σ_k of E_k with the following properties for all $k \geq K$:*
(1) $|\sigma_k - s_k|_{C^{r+1},g_k} < \delta$;
(2) *the jet $j^r\sigma_k$ of σ_k is η-transverse to the quasi-stratification $\mathcal{S}_k$.*

We start by introducing in §2 a general notion of ampleness over an almost-complex manifold. Then, in §3 we define the notion of approximately holomorphic quasi-stratification of a jet bundle. Theorem 1.1 and its one-parameter version are proved in §4. Finally, we discuss applications in §5.

2. Ample bundles over almost-complex manifolds

The most general setup in which one can try to define a notion of ampleness is the following. Let X be a compact $2n$-dimensional manifold (possibly with boundary), endowed with an almost-complex structure J. In order to make estimates, we also endow X with a Riemannian metric g compatible with J (i.e. J is g-antisymmetric).

Definition 2.1. *Given positive constants c and δ, a complex line bundle L over X endowed with a Hermitian metric and a connection ∇^L is* (c,δ)-ample *if its curvature 2-form F_L satisfies the inequalities $iF_L(v,Jv) \geq c\, g(v,v)$ for every tangent vector $v \in TX$, and $\sup |F_L^{0,2}| \leq \delta$.*

A sequence of complex line bundles L_k with metrics and connections is asymptotically very ample *if there exist fixed constants δ and $(C_r)_{r\geq 0}$, and a sequence $c_k \to +\infty$, such that the curvature F_k of L_k satisfies the following properties : (1) $iF_k(v,Jv) \geq c_k\, g(v,v)$ for every tangent vector $v \in TX$; (2) $\sup |F_k^{0,2}| \leq \delta\, c_k^{1/2}$; (3) $\sup |\nabla^r F_k| \leq C_r\, c_k\ \forall r \geq 0$.*

Most of this definition is a natural extension to the almost-complex setup of the classical notion of ampleness on a complex manifold. Because the notion of holomorphic bundle is not relevant in the case of a non-integrable complex structure, one should allow the curvature to contain a non-trivial $(0,2)$-part. However, because $F_L^{0,2}$ is an obstruction to the existence of holomorphic sections, we need uniform bounds on this quantity in order to hope for the existence of approximately holomorphic sections.

The last condition in the definition seems less natural and should largely be considered as a technical assumption needed to obtain some control over the behavior of sections; it is likely that a suitable argument, possibly involving plurisubharmonic techniques, could allow the bounds to be significantly weakened.

Observe that the curvature of a (c,δ)-ample line bundle over X defines, after multiplication by $\frac{i}{2\pi}$, a J-tame symplectic structure on X with integral cohomology class (J is compatible with this symplectic structure if and only if the curvature is of type $(1,1)$).

Conversely, assume that X carries a J-compatible symplectic form ω with integral cohomology class, and choose g to be the Riemannian metric induced by J and ω. Then there exists a line bundle L with first Chern class $c_1(L) = [\omega]$ and a connection with curvature $-2\pi i\omega$ on L. By construction the line bundle L is $(2\pi, 0)$-ample; moreover, the line bundles $L^{\otimes k}$ with the induced connections are $(2\pi k, 0)$-ample and define an asymptotically very ample sequence of line bundles. This example is by far the most interesting one for applications, but many other situations can be considered as well.

In the rest of this section, we consider an asymptotically very ample sequence of line bundles over X, and study the properties of L_k for large values of k. In order to make the estimates below easier to understand, we rescale the metric by setting $g_k = c_k\, g$, which amounts to dividing by $c_k^{r/2}$ the norm of all r-tensors; the Levi-Civita connection is not affected by this rescaling. The bounds of Definition 2.1 imply that : $iF_k(v,Jv) \geq g_k(v,v)$; $|F_k^{0,2}|_{g_k} = O(c_k^{-1/2})$; $|F_k|_{g_k} = O(1)$; $|\nabla^r F_k|_{g_k} = O(c_k^{-1/2})\ \forall r \geq 1$. Also observe that $|\nabla^r J|_{g_k} = O(c_k^{-1/2})\ \forall r \geq 1$. Better bounds on higher-order derivatives are trivially available but we won't need them.

Lemma 2.1. *Let L_k be a sequence of asymptotically very ample line bundles L_k over X, and denote by F_k the curvature of L_k. Let $\omega_k = iF_k$, and let c_k be the constants appearing in Definition 2.1. Denote by ∇ the Levi-Civita connection associated to g. Then, for large enough k there exist ω_k-compatible almost-complex structures $\tilde{J}_k$ such that $|\nabla^r(\tilde{J}_k - J)|_{g_k} = O(c_k^{-1/2})\ \forall r \geq 0$.*

Proof. We construct $\tilde{J}_k$ locally; patching together the local constructions in order to obtain a globally defined almost-complex structure still satisfying the same type of bounds is an easy task left to the reader (recall that the space of ω_k-compatible almost-complex structures is pointwise contractible).

Let e_1 be a local tangent vector field of unit g_k-length and with $|\nabla^r e_1|_{g_k} = O(c_k^{-1/2})\ \forall r \geq 1$ (observe that, because of the rescaling process, (X, g_k) is almost flat for large k). We define $e_1' = Je_1$, and observe that e_1' has unit g_k-length (J is g-unitary and hence g_k-unitary) and $\omega_k(e_1, e_1') \geq 1$. Next, we proceed inductively, assuming that we have defined local vector fields $e_1, e_1', \dots, e_m, e_m'$ with the following properties for all $i, j \leq m$: $e_1, \dots, e_m$ have unit g_k-length; $\omega_k(e_i, e_j) = \omega_k(e_i', e_j') = 0$; $\omega_k(e_i, e_j') = 0$ if $i \neq j$; $\omega_k(e_i, e_i') \geq 1$; $e_i' - Je_i \in \operatorname{span}(e_1, e_1', \dots, e_{i-1}, e_{i-1}')$; $|e_i' - Je_i|_{g_k} = O(c_k^{-1/2})$; $|\nabla^r e_i|_{g_k} = O(c_k^{-1/2})$ and $|\nabla^r e_i'|_{g_k} = O(c_k^{-1/2})\ \forall r \geq 1$.

We choose e_{m+1} to be a g_k-unit vector field which is ω_k-orthogonal to $e_1, e_1', \dots, e_m, e_m'$. The bound on $|\nabla^r \omega_k|_{g_k}$ implies that we can choose e_{m+1} in such a way that $|\nabla^r e_{m+1}|_{g_k} = O(c_k^{-1/2})$. Next, we define

$$e_{m+1}' = Je_{m+1} + \sum_{i=1}^{m} \frac{\omega_k(e_i', Je_{m+1})e_i - \omega_k(e_i, Je_{m+1})e_i'}{\omega_k(e_i, e_i')}.$$

By construction, $\omega_k(e_i, e_{m+1}') = \omega_k(e_i', e_{m+1}') = 0$ for all $i \leq m$. Moreover, we have $\omega_k(e_{m+1}, e_{m+1}') = \omega_k(e_{m+1}, Je_{m+1}) \geq 1$.

Since $\omega_k(Je_i, e_{m+1}) = 0$, and because $\omega_k(Je_i, e_{m+1}) - \omega_k(e_i, Je_{m+1})$ is a component of $\omega_k^{0,2}$, we have $\omega_k(e_i, Je_{m+1}) = O(c_k^{-1/2})$. Similarly, $\omega_k(e_i', Je_{m+1}) = \omega_k(Je_i, Je_{m+1}) + \omega_k(e_i' - Je_i, Je_{m+1})$; the first term differs from $\omega_k(e_i, e_{m+1}) = 0$ by a $(0,2)$-term and is therefore bounded by $O(c_k^{-1/2})$, while the bound on $e_i' - Je_i$ implies that the second term is also bounded by $O(c_k^{-1/2})$. Therefore we have $\omega_k(e_i', Je_{m+1}) = O(c_k^{-1/2})$. Finally, using the lower bound on $\omega_k(e_i, e_i')$ we obtain that $|e_{m+1}' - Je_{m+1}|_{g_k} = O(c_k^{-1/2})$. Finally, it is trivial that $|\nabla^r e_{m+1}'|_{g_k} = O(c_k^{-1/2})$; therefore we can proceed with the induction process.

We now define the almost-complex structure $\tilde{J}_k$ by the identities $\tilde{J}_k(e_i) = e_i'$ and $\tilde{J}_k(e_i') = -e_i$. By construction, $\tilde{J}_k$ is compatible with ω_k, and the corresponding Riemannian metric $\tilde{g}_k$ admits $e_1, e_1', \dots, e_n, e_n'$ as an orthonormal frame. The required bounds on $\tilde{J}_k$ immediately follow from the available estimates. □

Remark. As suggested by the referee, Lemma 2.1 can also be proved more efficiently using the following argument : $\tilde{J}_k$ is characterized by a linear map $\mu_k : TX_J^{(1,0)} \to TX_J^{(0,1)}$, such that $TX_{\tilde{J}_k}^{(1,0)} = \{v + \mu_k\, v,\ v \in TX_J^{(1,0)}\}$ and $TX_{\tilde{J}_k}^{(0,1)} = \{v + \bar{\mu}_k\, v,\ v \in TX_J^{(0,1)}\}$. The compatibility of $\tilde{J}_k$ with ω_k is expressed by the condition $\omega_k(u + \mu_k\, u, v + \mu_k\, v) = 0\ \forall u, v \in TX_J^{(1,0)}$, i.e.

$$\omega_k^{1,1}(\mu_k\, \cdot, \cdot) + \omega_k^{1,1}(\cdot, \mu_k\, \cdot) + O(|\mu_k|^2) = -\omega_k^{2,0}.$$

Since $\omega_k^{1,1}$ defines a non-degenerate pairing between $TX_J^{(1,0)}$ and $TX_J^{(0,1)}$, and because the first two terms of the left-hand side correspond to the antisymmetric part of $\omega_k^{1,1}(\mu_k\, \cdot, \cdot)$, the existence of a small solution μ_k to this equation follows directly from the smallness of $\omega_k^{2,0}$ and the implicit function theorem. □

Lemma 2.1 makes it possible to recover the main ingredients of Donaldson theory in the more general setting described here. We now introduce some basic definitions and results, imitating Donaldson's original work and subsequent papers [7, 3].

In what follows, L_k is an asymptotically very ample sequence of line bundles over X, c_k are the same constants as in Definition 2.1, and $g_k = c_k\, g$.

Lemma 2.2. *Near any point $x \in X$, and for any value of k, there exist local complex Darboux coordinates $(z_k^1, \dots, z_k^n) : (X, x) \to (C^n, 0)$ for the symplectic structure $\omega_k = iF_k$, such that, denoting by ψ_k the inverse of the coordinate map, the following bounds hold uniformly in x and k over a ball of fixed g-radius around x : $|z_k^i(y)| = O(dist_{g_k}(x, y))$; $|\nabla^r \psi_k|_{g_k} = O(1)\ \forall r \geq 1$; and, with respect to the almost-complex structure J on X and the canonical complex structure on $\mathbb{C}^n$, $|\bar\partial \psi_k(z)|_{g_k} = O(c_k^{-1/2} + c_k^{-1/2}|z|)$, and $|\nabla^r \bar\partial \psi_k(z)|_{g_k} = O(c_k^{-1/2})$ for all $r \geq 1$.*

Proof. The argument is very similar to that used by Donaldson [7], except that one needs to be slightly more careful in showing that the various bounds hold uniformly in k. Fix a point $x \in X$: then we can find a neighborhood U of x and a local coordinate map $\phi : U \to \mathbb{C}^n$, such that U contains a ball of fixed uniform g-radius around x, and such that the expressions of g and J in these local coordinates satisfy uniform bounds independently of x (these uniformity properties follow from the compactness of X). A linear transformation can be used to ensure that the differential of ϕ at the origin is $\mathbb{C}$-linear with respect to J. Next, we rescale the coordinates by $c_k^{1/2}$ to obtain a new coordinate map $\phi_k : U \to \mathbb{C}^n$, in which J coincides with the standard almost-complex structure at the origin and has derivatives bounded by $O(c_k^{-1/2})$, while the expression of g_k is bounded between fixed constants and has derivatives bounded by $O(c_k^{-1/2})$.

Next, we observe that the bound on $|\omega_k|$ and the lower bound on $\omega_k(v, Jv)$ imply that the expression of $\omega_k^{(1,1)}$ at the origin of the coordinate chart is bounded from above and below by uniform constants. Therefore, after composing ϕ_k with a suitable element of $GL(n, \mathbb{C})$, we can assume without affecting the bounds on J and g_k that $(\phi_k^{-1})^*(\omega_k^{(1,1)})$ coincides with the standard Kähler form ω_0 of $\mathbb{C}^n$ at the origin.

Define over $\phi_k(U) \subset \mathbb{C}^n$ the symplectic form $\omega_1 = (\phi_k^{-1})^*\omega_k$. By construction, $\omega_1(0) - \omega_0(0) = O(c_k^{-1/2})$. Observe that, in the chosen coordinates, the Levi-Civita connection of g_k differs from the trivial connection by $O(c_k^{-1/2})$; therefore, the bounds on $|\nabla^r \omega_k|_{g_k}$ imply that the derivatives of ω_1 are also bounded by $O(c_k^{-1/2})$, and that $|\omega_1(z) - \omega_0(z)| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$.

In particular, decreasing the size of U by at most a fixed factor if necessary, we obtain that the closed 2-forms $\omega_t = t\omega_1 + (1-t)\omega_0$ over $\phi_k(U)$ are all symplectic, and we can apply Moser's argument to construct in a controlled way a symplectomorphism between a subset of $(\phi_k(U), \omega_1)$ and a subset of $(\mathbb{C}^n, \omega_0)$. More precisely, it follows immediately from Poincaré's lemma that we can choose a 1-form α such that $\omega_1 - \omega_0 = d\alpha$, and such that $\alpha(0) = 0$, $|\alpha(z)| = O(c_k^{-1/2}|z| + c_k^{-1/2}|z|^2)$, $|\nabla\alpha(z)| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$ and $|\nabla^r \alpha(z)| = O(c_k^{-1/2})\ \forall r \geq 2$. Next, we define vector fields X_t by the identity $i_{X_t}\omega_t = \alpha$; clearly X_t and its derivatives satisfy the same bounds as α.

Integrating the flow of the vector fields X_t we obtain diffeomorphisms ρ_t, and it is a classical fact that the map $\tilde{\phi}_k = \rho_1 \circ \phi_k$ is a local symplectomorphism between (X, ω_k) and $(\mathbb{C}^n, \omega_0)$ and therefore defines Darboux coordinates. Because $|z| = O(c_k^{1/2})$ over a ball of fixed g-radius around x, the vector fields X_t satisfy a uniform bound of the type $|X_t(z)| \leq \lambda |z|$ for some constant λ, so that $|\rho_t(z)| \leq e^{\lambda t}|z|$, and therefore $\tilde{\phi}_k$ is well-defined over a ball of fixed g-radius around x. Moreover, the bounds $|\nabla(\rho_1 - \mathrm{Id})| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$, obtained by integrating the bounds on $\nabla\alpha$, and $|\bar{\partial}(\phi_k^{-1})| = O(c_k^{-1/2}|z|)$, obtained from the bounds on the expression on J in the local coordinates, imply that $|\bar{\partial}(\tilde{\phi}_k^{-1})| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$. Similarly, the bounds $|\nabla^{r+1}\rho_1| = O(c_k^{-1/2})$ and $|\nabla^r \bar{\partial}(\phi_k^{-1})| = O(c_k^{-1/2})$ for all $r \geq 1$ imply that $|\nabla^r \bar{\partial}(\tilde{\phi}_k^{-1})| = O(c_k^{-1/2})$. This completes the proof of Lemma 2.2. □

Definition 2.2. *A family of sections of L_k is* asymptotically J-holomorphic *for $k \to \infty$ if there exist constants $(C_r)_{r\geq 0}$ such that every section $s \in \Gamma(L_k)$ in the family satisfies at every point of X the bounds $|\nabla^r s|_{g_k} \leq C_r$ and $|\nabla^r \bar{\partial}_J s|_{g_k} \leq C_r\, c_k^{-1/2}$ for all $r \geq 0$, where $\bar{\partial}_J$ is the $(0,1)$-part of the connection on L_k.*

A family of sections of L_k has uniform Gaussian decay *properties if there exist a constant $\lambda > 0$ and polynomials $(P_r)_{r\geq 0}$ with the following property : for every section s of L_k in the family, there exists a point $x \in X$ such that for all $y \in X$ and for all $r \geq 0$, $|\nabla^r s(y)|_{g_k} \leq P_r(d_k(x,y)) \exp(-\lambda\, d_k(x,y)^2)$, where $d(.,.)$ is the distance induced by g_k.*

Lemma 2.3. *For all large enough values of k and for every point $x \in X$, there exists a section $s^{\mathrm{ref}}_{k,x}$ of L_k with the following properties : (1) the family of sections $(s^{\mathrm{ref}}_{k,x})_{x\in X, k\gg 0}$ is asymptotically J-holomorphic; (2) the family $(s^{\mathrm{ref}}_{k,x})_{x\in X,k\gg 0}$ has uniform Gaussian decay properties, each section $s^{\mathrm{ref}}_{k,x}$ being concentrated near the point x; (3) there exists a constant $\kappa > 0$ independent of x and k such that $|s^{\mathrm{ref}}_{k,x}| \geq \kappa$ at every point of the ball of g_k-radius 1 centered at x.*

Proof. The argument is a direct adaptation of the proof of Proposition 11 in Donaldson's paper [7]. Pick a value of k and a point $x \in X$. We work in the approximately J-holomorphic Darboux coordinates given by Lemma 2.2, and use a trivialization of L_k in which the connection 1-form becomes $\frac{1}{4}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$. Then, we define a local section of L_k by $s(z) = \exp(-\frac{1}{4}|z|^2)$ and observe that s is holomorphic with respect to the standard complex structure of $\mathbb{C}^n$. Multiplying s by a cut-off function which equals 1 over the ball of radius $c_k^{1/6}$ around the origin, we obtain a globally defined section of L_k; because of the estimates on the Darboux coordinates one easily checks that the families of sections constructed in this way are asymptotically holomorphic and have uniform Gaussian decay properties [7]. □

We are also interested in working with higher rank bundles. The definition of ampleness becomes the following :

Definition 2.3. *A sequence of complex vector bundles E_k with metrics and connections is* asymptotically very ample *if there exist constants δ, $(C_r)_{r\geq 0}$, and $c_k \to +\infty$, such that the curvature F_k of E_k satisfies the following properties :*

(1) *$\langle iF_k(v, Jv).u, u\rangle \geq c_k\, g(v,v)\, |u|^2$, $\forall v \in TX$, $\forall u \in E_k$;*

(2) *$\sup |F_k^{0,2}|_g \leq \delta_r\, c_k^{1/2}$;*

(3) $\sup |\nabla^r F_k|_g \le C_r\, c_k\ \forall r \ge 0$.

A sequence of asymptotically very ample complex vector bundles E_k with metrics $|.|_k$ and connections ∇_k is locally splittable *if, given any point $x \in X$, there exists over a ball of fixed g-radius around x a decomposition of E_k as a direct sum $L_{k,1} \oplus \cdots \oplus L_{k,m}$ of line bundles, such that the following properties hold :*

(1) *the $|.|_k$-determinant of a local frame consisting of unit length local sections of $L_{k,1}, \dots, L_{k,m}$ is bounded from below by a fixed constant independently of x and k;*

(2) *denoting by $\nabla_{k,i}$ the connection on $L_{k,i}$ obtained by projecting $\nabla_{k|L_{k,i}}$ to $L_{k,i}$, and by ∇'_k the direct sum of the $\nabla_{k,i}$, the 1-form $\alpha_k = \nabla_k - \nabla'_k$ (the non-diagonal part of ∇_k) satisfies the uniform bounds $|\nabla^r \alpha_k|_g = O(c_k^{r/2})\ \forall r \ge 0$ independently of x.*

For example, if E is a fixed complex vector bundle and L_k are asymptotically very ample line bundles, then the vector bundles $E \otimes L_k$ are locally splittable and asymptotically very ample; so are direct sums of vector bundles of this type.

Observe that, if E_k is an asymptotically very ample sequence of locally splittable vector bundles, then near any given point $x \in X$ the summands $L_{k,1}, \dots, L_{k,m}$ are asymptotically very ample line bundles. Therefore, by Lemma 2.3 they admit asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x,i}$ with uniform Gaussian decay away from x. Moreover, these sections, which define a local frame for E_k, are easily checked to be asymptotically J-holomorphic not only as sections of $L_{k,i}$ but also as sections of E_k.

3. Estimated transversality in jet bundles

3.1. **Asymptotically holomorphic stratifications.** Throughout this section, we will denote by F_k be a sequence of complex vector bundles over X, or more generally fiber bundles with almost-complex manifolds as fibers. We also fix, in a manner compatible with the almost-complex structures J^v of the fibers, metrics g^v on the fibers of F_k and a connection on F_k. Finally, we fix a sequence of constants $c_k \longrightarrow +\infty$.

The connection on F_k induces a splitting $TF_k = T^v F_k \oplus T^h F_k$ between horizontal and vertical tangent spaces ; this splitting makes it possible to define a metric $\hat{g}_k$ and an almost-complex structure $\hat{J}_k$ on the total space of F_k, obtained by orthogonal direct sum of g^v and J^v on $T^v F_k$ together with the pullbacks of J and $g_k = c_k\, g$ on $T^h F_k \simeq \pi^* TX$.

We want to consider approximately holomorphic stratifications of the fibers of F_k, depending in an approximately holomorphic way on the point in the base manifold X. For simplicity, we assume that the topological picture is the same in every fiber of F_k, i.e. we restrict ourselves to stratifications which are everywhere transverse to the fibers. We will denote the strata by $(S_k^a)_{a \in A_k}$; we assume that the number of strata is finite. Each S_k^a is a possibly non-closed submanifold in F_k, whose closure is obtained by adding other lower dimensional strata : writing $b \prec a$ iff S_k^b is contained in $\overline{S_k^a}$, we have

$$\partial S_k^a \stackrel{\text{def}}{=} \overline{S_k^a} - S_k^a = \bigcup_{b \prec a} S_k^b.$$

We only consider *Whitney stratifications* ; in particular, transversality to a given stratum S_k^a implies transversality over a neighborhood of S_k^a to all the strata whose closure contains S_k^a, i.e. all the S_k^b for $b \succ a$. Also note that we discard any open strata, as they are irrelevant for transversality purposes ; so each S_k^a has codimension at least 1.

Definition 3.1. *Let (M, J) be an almost-complex manifold, with a Riemannian metric, and let s be a complex-valued function over M or a section of an almost-complex bundle with metrics and connection. Given two constants C and c, we say that s is C^2-*approximately holomorphic with bounds (C, c)*, or C^2-*AH(C, c)*, if it satisfies the following estimates :*

$$|s| + |\nabla s| + |\nabla\nabla s| \le C, \qquad |\bar\partial s| + |\nabla\bar\partial s| \le C\, c^{-1/2}.$$

*Moreover, given constants $c_k \to +\infty$, we say that a sequence $(s_k)_{k\gg 0}$ of functions or sections is C^2-*asymptotically holomorphic*, or C^2-*AH*, if there exists a fixed constant C such that each section s_k is C^2-AH(C, c_k).*

Definition 3.2. *Let F_k be a sequence of almost-complex bundles over X, endowed with metrics and connections as above. For all values of k, let $(S_k^a)_{a\in A_k}$ be finite Whitney stratifications of F_k ; assume that the total number of strata is bounded by a fixed constant independently of k, and that all strata are transverse to the fibers of F_k.*

We say that this sequence of stratifications is asymptotically holomorphic *if, given any bounded subset $U_k \subset F_k$, and for every $\epsilon > 0$, there exist positive constants C_ϵ and ρ_ϵ depending only on ϵ and on the size of the subset U_k but not on k, with the following property. For every point $x \in U_k$ lying in a certain stratum S_k^a and at $\hat g_k$-distance greater than ϵ from $\partial S_k^a = \overline{S_k^a} - S_k^a$, there exist complex-valued functions $f_1, \dots, f_p$ over the ball $B = B_{\hat g_k}(x, \rho_\epsilon)$ with the following properties :*

(1) a local equation of S_k^a over B is $f_1 = \dots = f_p = 0$;

(2) $|df_1 \wedge \dots \wedge df_p|_{\hat g_k}$ is bounded from below by ρ_ϵ at every point of B ;

(3) the restrictions of f_i to each fiber of F_k near x are C^2-AH(C_ϵ, c_k) ;

(4) for any constant $\lambda > 0$, and for any local section s of F_k which is C^2-AH(λ, c_k) with respect to the metric g_k on X and which intersects non-trivially the ball B, the function $f_i \circ s$ is C^2-AH$(\lambda C_\epsilon, c_k)$; moreover, given a local C^2-AH(λ, c_k) section θ of $s^*T^vF_k$, the functions $df_i \circ \theta$ are C^2-AH$(\lambda C_\epsilon, c_k)$;

(5) at every point $y \in B$ belonging to a stratum S_k^b such that $S_k^a \subset \partial S_k^b$, the norm of the orthogonal projection onto the normal space $N_y S_k^b$ of any unit length vector $v \in T_yF_k$ such that $df_1(v) = \dots = df_p(v) = 0$ is bounded by $C_\epsilon \operatorname{dist}_{\hat g_k}(y, S_k^a)$.

These conditions on the stratification can be reformulated more geometrically as follows. First, the strata must be uniformly transverse to the fibers of F_k, i.e. one requires the minimum angle [11] between TS_k^a and T^vF_k to be bounded from below. Second, the submanifolds $S_k^a \subset F_k$ must be asymptotically $\hat J_k$-holomorphic, i.e. $\hat J_k(TS_k^a)$ and TS_k^a lie within $O(c_k^{-1/2})$ of each other. Third, the curvature of S_k^a as a submanifold of F_k must be uniformly bounded. Finally, the quantity measuring the lack of $\hat J_k$-holomorphicity of S_k^a must similarly vary in a controlled way.

We finish this section by introducing the notion of estimated transversality between a section and a stratification. Observe that, given any submanifold $N \subset M$, we can define over a neighborhood of N a "parallel" distribution $D_N \subset TM$ by parallel transport of TN in the normal direction to N. Also recall that the *minimum angle* between two linear subspaces U and V is defined as the minimum angle between a vector orthogonal to U and a vector orthogonal to V [11]. The minimum angle between U and V is non-zero if and only if they are transverse to each other, and in that case it can also be defined as the minimum angle between non-zero vectors orthogonal to $U \cap V$ in U and V.

Definition 3.3. *Given a constant $\eta > 0$, we say that a section s of a vector bundle carrying a metric and a connection is η-transverse to 0 if, at every point x such that $|s(x)| \leq \eta$, the covariant derivative $\nabla s(x)$ is surjective and admits a right inverse of norm less than η^{-1}.*

Fix a constant $\eta > 0$, and a section s of a bundle carrying a metric and a finite Whitney stratification $\mathcal{S} = (S^a)_{a \in A}$ everywhere transverse to the fibers. We say that s is η-transverse to the stratification $\mathcal{S}$ if, at every point where s lies at distance less than η from some stratum S^a, the graph of s is transverse to the parallel distribution D_{S^a}, with a minimum angle greater than η.

Finally, we say that a sequence of sections is uniformly tranverse to 0 (resp. to a sequence of stratifications) if there exists a fixed constant $\eta > 0$ such that all sections in the sequence are η-transverse to 0 (resp. the stratifications).

Note that the above condition of transversality of the section s to each stratum S^a is only well-defined outside of a small neighborhood of the lower-dimensional strata contained in ∂S^a ; however, near these strata the assumption that $\mathcal{S}$ is Whitney makes transversality to S^a a direct consequence of the η-transversality to the lower-dimensional strata.

Another way in which uniform transversality to a stratification can be formulated is to use local equations of the strata, as in Definition 3.2. One can then define η-transversality as follows : at every point where s lies at distance less than η from S^a, and considering local equations $f_1 = \cdots = f_p = 0$ of S^a such that each $|df_i|$ is bounded by a fixed constant and $|df_1 \wedge \cdots \wedge df_p|$ is bounded from below by a fixed constant, the function $(f_1 \circ s, \ldots, f_p \circ s)$ with values in $\mathbb{C}^p$ must be η-transverse to 0. The two definitions are equivalent up to changing the constant η by at most a bounded factor.

3.2. **Quasi-stratifications in jet bundles.** Let E_k be an asymptotically very ample sequence of locally splittable rank m vector bundles over the compact almost-complex manifold (X, J). We can introduce the *holomorphic jet bundles*

$$\mathcal{J}^r E_k = \bigoplus_{j=0}^{r} \big(T^*X^{(1,0)}\big)^{\otimes j}_{\text{sym}} \otimes E_k.$$

More precisely, the holomorphic part of the r-jet of a section s of E_k is defined inductively as follows : $T^*X^{(1,0)}$ and E_k, as complex vector bundles carrying a connection over an almost-complex manifold, are endowed with ∂ operators (the $(1,0)$ part of the connection) ; the r-jet of s is $j^r s = (s, \partial_{E_k} s, \partial_{T^*X^{(1,0)} \otimes E_k}(\partial_{E_k} s)_{\text{sym}}, \ldots)$.

Observe that, because the almost-complex structure J is not integrable and because the curvature of E_k is not necessarily of type $(1,1)$, the derivatives of order ≥ 2 are not symmetric tensors, but rather satisfy equality relations involving curvature terms and lower-order derivatives. However, we will only consider the symmetric part of the jet ; for example, the 2-tensor component of $j^r s$ is defined by $(\partial\partial s)_{\text{sym}}(u, v) = \frac{1}{2}(\langle \partial(\partial s), u \otimes v\rangle + \langle \partial(\partial s), v \otimes u\rangle)$. Note that, anyway, in the case of asymptotically holomorphic sections, the antisymmetric terms are bounded by $O(c_k^{-1/2})$, because the $(2,0)$ curvature terms and Nijenhuis tensor are bounded by $O(c_k^{-1/2})$.

The metrics and connections on TX and on E_k naturally induce Hermitian metrics and connections on $\mathcal{J}^r E_k$ (to define the metric we use the rescaled metric g_k on X). In fact, it is easy to see that the vector bundles $\mathcal{J}^r E_k$ are asymptotically very ample.

Recall that, near any given point $x \in X$, there exist local approximately holomorphic coordinates ; besides a local identification of X with $\mathbb{C}^n$, these coordinates also provide an identification of $T^*X^{(1,0)}$ with $T^*\mathbb{C}^{n(1,0)}$. Moreover, by Lemma 2.3 there exist asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x,i}$ of E_k with Gaussian decay away from x and defining a local frame in E_k. Using these sections to trivialize E_k, we can locally identify $\mathcal{J}^r E_k$ with a space of jets of holomorphic $\mathbb{C}^m$-valued maps over $\mathbb{C}^n$. Observe however that, when we consider the holomorphic parts of jets of approximately holomorphic sections of E_k, the integrability conditions normally satisfied by jets of holomorphic functions only hold in an approximate sense.

In general, the various possible choices of trivializations of $\mathcal{J}^r E_k$ differ by approximately holomorphic diffeomorphisms of $\mathbb{C}^n$ and also by the action of approximately holomorphic local sections of the automorphism bundle $\mathrm{GL}(E_k)$. However, when E_k is of the form $\mathbb{C}^m \otimes L_k$ where L_k is a line bundle, the only automorphisms of E_k which we need to consider are multiplications by complex-valued functions.

Denote by $\mathcal{J}^r_{n,m}$ the space of r-jets of holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$: pointwise, the identifications of the fibers of $\mathcal{J}^r E_k$ with $\mathcal{J}^r_{n,m}$ given by local trivializations differ from each other by the action of $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ (or $GL_n(\mathbb{C}) \times \mathbb{C}^*$ when $E_k = \mathbb{C}^m \otimes L_k$), where $GL_n(\mathbb{C})$ corresponds to changes in the identification of $T^*X^{(1,0)}$ with $T^*\mathbb{C}^{n(1,0)}$ and $GL_m(\mathbb{C})$ or $\mathbb{C}^*$ corresponds to changes in the trivialization of E_k. Some stratifications of $\mathcal{J}^r_{n,m}$ are invariant under the actions of $GL_n(\mathbb{C})$ and $GL_m(\mathbb{C})$ (resp. $\mathbb{C}^*$). Given such a stratification it becomes easy to construct an asymptotically holomorphic sequence of finite Whitney stratifications of $\mathcal{J}^r E_k$, modelled in each fiber on the given stratification of $\mathcal{J}^r_{n,m}$. Many important examples of asymptotically holomorphic stratifications, and in a certain sense all the geometrically relevant ones, are obtained by this construction (see Proposition 3.1 below).

We also wish to consider cases where the available structure is not exactly a Whitney stratification but behaves in a similar manner with respect to transversality. We call such a structure a "Whitney quasi-stratification". Given a submanifold $S \subset \mathcal{J}^r_{n,m}$, one can introduce the subset Θ_S of all points $\sigma \in S$ such that there exists a holomorphic $(r+1)$-jet whose r-jet component is σ and which, considered as a 1-jet of r-jets, intersects S transversely at σ. For example, if S is the subset of all jets $(\sigma_0, \dots, \sigma_r)$ such that $\sigma_0 = 0$, the subset Θ_S consists of those jets such that $\sigma_0 = 0$ and σ_1 is surjective.

Similarly, when S is a submanifold in $\mathcal{J}^r E_k$, we can view an element of $\mathcal{J}^{r+1} E_k$ as the holomorphic 1-jet of a section of $\mathcal{J}^r E_k$. More precisely, for any point $x \in X$, we can associate to any $\sigma = (\sigma_0, \dots, \sigma_{r+1}) \in (\mathcal{J}^{r+1} E_k)_x$ the 1-jet at x of a local section $\tilde{\sigma}$ of $\mathcal{J}^r E_k$, such that $\tilde{\sigma}(x) = (\sigma_0, \dots, \sigma_r)$, $(\partial\tilde{\sigma}(x))^{\mathrm{sym}} = (\sigma_1, \dots, \sigma_{r+1})$, $(\partial\tilde{\sigma}(x))^{\mathrm{antisym}} = 0$, and $\bar\partial\tilde{\sigma}(x) = 0$ (in this definition, $\partial\tilde{\sigma}(x) \in$

$T^*X^{1,0} \otimes (\bigoplus (T^*X^{1,0})^{\otimes j}_{\mathrm{sym}} \otimes E)$ is decomposed into a symmetric part and an antisymmetric part). Then, we define Θ_S as the set of points of S for which there exists an element $\sigma \in \mathcal{J}^{r+1}E_k$ such that the corresponding 1-jet $\tilde{\sigma}$ in $\mathcal{J}^r E_k$ intersects S transversely at the given point. For example, if S is the set of r-jets $(\sigma_0, \dots, \sigma_r)$ such that $\sigma_0 = 0$, then Θ_S is the set of r-jets such that $\sigma_0 = 0$ and σ_1 is surjective. Also observe that Θ_S is always empty when the codimension of S is greater than n.

Definition 3.4. *Given a finite set $(A, \prec)$ carrying a binary relation without cycles (i.e., $a_1 \prec \dots \prec a_p \Rightarrow a_p \not\prec a_1$), a finite Whitney quasi-stratification of $\mathcal{J}^r_{n,m}$ indexed by A is a collection $(S^a)_{a\in A}$ of smooth submanifolds of $\mathcal{J}^r_{m,n}$, not necessarily mutually disjoint, with the following properties : (1) $\partial S^a = \overline{S^a} - S^a \subseteq \bigcup_{b\prec a} S^b$; (2) given any point $p \in \partial S^a$, there exists $b \prec a$ such that $p \in S^b$ and such that either $p \notin \Theta_{S^b}$ or $S^b \subset \partial S^a$ and the Whitney regularity condition is satisfied at all points of S^b.*

Similarly, we can define the notion of asymptotically holomorphic finite Whitney quasi-stratifications of $\mathcal{J}^r E_k$. This is similar to Definition 3.2, except that the collections (S^a_k) are quasi-stratifications rather than stratifications, i.e. $\partial S^a_k \subseteq \bigcup_{b\prec a} S^b_k$, and for every $p \in \partial S^a_k$ there exists $b \prec a$ such that either $p \in S^b_k - \Theta_{S^b_k}$ or $p \in S^b_k \subset \partial S^a_k$; in the latter case the Whitney condition is required. Also, observe that condition (5) in Definition 3.2 is only required in the second case, and not for all b such that $a \prec b$.

It is important to understand that the notion of quasi-stratification is merely an attempt at simplifying the framework for applications of Theorem 1.1. In fact, most quasi-stratifications can be refined into genuine stratifications by suitably subdividing the strata into smaller pieces. However, by definition these modifications occur at points of $\mathcal{J}^r E_k$ that no generic jet can hit, thus making them utterly irrelevant to transversality.

Proposition 3.1. *Let $\mathcal{S} = (S^a)_{a\in A}$ be a finite Whitney quasi-stratification of $\mathcal{J}^r_{n,m}$ by complex submanifolds, invariant under the action of $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ or $GL_n(\mathbb{C}) \times \mathbb{C}^*$. Let E_k be an asymptotically very ample sequence of rank m complex vector bundles over X, trivialized near every point by suitable choices of local asymptotically holomorphic coordinates and sections. Assume that $\mathcal{S}_k = (S^a_k)_{a\in A}$ are quasi-stratifications of $\mathcal{J}^r E_k$ such that, in each local trivialization, the intersection of S^a_k with every fiber becomes identified with S^a. Then the sequence of quasi-stratifications $\mathcal{S}_k$ is asymptotically holomorphic.*

The proof of this result is easy and left to the reader ; the independence on k of the model holomorphic quasi-stratification of $\mathcal{J}^r_{n,m}$ and the availability of asymptotically holomorphic local trivializations of $\mathcal{J}^r E_k$ (Lemma 2.2 and Lemma 2.3) immediately yield the necessary estimates on the strata of $\mathcal{S}_k$. The only important point to observe is that, because the strata of $\mathcal{S}$ are $GL_m(\mathbb{C})$-invariant (resp. $\mathbb{C}^*$-invariant), the local trivializations identifying S^a_k with S^a also identify $\Theta_{S^a_k}$ with Θ_{S^a}. This is e.g. due to the fact that, up to a suitable change in the choice of the local coordinates on X and local reference sections of L_k, i.e. up to a local gauge transformation, we can assume that the connection on $\mathcal{J}^r E_k$ agrees at a given point $x \in X$ with the trivial connection on $\mathcal{J}^r_{n,m}$; above x, the identification of $(r+1)$-jets with 1-jets of

r-jets then becomes the same in $\mathcal{J}^r E_k$ as in $\mathcal{J}^r_{n,m}$, so that the definitions of Θ_S in $\mathcal{J}^r_{n,m}$ and in $\mathcal{J}^r E_k$ agree with each other.

Various examples of applications of Proposition 3.1 will be given in §5.

Finally, we state a one-parameter version of Theorem 1.1. Consider a continuous one-parameter family $(J_t)_{t\in[0,1]}$ of almost-complex structures on X, and a one-parameter family of asymptotically holomorphic finite Whitney (quasi)-stratifications $(\mathcal{S}_{k,t})_{k\gg 0,t\in[0,1]}$ of almost-complex bundles $F_{k,t}$ over (X, J_t). We say that the (quasi)-stratifications $\mathcal{S}_{k,t}$ depend continuously on t if the following one-parameter version of Definition 3.2 is true : for every $\epsilon > 0$, there exist constants ρ_ϵ and C_ϵ with the following property. Given any continuous path $(x_t)_{t\in[t_1,t_2]}$ of points all belonging to the fibers of $F_{k,t}$ above a same point in X, and assuming that all the points x_t belong to certain strata $S^a_{k,t}$ while lying at distance more than ϵ from $\partial S^a_{k,t}$, there exist for all $t \in [t_1, t_2]$ complex-valued functions $f_{1,t}, \dots, f_{p,t}$ defined over the ball $B_{\hat{g}_{k,t}}(x_t, \rho_\epsilon)$ and depending continuously on t, satisfying the various properties of Definition 3.2 for all values of t.

With this understood, the result is the following :

Theorem 3.2. *Let $(J_t)_{t\in[0,1]}$ be a continuous one-parameter family of almost-complex structures on the compact manifold X, and let $(E_{k,t})_{k\gg 0,t\in[0,1]}$ be a family of complex vector bundles over X endowed with metrics and connections depending continuously on t and such that the sequence $E_{k,t}$ is asymptotically very ample and locally splittable over (X, J_t) for all t. Let $\mathcal{S}_{k,t}$ be asymptotically holomorphic finite Whitney quasi-stratifications of $\mathcal{J}^r E_{k,t}$ depending continuously on t. Finally, let $\delta > 0$ be a fixed constant. Then there exist constants K and η such that, given any one-parameter family of asymptotically holomorphic sections $s_{k,t}$ of $E_{k,t}$ over X depending continuously on t, there exist asymptotically holomorphic sections $\sigma_{k,t}$ of $E_{k,t}$, depending continuously on t, with the following properties for all $k \ge K$ and for all $t \in [0, 1]$:*

(1) $|\sigma_{k,t} - s_{k,t}|_{C^{r+1},g_k} < \delta$ *;*

(2) *the jet $j^r\sigma_{k,t}$ of $\sigma_{k,t}$ is η-transverse to $\mathcal{S}_{k,t}$.*

4. Proof of the main result

The proof of Theorem 1.1 is quite similar to the arguments in previous papers [3, 4, 7, 8, 11]. It relies heavily on the fact that the estimated transversality of the r-jet of a section to a given submanifold of the jet bundle is a local and C^{r+1}-open property in the following sense [3]. Given a submanifold S of $\mathcal{J}^r E_k$, a constant $\eta > 0$ and a point $x \in X$, say that a section s of E_k satisfies the property $\mathcal{P}(S, \eta, x)$ if either the r-jet $j^r s(x)$ lies at distance more than η from S or $j^r s$ is η-transverse to S at x (in the sense of Definition 3.3, i.e. the minimum angle between the graph of $j^r s$ and the parallel distribution to S is at least η). The property $\mathcal{P}(S, \eta, x)$ depends only on the $(r+1)$-jet of s at x ("locality"). Moreover, if s satisfies $\mathcal{P}(S, \eta, x)$, then any section σ such that $|j^{r+1}\sigma(x) - j^{r+1}s(x)| < \epsilon$ satisfies $\mathcal{P}(S, \eta - C\epsilon, x)$, where C is some fixed constant involving only the curvature bounds of S ("openness").

A first consequence is that Theorem 1.1 can be proved by successively perturbing the given sections s_k in order to ensure transversality to the various strata. To show this, we first remark that, given any index b, the uniform transversality of $j^r s_k$ to all the strata S^a_k with $a \prec b$ implies its uniform transversality to S^b_k over a neighborhood of ∂S^b_k.

Indeed, first consider a pair of indices $a \prec b$ such that $S_k^a \subset \partial S_k^b$. By condition (5) of Definition 3.2, near a point of S_k^a the tangent space to S_k^b almost contains the parallel distribution to TS_k^a ; therefore, there exists a constant κ (independent of a and b) such that, for any small $\alpha > 0$, the α-transversality of $j^r s_k$ to S_k^a implies its $\frac{\alpha}{4}$-transversality to S_k^b over the $\kappa\alpha$-neighborhood of S_k^a. Next, consider a pair of indices $a \prec b$ and a point $p \in \partial S_k^b \cap (S_k^a - \Theta_{S_k^a})$: in this case, if the graph of $j^r s_k$ is α-transverse to S_k^a but intersects the ball of radius $\frac{\alpha}{2}$ around p, we can find an approximately holomorphic section σ_k of E_k differing from s_k by less than $\frac{3\alpha}{4}$ and whose jet goes through p. By definition of $\Theta_{S_k^a}$, all lifts of p in $\mathcal{J}^{r+1}E_k$, including $j^{r+1}\sigma_k$, correspond to local sections which intersect S_k^a non-transversely; because the antisymmetric and antiholomorphic terms in $\nabla(j^r\sigma_k)$ are smaller than $O(c_k^{-1/2})$, the minimum angle between $j^r\sigma_k$ and S_k^a at p is bounded by $O(c_k^{-1/2})$. However, since σ_k is close to s_k, its r-jet should be $\frac{\alpha}{4}$-transverse to S_k^a, which gives a contradiction. Therefore, $j^r s_k$ remains at distance more than $\frac{\alpha}{2}$ from p; this implies the $\frac{\alpha}{4}$-transversality to S_k^b of $j^r s_k$ over the $\frac{\alpha}{4}$-neighborhood of every point of $(S_k^a - \Theta_{S_k^a}) \cap \partial S_k^b$. Since these are the only two possible cases near the boundary of S_k^b, the uniform transversality of $j^r s_k$ to S_k^a for all $a \prec b$ implies its uniform transversality to S_k^b near ∂S_k^b.

Now, extend the binary relation $\prec$ on the set of strata of each $\mathcal{S}_k$ into a total order relation $<$, so that the indices in A_k can be identified with integers and the closure of a given stratum consists only of strata appearing before it. Assume that a first perturbation by less than $\delta_0 = \frac{\delta}{2}$ makes it possible to obtain for large k the η_1-transversality of $j^r s_k$ to the first stratum S_k^1, for some constant η_1 independent of k. Next, let δ_1 be a constant sufficiently smaller than δ and η_1 (but independent of k), and assume that a perturbation by at most δ_1 allows us to obtain the η_2-transversality of $j^r s_k$ to the second stratum S_k^2 outside of the $\frac{1}{4}\kappa\eta_1$-neighborhood of ∂S_k^2, for some constant η_2. Because this new perturbation is small enough, the resulting sections remain $\frac{\eta_1}{2}$-transverse to S_k^1 ; also, by the above observation this automatically implies the estimated transversality to S_k^2 of $j^r s_k$ near the points of $\partial S_k^2 \subseteq S_k^1$.

We can continue in this way until all strata have been considered ; each perturbation added to ensure estimated transversality to a new stratum outside of a small fixed size neighborhood of its boundary is chosen small enough in order not to affect the previously obtained transversality properties.

The fact that estimated transversality is local and open also makes it possible to reduce to a purely local setup, using a globalization principle due to Donaldson [7] and which can be formulated as follows (Proposition 3 of [3]) :

Proposition 4.1. *Let $\mathcal{P}_k(\eta, x)_{x\in X,\eta>0,k\gg 0}$ be local and C^{r+1}-open properties of sections of E_k over X. Assume that there exist constants c, c' and ν such that, given any $x \in X$, any small enough $\delta > 0$, and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections $\tau_{k,x}$ of E_k with the following properties :* (a) $|\tau_{k,x}|_{C^{r+1},g_k} < \delta$, (b) *the sections $\frac{1}{\delta}\tau_{k,x}$ have uniform Gaussian decay away from x, and* (c) *the sections $s_k + \tau_{k,x}$ satisfy the property $\mathcal{P}_k(\eta, y)$ for all $y \in B_{g_k}(x, c)$, with $\eta = c'\delta\log(\delta^{-1})^{-\nu}$.*

Then, given any $\alpha > 0$ and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections σ_k of E_k such that

$|s_k - \sigma_k|_{C^{r+1},g_k} < \alpha$ *and the sections* σ_k *satisfy* $\mathcal{P}_k(\epsilon, x)$ $\forall x \in X$ *for some* $\epsilon > 0$ *independent of* k.

Proposition 4.1 is in fact slightly stronger than the previous results, as the notion of asymptotic holomorphicity has been extended to a more general framework in §2, but the argument remains strictly the same.

With this result, we are reduced to the problem of finding a localized perturbation of s_k near a given point x in order to ensure transversality to a given stratum. More precisely, fix an index $a \in A_k$ in each stratification, and remember that, from the previous steps of the inductive argument, we can restrict ourselves to considering only asymptotically holomorphic sections whose jet is γ-transverse to the strata S_k^b for $b < a$, for some fixed constant γ (this constant γ is half of the transversality estimate obtained in the previous step ; by assumption we only consider perturbations which are small enough to preserve γ-transversality to the previous strata). With this understood, say that a section s_k satisfies $\mathcal{P}_k(\eta, x)$ if either $j^r s_k(x)$ lies at distance more than η from S_k^a, or $j^r s_k(x)$ lies at distance less than $\frac{1}{4}\kappa\gamma - \eta$ from ∂S_k^a, or $j^r s_k$ is η-transverse to S_k^a at x. We want to show that the assumptions of Proposition 4.1 are satisfied by these properties.

Fix a point $x \in X$ and a constant $0 < \delta < \frac{1}{20}\kappa\gamma$, and consider asymptotically holomorphic sections s_k of E_k. First, if $j^r s_k(x)$ lies at distance less than $\frac{3}{20}\kappa\gamma$ from a point of $\partial S_k^a \cap S_k^b$ for some $b \prec a$, then the uniform bounds on covariant derivatives of s_k imply that the graph of $j^r s_k$ remains within distance less than $\frac{1}{5}\kappa\gamma$ of this point over a ball of fixed radius c_1 (independent of k, x or δ) around x. So, the property $\mathcal{P}_k(\frac{1}{20}\kappa\gamma, y)$ holds at every point $y \in B_{g_k}(x, c_1)$, and no perturbation is needed. In the rest of the argument, we can therefore assume that $j^r s_k(x)$ lies at distance at least $\frac{3}{20}\kappa\gamma$ from ∂S_k^a.

Let $\epsilon = \frac{1}{10}\kappa\gamma$, and let ρ_ϵ be the radius appearing in Definition 3.2. Without loss of generality we can assume that $\rho_\epsilon < \epsilon$. Assume that $j^r s_k(x)$ lies at distance more than $\frac{1}{2}\rho_\epsilon$ from S_k^a. Then, the bounds on covariant derivatives of s_k imply that the graph of $j^r s_k$ remains at distance more than $\frac{1}{4}\rho_\epsilon$ from S_k^a over a ball of fixed radius c_2 around x, and therefore that s_k satisfies $\mathcal{P}_k(\frac{1}{4}\rho_\epsilon, y)$ at every point $y \in B_{g_k}(x, c_2)$. No perturbation is needed.

Therefore, we may assume that $j^r s_k(x)$ lies at distance less than $\frac{1}{2}\rho_\epsilon$ from a certain point $u_0 \in S_k^a$. We may also safely assume that $\delta < \frac{1}{4}\rho_\epsilon$. One easily checks that u_0 lies at distance more than ϵ from ∂S_k^a. So we can find complex-valued functions $f_1, \dots, f_p$ over the ball $B_{\hat{g}_k}(u_0, \rho_\epsilon)$ such that a local equation of S_k^a is $f_1 = \dots = f_p = 0$ and satisfying the various properties listed in Definition 3.2. Let c_3 be a fixed positive constant (independent of k, x and δ) such that the graph of $j^r s_k$ over $B_{g_k}(x, c_3)$ is contained in $B_{\hat{g}_k}(u_0, \frac{3}{4}\rho_\epsilon)$, and define the $\mathbb{C}^p$-valued function $h = (f_1 \circ j^r s_k, \dots, f_p \circ j^r s_k)$ over $B_{g_k}(x, c_3)$. By property (4) of Definition 3.2, the function h is C^2-approximately holomorphic.

Recall from Lemma 2.2 that there exist local approximately holomorphic ω_k-Darboux coordinates $z_1, \dots, z_n$ over a neighborhood of x in X. Also recall from Lemma 2.3 that there exist approximately holomorphic sections $s_{k,x,i}^{\mathrm{ref}}$ of E_k with Gaussian decay away from x and defining a local frame in E_k. For any $(n+1)$-tuple $I = (i_0, i_1, \dots, i_n)$ with $1 \le i_0 \le m$, $i_1, \dots, i_n \ge 0$, and $i_1 + \dots + i_n \le r$, we define $s^{\text{ref}}_{k,x,I} = z_1^{i_1} \dots z_n^{i_n} s^{\text{ref}}_{k,x,i_0}$. Clearly, these sections of E_k are asymptotically holomorphic and have uniform Gaussian decay away from x ; moreover it is easy to check that their r-jets define a local frame in $\mathcal{J}^r E_k$ near x. After multiplication by a suitable fixed constant factor, we can also assume that $|s^{\text{ref}}_{k,x,I}|_{C^{r+1},g_k} \le \frac{1}{p}$. For each tuple I, define a $\mathbb{C}^p$-valued function Θ_I by $\Theta_I(y) = (df_1(j^r s_k(y)).j^r s^{\text{ref}}_{k,x,I}(y), \dots, df_p(j^r s_k(y)).j^r s^{\text{ref}}_{k,x,I}(y))$. The functions Θ_I measure the variations of the function h when small multiples of the localized perturbations $s^{\text{ref}}_{k,x,I}$ are added to s_k ; by condition (4) of Definition 3.2, they are C^2-asymptotically holomorphic.

The fact that the jets of $s^{\text{ref}}_{k,x,I}$ define a frame of $\mathcal{J}^r E_k$ near x implies, by condition (2) of Definition 3.2, that the values $\Theta_I(x)$ generate all of $\mathbb{C}^p$. Moreover, for $1 \le i \le p$ there exist complex constants $\lambda_{I,i}$ with $\sum_I |\lambda_{I,i}| \le 1$ such that, defining the linear combinations $\sigma_{k,x,i} = \sum_I \lambda_{I,i} s^{\text{ref}}_{k,x,I}$ and $\Theta_i = \sum_I \lambda_{I,i}\Theta_I$, the quantity $|\Theta_1(x) \wedge \dots \wedge \Theta_p(x)|$ is larger than some fixed positive constant $\beta > 0$ depending only on ϵ (and not on k, x or δ). The uniform bounds on derivatives imply that, for some fixed constant $0 < c_4 < c_3$, the norm of $\Theta_1 \wedge \dots \wedge \Theta_p$ remains larger than $\frac{1}{2}\beta$ at every point of $B_{g_k}(x, c_4)$. Therefore, over this ball we can express h in the form $h = \mu_1\Theta_1 + \dots + \mu_p\Theta_p$, and the $\mathbb{C}^p$-valued function $\mu = (\mu_1, \dots, \mu_p)$ is easily checked to be C^2-AH as well.

Finally, use once more the local approximately holomorphic coordinates to identify $B_{g_k}(x, c_4)$ with a neighborhood of the origin in $\mathbb{C}^n$. After rescaling the coordinates by a fixed constant factor, we can assume that this neighborhood contains the ball B^+ of radius $\frac{11}{10}$ around the origin in $\mathbb{C}^n$, and that there exists a fixed constant $0 < c_5 < c_4$ such that the inverse image of the unit ball B in $\mathbb{C}^n$ contains $B_{g_k}(x, c_5)$. Composing μ with the coordinate map, we obtain a $\mathbb{C}^p$-valued function $\tilde{\mu}$ over B^+ ; by construction $\tilde{\mu}$ is C^2-AH.

We may now use the following local result, due to Donaldson [8] (the case $p = 1$ is an earlier result of Donaldson [7]; the comparatively much easier case $p > n$ is handled in [3]) :

Proposition 4.2 (Donaldson [8]). *Let f be a function with values in $\mathbb{C}^p$ defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$. Let δ be a constant with $0 < \delta < \frac{1}{2}$, and let $\eta = \delta \log(\delta^{-1})^{-\nu}$ where ν is a suitable fixed integer depending only on n and p. Assume that f satisfies the following bounds over B^+:*

$$|f| \le 1, \qquad |\bar\partial f| \le \eta, \qquad |\nabla\bar\partial f| \le \eta.$$

Then, there exists $w \in \mathbb{C}^p$, with $|w| \le \delta$, such that $f - w$ is η-transverse to 0 *over the interior ball B of radius* 1.

Let $\eta = \delta \log(\delta^{-1})^{-\nu}$ as in the statement of the proposition, and observe that, if k is large enough, the antiholomorphic derives of $\tilde{\mu}$, which are bounded by a fixed multiple of $c_k^{-1/2}$, are smaller than η. Therefore, if k is large enough we can apply Proposition 4.2 (after a suitable rescaling to ensure that $\tilde{\mu}$ is bouded by 1) and find a constant $w = (w_1, \dots, w_p) \in \mathbb{C}^p$, smaller than δ, such that $\tilde{\mu} - w$ is η-transverse to 0 over the unit ball B. Going back through the coordinate map, this implies that $\mu - w$ is $c'_1\eta$-transverse to 0 over $B_{g_k}(x, c_5)$ for some fixed constant c'_1. Multiplying by the functions $\Theta_1, \dots, \Theta_p$, we obtain that $h - (w_1\Theta_1 + \dots + w_p\Theta_p)$ is $c'_2\eta$-transverse to 0 over $B_{g_k}(x, c_5)$ for some fixed constant c'_2.

Let $\tau_{k,x} = -(w_1\sigma_{k,x,1}+\dots+w_p\sigma_{k,x,p})$: by construction, the sections $\tau_{k,x}$ of E_k are asymptotically holomorphic, their norm is bounded by δ, and they have uniform Gaussian decay properties. Let $\tilde{s}_k = s_k + \tau_{k,x}$, and observe that by construction the graph of $j^r\tilde{s}_k$ over $B_{g_k}(x, c_3)$ is contained in $B_{\hat{g}_k}(u_0, \rho_\epsilon)$. Define $\tilde{h} = (f_1 \circ j^r\tilde{s}_k, \dots, f_p \circ j^r\tilde{s}_k)$; by construction, and because of the bounds on second derivatives of $f_1, \dots, f_p$, we have the equality $\tilde{h} = h - (w_1\Theta_1 + \dots + w_p\Theta_p) + O(\delta^2)$. If δ is assumed to be small enough, the quadratic term in this expression is much smaller than η ; therefore, under this assumption we get that $\tilde{h}$ is $c_3'\eta$-transverse to 0 over $B_{g_k}(x, c_5)$ for some fixed constant c_3'. Finally, recalling the characterization of estimated transversality to a submanifold defined by local equations given at the end of §3.1, we conclude that the graph of $j^r\tilde{s}_k$ is $c_4'\eta$-transverse to S_k^a over $B_{g_k}(x, c_5)$ for some fixed constant c_4', i.e. $\tilde{s}_k$ satisfies the property $\mathcal{P}_k(c_4'\delta\log(\delta^{-1})^{-\nu}, y)$ at every point $y \in B_{g_k}(x, c_5)$.

Putting together the various possible cases (according to the distance between $j^r s_k(x)$ and S_k^a or its boundary), we obtain that the properties $\mathcal{P}_k$ satisfy the assumptions of Proposition 4.1. Therefore, for all large values of k a small perturbation can be added to s_k in order to achieve uniform transversality to S_k^a away from ∂S_k^a. The inductive argument described at the beginning of this section then makes it possible to complete the proof of Theorem 1.1.

The proof of Theorem 3.2 follows the same argument, but for one-parameter families of sections. One easily checks that the various results of §2 (Lemma 2.1, 2.2, 2.3) remain valid for families of objects depending continuously on a parameter $t \in [0, 1]$. Moreover, Propositions 4.1 and 4.2 also extend to the one-parameter case [8, 3]. So we only need to check that the argument used above to verify that the properties $\mathcal{P}_k$ satisfy the assumptions of Proposition 4.1 extends to the case of one-parameter families.

As before, fix a stratum $S_{k,t}^a$ in each stratification, a constant $\delta > 0$, a point $x \in X$, and asymptotically holomorphic sections $s_{k,t}$ of $E_{k,t}$. With the same notations as above, let $\Omega_k \subset [0, 1]$ be the set of values of t such that $j^r s_{k,t}(x)$ lies at distance more than $\frac{3}{20}\kappa\gamma$ from $\partial S_{k,t}^a$, and within distance $\frac{1}{2}\rho_\epsilon$ from $S_{k,t}^a$. Let $\Omega_k^- \subset \Omega_k$ be the set of values of t such that $j^r s_{k,t}(x)$ lies at distance more than $\frac{1}{5}\kappa\gamma$ from $\partial S_{k,t}^a$ and within distance $\frac{1}{4}\rho_\epsilon$ from $S_{k,t}^a$. Observe that, if $t \notin \Omega_k^-$, a certain uniform transversality property with respect to $S_{k,t}^a$ is already satisfied by $j^r s_{k,t}$ over a small ball centered at x, and therefore no specific perturbation is needed : if x lies within distance $\frac{1}{5}\kappa\gamma$ from $\partial S_{k,t}^a$, then $\mathcal{P}_k(\frac{1}{40}\kappa\gamma, y)$ is satisfied at every point of a ball of fixed radius, while if x lies at distance more than $\frac{1}{4}\rho_\epsilon$ from $S_{k,t}^a$ then $\mathcal{P}_k(\frac{1}{8}\rho_\epsilon, y)$ holds over a ball of fixed radius around x. Even better, if δ is small enough compared to γ and ρ_ϵ, then any perturbation of $s_{k,t}$ by less than δ still satisfies a similar transversality property (with decreased estimates).

For t in Ω_k, the proximity of $j^r s_{k,t}(x)$ to $S_{k,t}^a$ makes it possible to locally define complex-valued functions $f_{1,t}, \dots, f_{p,t}$ depending continuously on t and such that a local equation of $S_{k,t}^a$ is $f_{1,t} = \dots = f_{p,t} = 0$ (recall the definition of the continuous dependence of the stratifications $\mathcal{S}_{k,t}$ upon the parameter t given in §3.2). This lets us define as above the function $h_t = (f_{1,t} \circ j^r s_{k,t}, \dots, f_{p,t} \circ j^r s_{k,t})$, depending continuously on t. As in the non-parametric case, we can construct asymptotically holomorphic sections $s_{k,x,t,I}^{\mathrm{ref}}$ of $E_{k,t}$, with Gaussian decay away from x and defining local frames in $\mathcal{J}^r E_{k,t}$, simply by multiplying the sections of Lemma 2.3 by polynomials of degree at most r in the local coordinates (all these sections depend continuously on t). We can then find linear combinations $\sigma_{k,x,t,1},\dots,\sigma_{k,x,t,p}$ of the sections $s^{\mathrm{ref}}_{k,x,t,I}$, with constant coefficients depending continuously on t, such that, denoting by $\Theta_{t,i}$ the $\mathbb{C}^p$-valued functions expressing the variations of h_t upon adding small multiples of $\sigma_{k,x,t,i}$ to $s_{k,t}$, the norm of $\Theta_{t,1}\wedge\dots\wedge\Theta_{t,p}$ is bounded from below at x and over a small ball surrounding it.

Constructing the functions $\tilde{\mu}_t$ as in the proof of Theorem 1.1 and applying the one-parameter version of Proposition 4.2, we obtain, if k is large enough, a continuous one-parameter family of constants $w_t\in\mathbb{C}^p$, depending continuously on $t\in\Omega_k$ and bounded by δ for all t, such that $\tilde{\mu}_t-w_t$ is η-transverse to 0 over the unit ball in $\mathbb{C}^n$. It follows that, denoting by $\tau_{k,x,t}$ the asymptotically holomorphic perturbations $-(w_{t,1}\sigma_{k,x,t,1}+\dots+w_{t,p}\sigma_{k,x,t,p})$, bounded by δ, with Gaussian decay away from x, and depending continuously on $t\in\Omega_k$, the sections $s_{k,t}+\tau_{k,x,t}$ satisfy the desired transversality property over a small ball centered at x. However these perturbations are only well-defined for $t\in\Omega_k$. In order to extend their definition to all values of t, let $\chi_k:[0,1]\to[0,1]$ be a continuous cut-off function such that $\chi_k(t)=1$ for all $t\in\Omega_k^-$ and $\chi_k(t)=0$ for all $t\notin\Omega_k$, and let $\tilde{\tau}_{k,x,t}=\chi_k(t)\tau_{k,x,t}$ (for $t\notin\Omega_k$ we set $\tilde{\tau}_{k,x,t}=0$). For $t\in\Omega_k^-$ we have $\tilde{\tau}_{k,x,t}=\tau_{k,x,t}$, so the sections $s_{k,t}+\tilde{\tau}_{k,x,t}$ satisfy the required transversality property ; for $t\notin\Omega_k^-$, the sections $s_{k,t}$ already satisfy such a property and, because we have assumed δ to be small enough, transversality is not affected by adding $\tilde{\tau}_{k,x,t}$. Therefore, the assumptions of Proposition 4.1 are satisfied even in the one-parameter setting, and we can conclude the argument in the same way as in the non-parametric case.

Remark. In many cases, Theorems 1.1 and 3.2 can be proved without using Proposition 4.2 (estimated Sard lemma) in its full generality. Indeed, given suitable asymptotically holomorphic quasi-stratifications $\mathcal{S}_k$ of $\mathcal{J}^rE_k$, we can define quasi-stratifications $\tilde{\mathcal{S}}_k$ of $\mathcal{J}^{r+1}E_k$ in the following way. View each element of $\mathcal{J}^{r+1}E_k$ as a 1-jet in $\mathcal{J}^rE_k$, as in §3.2; for each stratum S_k^a of $\mathcal{S}_k$ with codimension greater than n in $\mathcal{J}^rE_k$, let $\tilde{S}_k^a$ be the set of points in $\mathcal{J}^{r+1}E_k$ whose r-jet component belongs to S_k^a. For each stratum S_k^a of $\mathcal{S}_k$ with codimension $p\le n$, and for each value $0\le i\le p-1$, let $\tilde{S}_k^{a,i}$ be the set of points in $\mathcal{J}^{r+1}E_k$ whose r-jet component belongs to S_k^a and such that the corresponding element in $T^*X^{(1,0)}\otimes\mathcal{J}^rE_k$, after projection to the normal space to TS_k^a, has rank equal to i. In other terms, the union of $\tilde{S}_k^{a,i}$ is the set of $(r+1)$-jets which intersect S_k^a non-transversely.

In a large number of examples, those of the $\tilde{S}_k^{a,i}$ which are not empty are approximately holomorphic submanifolds of $\mathcal{J}^{r+1}E_k$, transverse to the fibers and of codimension at least $n+1$. These submanifolds determine finite Whitney quasi-stratifications $\tilde{\mathcal{S}}_k$ of $\mathcal{J}^{r+1}E_k$, satisfying properties similar to those of Definition 3.2 but with C^1 estimates only instead of C^2 bounds. Still, the same argument as in the proof of Theorem 1.1 shows that, given asymptotically holomorphic sections s_k of E_k, small perturbations can be added for large enough k in order to ensure the uniform transversality of $j^{r+1}s_k$ to $\tilde{\mathcal{S}}_k$; the argument only uses Proposition 4.2 in the case $p>n$, where the proof becomes much easier [3] and C^1 bounds are sufficient. Because all the strata are of codimension greater than n, the η-transversality of $j^{r+1}s_k$ to $\tilde{\mathcal{S}}_k$ simply means that the graph of $j^{r+1}s_k$ remains at distance more than η from the strata of $\tilde{\mathcal{S}}_k$. By definition of $\tilde{\mathcal{S}}_k$, this is equivalent to the uniform transversality of j^rs_k to $\mathcal{S}_k$, which was the desired result.

5. Examples and applications

We now consider various examples of (quasi)-stratifications to which we can apply Theorems 1.1 and 3.2. The fact that they are asymptotically holomorphic is in all cases a direct consequence of Proposition 3.1.

To make things more topological, we place ourselves in the case where the almost-complex structure J on X is tamed by a given symplectic form ω. In this context, the various approximately J-holomorphic submanifolds of X appearing in the constructions are automatically symplectic with respect to ω. Moreover, remember that the space of ω-tame or ω-compatible almost-complex structures on X is contractible. In most applications, asymptotically very ample bundles are constructed from line bundles with first Chern class proportional to $[\omega]$; in that situation, the ampleness properties of these bundles do not depend on the choice of an ω-compatible almost-complex structure J. Theorem 3.2 then implies that all the constructions described below are, for large enough values of k, canonical up to isotopy, independently of the choice of J. In the general case, the constructions are still canonical up to isotopy, but the space of possible choices for J is constrained by the necessity for the bundles E_k to be ample.

The first application is the construction of symplectic submanifolds as zero sets of asymptotically holomorphic sections of vector bundles over X, as initially obtained by Donaldson [7] and later extended to a slightly more general setting [2].

Corollary 5.1. *Let (X,ω) be a compact symplectic manifold endowed with an ω-tame almost-complex structure J, and let E_k be an asymptotically very ample sequence of locally splittable vector bundles over (X,J). Then, for all large enough values of k there exist asymptotically holomorphic sections s_k of E_k which are uniformly transverse to 0 and whose zero sets are smooth symplectic manifolds in X. Moreover these sections and submanifolds are, for large k, canonical up to isotopy, indepedently of the chosen almost-complex structure on X.*

Proof. Let $\mathcal{S}_k$ be the stratification of $\mathcal{J}^0 E_k = E_k$ in which the only stratum is the zero section of E_k (these stratifications are obviously asymptotically holomorphic). By Theorem 1.1, starting from any asymptotically holomorphic sections of E_k (e.g. the zero sections) we can obtain for large k asymptotically holomorphic sections of E_k which are uniformly transverse to $\mathcal{S}_k$, i.e. uniformly transverse to 0. It is then a simple observation that the zero sets of these sections are, for large k, smooth approximately J-holomorphic (and therefore symplectic) submanifolds of X [7]. Finally, the uniqueness of the construction up to isotopy is a direct consequence of the one-parameter result Theorem 3.2 [2]. □

The next example is that of determinantal submanifolds as constructed by Muñoz, Presas and Sols [11].

Corollary 5.2. *Let (X,ω) be a compact symplectic manifold endowed with an ω-tame almost-complex structure J, let L_k be an asymptotically very ample sequence of line bundles over (X,J), and let E and F be complex vector bundles over X. Then, for all large enough values of k there exist asymptotically holomorphic sections s_k of $E^* \otimes F \otimes L_k$ such that the determinantal loci $\Sigma_i(s_k) = \{x \in X,\ \mathrm{rk}(s_k(x)) = i\}$ are stratified symplectic submanifolds in X. Moreover these sections and submanifolds* *are, for large k, canonical up to isotopy, indepedently of the chosen almost-complex structure on X.*

Proof. Let $E_k = E^* \otimes F \otimes L_k$, and let $\mathcal{S}_k$ be the stratification of $\mathcal{J}^0 E_k = E_k$ consisting of strata S_k^i, $0 \le i < \min(\mathrm{rk}\, E, \mathrm{rk}\, F)$, defined as follows : viewing the points of E_k as elements of $\mathrm{Hom}(E, F)$ with coefficients in L_k, each S_k^i is the set of all elements in E_k whose rank is equal to i. By Proposition 3.1, the stratifications $\mathcal{S}_k$ are asymptotically holomorphic. Applying Theorem 1.1 to these stratifications and starting from the zero sections, we obtain asymptotically holomorphic sections of E_k which are uniformly transverse to $\mathcal{S}_k$. The determinantal locus $\Sigma_i(s_k)$ is precisely the set of points where the graph of s_k intersects the stratum S_k^i. The result of uniqueness up to isotopy is obtained by applying Theorem 3.2. □

However, our main application is that of maps to projective spaces. Observe that, given a section $s = (s_1, \dots, s_{m+1})$ of a vector bundle of the form $\mathbb{C}^{m+1} \otimes L$, where L is a line bundle over X, we can construct away from its zero set a projective map $\mathbb{P}s = (s_1 : \ldots : s_{m+1}) : X - s^{-1}(0) \to \mathbb{CP}^m$.

Recall that the space of jets of holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ carries a natural partition into submanifolds, the Boardman "stratification" [1, 6]. Restricting oneself to *generic* r-jets, the strata Σ_I, labelled by r-tuples $I = (i_1, \dots, i_r)$ with $i_1 \ge \cdots \ge i_r \ge 0$, are defined in the following way. Given a generic holomorphic map f, call $\Sigma_i(f)$ the set of points where $\dim \mathrm{Ker}\, df = i$, and denote by Σ_i the set of holomorphic 1-jets corresponding to such points (i.e., Σ_i is the set of 1-jets (σ_0, σ_1) such that $\dim \mathrm{Ker}\, \sigma_1 = i$). The submanifolds Σ_i determine a stratification of $\mathcal{J}^1_{n,m}$. For a generic holomorphic map f the critical loci $\Sigma_I(f)$ are smooth submanifolds defining a partition of $\mathbb{C}^n$. Therefore, we can define inductively $\Sigma_{i_1,\dots,i_r}(f)$ as the set of points of $\Sigma_{i_1,\dots,i_{r-1}}(f)$ where the kernel of the restriction of df to $T\Sigma_{i_1,\dots,i_{r-1}}(f)$ has dimension i_r (in particular, $\Sigma_{i_1,\dots,i_{r-1},0}(f)$ is open in $\Sigma_{i_1,\dots,i_{r-1}}(f)$ and corresponds to the set of points where f restricts to $\Sigma_{i_1,\dots,i_{r-1}}(f)$ as an immersion).

It is easy to check that the r-jet of f at a given point of $\mathbb{C}^n$ completely determines in which $\Sigma_I(f)$ it lies ; therefore, one can define $\Sigma_I \subset \mathcal{J}^r_{n,m}$ as the set of r-jets $j^r f(x)$ of generic holomorphic maps $f : \mathbb{C}^n \to \mathbb{C}^m$ at points $x \in \Sigma_I(f)$. In other terms, $\Sigma_I(f) = \{x \in \mathbb{C}^n,\ j^r f(x) \in \Sigma_I\}$. It is a classical result [6] that the Σ_I's are smooth submanifolds and define a partition of the space of generic holomorphic r-jets (an open subset in $\mathcal{J}^r_{n,m}$ whose complement has codimension $\ge n+1$), which can be extended into a partition of $\mathcal{J}^r_{n,m}$ by smooth submanifolds.

The Boardman classes Σ_I play a fundamental role in singularity theory, and they completely determine the classification of singularities in certain dimensions. For low enough values of r, m or n, the submanifolds Σ_I define a genuine stratification of the jet space $\mathcal{J}^r_{n,m}$. However, as observed by Boardman, things become more complicated as the dimension increases, and the boundary of Σ_I is in general not a union of entire strata ; in high dimensions Boardman classes do not even define a quasi-stratification.

Still, there exist well-known methods that allow Boardman's partitions to be refined into finite Whitney stratifications of $\mathcal{J}^r_{n,m}$. An example of such a construction can be found in the work of Mather [10] (the constructed object is tautologically a finite Whitney stratification, and one easy checks that each Boardman class is a union of several of its strata).

We now consider the case of maps to projective spaces defined by asymptotically holomorphic sections of $E_k = \mathbb{C}^{m+1} \otimes L_k$ over X. We want to construct a natural approximately holomorphic analogue of the Thom-Boardman stratifications, by defining certain submanifolds in $\mathcal{J}^r E_k$. In order to make things easier by avoiding a lengthy analysis of the boundary structure at the points where the vanishing of the section prevents the definition of a projective map, our aim will only be to construct quasi-stratifications of $\mathcal{J}^r E_k$ rather than genuine stratifications.

We first define $Z = \{(\sigma_0, \dots, \sigma_r) \in \mathcal{J}^r E_k,\ \sigma_0 = 0\}$, i.e. Z is the set of r-jets of sections which vanish at the considered point. As observed in §3.2, Θ_Z consists of all points of Z such that σ_1 is surjective. Next, observe that any point $(\sigma_0, \dots, \sigma_r) \in \mathcal{J}^r E_k$ which does not belong to Z determines the (symmetric) holomorphic r-jet $(\phi_0, \dots, \phi_r)$ of a map to $\mathbb{CP}^m$: $\phi_0 \in \mathbb{CP}^m$, $\phi_1 \in T_x^* X^{1,0} \otimes T_{\phi_0}\mathbb{CP}^m$, $\dots$, $\phi_r \in (T_x^* X^{1,0})^{\otimes r}_{\mathrm{sym}} \otimes T_{\phi_0}\mathbb{CP}^m$ are defined in terms of $\sigma_0, \dots, \sigma_r$ by expressions involving the projection map from $\mathbb{C}^{m+1} - \{0\}$ to $\mathbb{CP}^m$ and its derivatives. In fact, one easily checks that, if $(\sigma_0, \dots, \sigma_r) = j^r s$ is the symmetric holomorphic part of the r-jet of a section of E_k, then $(\phi_0, \dots, \phi_r) = j^r f$ is the symmetric holomorphic part of the r-jet of the corresponding projective map. Using this notation, define

$$\Sigma_i = \{(\sigma_0, \dots, \sigma_r) \in \mathcal{J}^r E_k,\ \sigma_0 \neq 0,\ \dim \operatorname{Ker} \phi_1 = i\}.$$

For $\max(0, n-m) < i \leq n$, one easily checks that Σ_i is a smooth submanifold of $\mathcal{J}^r E_k$, and that $\partial\Sigma_i$ is the union of $\bigcup_{j>i} \Sigma_j$ and a subset of $Z - \Theta_Z$: indeed, observe that if $n \geq m$, then for any $(\sigma_0, \dots, \sigma_r) \in \overline{\Sigma}_i \cap Z$ we have $\dim \operatorname{Ker} \sigma_1 \geq i - 1 > n - (m+1)$ and therefore σ_1 is not surjective, while in the case $n < m$ dimensional reasons prevent σ_1 from being surjective.

Next, we assume that $r \geq 2$, and observe that Θ_{Σ_i} is the set of points $(\sigma_0, \dots, \sigma_r) \in \Sigma_i$ such that

$$\Xi_{i;(\sigma_0,\dots,\sigma_r)} = \{u \in T_x X^{1,0},\ (\iota_u \sigma_1, \dots, \iota_u \sigma_r, 0) \in T_{(\sigma_0,\dots,\sigma_r)}\Sigma_i\}$$

has the expected codimension in $T_x X^{1,0}$ (i.e., the same codimension as Σ_i in $\mathcal{J}^r E_k$). Indeed, by definition $(\sigma_0, \dots, \sigma_r)$ belongs to Θ_{Σ_i} if and only if the $(r+1)$-jet $(\sigma_0, \dots, \sigma_r, 0)$, viewed as a 1-jet in $\mathcal{J}^r E_k$, intersects Σ_i transversly (because the definition of Σ_i involves only σ_0 and σ_1, the choice of a lift in $\mathcal{J}^{r+1} E_k$ does not matter, so we can choose the $(r+1)$-tensor component to be zero). By convention (see §3.2), this element of $\mathcal{J}^{r+1} E_k$ corresponds to the 1-jet of a local section σ of $\mathcal{J}^r E_k$ satisfying, at the given point $x \in X$, $\sigma(x) = (\sigma_0, \dots, \sigma_r)$ and $\nabla\sigma(x) = (\sigma_1, \dots, \sigma_{r+1})$: the covariant derivative contains no antiholomorphic or antisymmetric terms. The graph of σ intersects Σ_i transversely if and only if $\{u \in TX,\ \nabla\sigma(x).u \in T_{\sigma(x)}\Sigma_i\}$ has the expected dimension, hence the above criterion.

With this understood, we can define inductively, for $p + 1 \leq r$,

$$\Sigma_{i_1,\dots,i_{p+1}} = \{\sigma \in \Theta_{\Sigma_{i_1,\dots,i_p}},\ \dim(\operatorname{Ker} \phi_1 \cap \Xi_{(i_1,\dots,i_p);\sigma}) = i_{p+1}\},$$

where $\Xi_{I;\sigma} = \{u \in T_x X^{1,0},\ (\iota_u \sigma_1, \dots, \iota_u \sigma_r, 0) \in T_{(\sigma_0,\dots,\sigma_r)}\Sigma_I\}$ as above, and Θ_{Σ_I} again consists of all points $\sigma \in \Sigma_I$ such that $\Xi_{I;\sigma}$ has the same codimension in $T_x X^{1,0}$ as Σ_I in $\mathcal{J}^r E_k$.

For $i_1 \geq \dots \geq i_{p+1} \geq 1$, $\Sigma_{i_1,\dots,i_{p+1}}$ is a smooth submanifold in $\mathcal{J}^r E_k$, and its closure inside $\Sigma_{i_1,\dots,i_p}$ is obtained by adding $\bigcup_{j>i_{p+1}} \Sigma_{i_1,\dots,i_p,j}$ and a subset of $\Sigma_{i_1,\dots,i_p} - \Theta_{\Sigma_{i_1,\dots,i_p}}$. However, it is quite difficult to fully understand the boundary structure of $\Sigma_{i_1,\dots,i_{p+1}}$; the situation is exactly the same as in standard Boardman theory for holomorphic jets, except that, besides pieces of $\Sigma_{j_1,\dots,j_q}$ where $q \le p+1$ and $(j_1,\dots,j_q) \ge (i_1,\dots,i_q)$ for the lexicographic order, the boundary of $\Sigma_{i_1,\dots,i_{p+1}}$ also contains a subset of $Z - \Theta_Z$.

In low dimensions and/or for low values of r, it can be checked that the submanifolds $Z, \Sigma_i, \Sigma_{i_1,i_2}, \dots, \Sigma_{i_1,\dots,i_r}$ determine a finite Whitney quasi-stratification of $\mathcal{J}^r E_k$; for example when $r = 1$ this is an immediate consequence of the above discussion.

However, in larger dimensions it is necessary to refine Boardman's construction as in the holomorphic case. The important observation is that, when $\mathcal{J}^r E_k$ is trivialized by choosing local asymptotically holomorphic coordinates and sections, the partition of $\mathcal{J}^r E_k - Z$ described above corresponds exactly to the partition of the space of r-jets of maps to $\mathbb{CP}^m$ given by Boardman classes. Therefore, we can circumvent the problem by refining Boardman's partition of $\mathcal{J}^r_{n,m}$ into a genuine stratification as explained above, lifting it by the projectivization map to a stratification of the space of non-vanishing jets in $\mathcal{J}^r_{n,m+1}$, and finally pull it back to obtain a stratification of $\mathcal{J}^r E_k - Z$. As in the holomorphic case, the Σ_I classes are realized as unions of strata ; therefore, transversality to this stratification implies transversality to the Σ_I's. Moreover, all strata (except for the open one which we discard anyway) are contained in the closure of Σ_1, so that the boundary structures near Z are entirely contained in $Z - \Theta_Z$; therefore adding Z to this stratification yields a quasi-stratification of $\mathcal{J}^r E_k$.

Definition 5.1. *Given asymptotically very ample line bundles L_k over (X^{2n}, J), and setting $E_k = \mathbb{C}^{m+1} \otimes L_k$, the* Boardman stratification *of $\mathcal{J}^r E_k$ is the quasi-stratification given by the submanifold Z and by a refined Thom-Boardman stratification of $\mathcal{J}^r E_k - Z$.*

Corollary 5.3. *Let (X, ω) be a compact symplectic manifold endowed with an ω-tame almost-complex structure J, let L_k be an asymptotically very ample sequence of line bundles over (X, J), and let $E_k = \mathbb{C}^{m+1} \otimes L_k$. Then, for all large enough values of k there exist asymptotically holomorphic sections s_k of E_k such that the r-jets $j^r s_k$ are uniformly transverse to the Boardman stratifications of $\mathcal{J}^r E_k$.*

In particular, the zero sets $Z_k = s_k^{-1}(0)$ are smooth symplectic codimension $2m$ submanifolds in X, and the holomorphic r-jets of the projective maps $f_k = \mathbb{P}s_k : X - Z_k \to \mathbb{CP}^m$ behave at every point in a manner similar to those of generic holomorphic maps from a complex n-fold to $\mathbb{CP}^m$. Moreover, the singular loci $\Sigma_I(f_k) = \{x \in X - Z_k,\ j^r f_k(x) \in \Sigma_I\}$ are smooth symplectic submanifolds of the expected codimension and define a partition of $X - Z_k$. Finally, the sections s_k and the maps f_k are, for large k, canonical up to isotopy, independently of the chosen almost-complex structure on X.

Proof. By construction the Boardman stratifications of $\mathcal{J}^r E_k$ satisfy the assumptions of Proposition 3.1, as in every fiber of $\mathcal{J}^r E_k$ they can be identified with the same holomorphic quasi-stratification of $\mathcal{J}^r_{n,m+1}$. As a consequence, they are asymptotically holomorphic, and the existence of asymptotically holomorphic sections of E_k with the desired transversality properties is an immediate consequence of Theorem 1.1. The properties of Z_k follow immediately from the uniform transversality to the stratum Z of vanishing sections, while the properties of f_k are direct consequences of the uniform transversality to the Boardman strata (recall that each Σ_I is smooth and is a union of strata). Finally, the uniqueness result is obtained by applying Theorem 3.2. □

Corollary 5.3 is, in a certain sense, a fundamental result of asymptotically holomorphic singularity theory. Still, it falls short of the natural goal that one may have in mind at this point, namely the construction of approximately holomorphic projective maps which are near every point of X topologically conjugate in approximately holomorphic coordinates to generic holomorphic maps between complex manifolds.

Indeed, in order to achieve such a result, one needs to obtain some control on the antiholomorphic part of the jet of f_k at the points of the singular loci $\Sigma_I(f_k)$: roughly speaking, $\bar\partial f_k$ must be much smaller than ∂f_k in every direction and at every point, and when ∂f_k is singular this is no longer an immediate consequence of asymptotic holomorphicity and transversality. Note however that the behavior of f_k near the set of base points Z_k is always the expected one.

In many cases, it is possible to perturb slightly the sections s_k (by less than a fixed multiple of $c_k^{-1/2}$, which affects neither holomorphicity nor transversality properties) along the singular loci in order to obtain the proper topological picture for f_k.

The easiest case is $m \geq 2n$, where it is enough to consider 1-jets, and all the strata turn out to be of codimension greater than n ; the uniform transversality of s_k to the Boardman stratification then implies that the maps f_k are approximately holomorphic immersions. Moreover, when $m \geq 2n+1$ an arbitrarily small perturbation is enough to get rid of multiple points, thus giving approximately holomorphic embeddings into projective spaces, a result already obtained by Muñoz, Presas and Sols [11].

Next, we can consider the case $m = 1$, where 1-jets are again sufficient, and the only interesting Boardman stratum is Σ_n, of complex codimension n, corresponding to critical points of $\mathbb{CP}^1$-valued maps. The sections of $\mathbb{C}^2 \otimes L_k$ given by Corollary 5.3 vanish along smooth codimension 4 base loci ; moreover, the differential ∂f_k of the $\mathbb{CP}^1$-valued map f_k only vanishes at isolated points, and does so in a non-degenerate way. These transversality properties are precisely those imposed by Donaldson in his construction of symplectic Lefschetz pencils [8]; the only missing ingredient is an extra perturbation near the zeroes of ∂f_k in order to get rid of the antiholomorphic terms and therefore ensure that they are genuine non-degenerate critical points, thus making f_k a complex Morse function.

The last case we will consider is when $m = 2$. In this case, we need to consider 2-jets, and the relevant Boardman strata are Σ_{n-1}, of complex codimension $n-1$, and $\Sigma_{n-1,1}$, of complex codimension n (the other strata have codimension greater than n). The sections of $\mathbb{C}^3 \otimes L_k$ constructed by Corollary 5.3 vanish along smooth codimension 6 base loci. The $\mathbb{CP}^2$-valued maps f_k are submersions outside of the smooth symplectic curves $R_k = \Sigma_{n-1}(f_k)$, and the restriction of f_k to R_k is an immersion except at the points of $C_k = \Sigma_{n-1,1}(f_k)$. After a suitable perturbation in order to ensure the vanishing of some antiholomorphic derivatives of f_k along R_k, one obtains a situation similar to that described in previous papers [3, 4]: at every point of $R_k - C_k$, a local model for f_k in approximately holomorphic coordinates is $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$, while at the points of C_k the local model becomes $(z_1, \dots, z_n) \mapsto (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$ and the symplectic curve $f_k(R_k) \subset \mathbb{CP}^2$ presents an isolated cusp singularity.

In the general case, the most promising strategy to achieve topological conjugacy to generic holomorphic local models is to perturb the sections s_k in order to make sure that, along each stratum $\Sigma_I(f_k)$, the germ of f_k is holomorphic along the normal directions to $\Sigma_I(f_k)$. Such perturbations should be relatively easy to construct by methods similar to those in the above-mentioned papers [3, 4], provided that one starts from the strata of lowest dimension. This approach will be developped in a forthcoming paper.

Finally, let us formulate some natural extensions of Corollary 5.3 to more general situations. First, we mention the case when the asymptotically very ample line bundles L_k are replaced by vector bundles of rank $\nu \geq 2$. In that case, and provided that $m \geq \nu$, the projective maps defined by sections s_k of $\mathbb{C}^{m+1} \otimes L_k$ are replaced by maps $\mathrm{Gr}(s_k)$ taking values in the Grassmannian $\mathrm{Gr}(\nu, m+1)$ of ν-planes in $\mathbb{C}^{m+1}$, defined at every point of X where the $m+1$ chosen sections generate the whole fiber of L_k. More precisely, at every such point there exist $m+1-\nu$ independent linear relations between the $m+1$ components $s_k^1, \dots, s_k^{m+1}$, and these $m+1-\nu$ linear equations in $m+1$ variables determine a ν-dimensional complex subspace $\mathrm{Gr}(s_k)$ in $\mathbb{C}^{m+1}$. By adapting Corollary 5.3 to this situation, it is for example possible to recover the Grassmannian embedding result of Muñoz, Presas and Sols [11].

Another direction in which Corollary 5.3 can be improved is by adding extra transversality requirements to the projective maps f_k. For example, given a stratified holomorphic submanifold $\mathcal{D} = (D_a)_{a \in A}$ in $\mathbb{CP}^m$, we can require the transversality of the map f_k to $\mathcal{D}$. Indeed, $\mathcal{D}$ induces a stratification $\tilde{\mathcal{D}}_k$ of $\mathcal{J}^r E_k$, in which each stratum consists of the jets $(\sigma_0, \dots, \sigma_r)$ such that $\mathbb{P}\sigma_0$ belongs to a certain stratum D_a of $\mathcal{D}$ (in fact, this stratification only involves the 0-jet part). Starting from sections s_k of E_k given by Corollary 5.3, we can apply Theorem 1.1 to the stratifications $\tilde{\mathcal{D}}_k$ (which one easily shows to to be asymptotically holomorphic by Proposition 3.1) ; this yields asymptotically holomorphic sections $\tilde{s}_k$ which are uniformly transverse to $\tilde{\mathcal{D}}_k$ but differ from s_k by an amount small enough to ensure that the transversality of the jets to the Boardman stratification is preserved. In this way, one obtains projective maps which have the same properties as in Corollary 5.3 and additionally are uniformly transverse to the stratified submanifold $\mathcal{D}$. This extends a result of Muñoz, Presas and Sols [11] where asymptotically holomorphic embeddings are made transverse to a given submanifold of $\mathbb{CP}^m$.

Another class of stratifications of $\mathcal{J}^r E_k$ that we can consider are those obtained from lower-dimensional Boardman stratifications by linear projections. Namely, fix $q < m$, and let $\pi : \mathbb{C}^{m+1} \to \mathbb{C}^{q+1}$ be a linear projection ; π induces maps $\tilde{\pi} : \mathcal{J}^r(\mathbb{C}^{m+1} \otimes L_k) \to \mathcal{J}^r(\mathbb{C}^{q+1} \otimes L_k)$, and the inverse images by $\tilde{\pi}$ of the Boardman stratifications of $\mathcal{J}^r(\mathbb{C}^{q+1} \otimes L_k)$ are asymptotically holomorphic quasi-stratifications of $\mathcal{J}^r(\mathbb{C}^{m+1} \otimes L_k)$. The transversality of $j^r s_k$ to these quasi-stratifications is equivalent to that of $j^r(\pi(s_k))$ to the Boardman stratifications ; denoting by $\bar{\pi}$ the map from $\mathbb{CP}^m$ to $\mathbb{CP}^q$ induced by π, this is also equivalent to the genericity of the holomorphic jets of the projective maps $\bar{\pi} \circ f_k$. Therefore, by applying Theorem 1.1 as in the previous example, we can obtain projective maps f_k with the same genericity properties as in Corollary 5.3 and such that the maps $\bar{\pi} \circ f_k$ also enjoy similar properties. Even better, by iteratedly applying Theorem 1.1 we can obtain the same property for any given finite family of linear projections. For example, when $m = 2$ and considering projections of $\mathbb{C}^3$ to $\mathbb{C}^2$ along coordinate axes, one obtains exactly the transversality properties which are needed in order to extend Moishezon-Teicher braid group techniques to the study of symplectic manifolds [5, 4].

To conclude, let us mention a different class of potential applications of Theorem 1.1, following the ideas of Donaldson and Smith. As shown by Donaldson [8], any compact symplectic 4-manifold carries structures of symplectic Lefschetz pencils obtained from pairs of sections of asymptotically very ample line bundles L_k ; after blowing up the base points, we obtain Lefschetz fibrations over $\mathbb{CP}^1$, which may also be thought of as maps from $\mathbb{CP}^1$ to the moduli space $\bar{M}_g$ of stable curves of a certain genus g. These maps become asymptotically holomorphic as one considers pencils given by sections of L_k for $k \to +\infty$. In a largely unexplored class of constructions, one considers certain vector bundles over $\mathbb{CP}^1$ naturally arising from the Lefschetz fibrations : for example, spaces of holomorphic sections of certain bundles over each fiber, or pull-backs by the maps from $\mathbb{CP}^1$ to $\bar{M}_g$ of vector bundles over $\bar{M}_g$. It often turns out that these bundles over $\mathbb{CP}^1$ either are naturally asymptotically very ample or become so after tensor product by the line bundles $O(k)$. Theorem 1.1 can then used in order to obtain sections with suitable genericity properties, which in turn give rise to interesting geometric or topological structures. In some cases the objects naturally arising are sheaves rather than bundles, but the same type of argument should remain valid. It is to be expected that some interesting results about symplectic 4-manifolds and Lefschetz pencils can be obtained in this way, as similar considerations (but at a much more sophisticated level) have for example led to Donaldson and Smith's proof of the existence of a pseudo-holomorphic curve realizing the canonical class via Lefschetz fibrations [9].

References

[1] V. I. Arnold, S. M. Gusein-Zadeh, A. N. Varchenko, *Singularities of smooth mappings*, Monographs in Mathematics, Birkhäuser (1982).

[2] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[3] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$, Invent. Math. **139** (2000), 551–602.

[4] D. Auroux, *Symplectic maps to projective spaces and symplectic invariants*, to appear in Proc. 7th Gökova Geometry-Topology Conference (2000), International Press.

[5] D. Auroux, L. Katzarkov, *Branched coverings of* $\mathbb{CP}^2$ *and invariants of symplectic 4-manifolds*, to appear in Invent. Math.

[6] J. M. Boardman, *Singularities of differentiable maps*, Publ. Math. IHES, **33** (1967), 21–57.

[7] S. K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. **44** (1996), 666–705.

[8] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, preprint.

[9] S. K. Donaldson, I. Smith, in preparation.

[10] J. N. Mather, *How to stratify mappings and jet spaces*, Singularités d'Applications Différentiables (Plans-sur-Bex 1975), Lecture Notes in Math. **535**, Springer (1976), 128–176.

[11] V. Muñoz, F. Presas, I. Sols, *Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds*, preprint.

SYMPLECTIC HYPERSURFACES IN THE COMPLEMENT OF AN ISOTROPIC SUBMANIFOLD

DENIS AUROUX, DAMIEN GAYET, AND JEAN-PAUL MOHSEN

Abstract. Using Donaldson's approximately holomorphic techniques, we construct symplectic hypersurfaces lying in the complement of any given compact isotropic submanifold of a compact symplectic manifold. We discuss the connection with rational convexity results in the Kähler case and various applications.

1. Introduction

It was first observed by Duval (see e.g. [Du]) that, in Kähler geometry, the notions of isotropy and rational convexity are tightly related to each other. Recall that a compact subset N of $\mathbb{C}^n$ or more generally of a complex algebraic manifold is said to be *rationally convex* if there exists a complex algebraic hypersurface passing through any given point in the complement of N and avoiding N. Among the results motivating the interest in this notion, one can mention the classical theorem of Oka and Weil (further improved by subsequent work) stating that every holomorphic function over a neighborhood of a rationally convex compact subset $N \subset \mathbb{C}^n$ can be uniformly approximated over N by rational functions.

It was shown in 1995 by Duval and Sibony that, if a smooth compact submanifold of $\mathbb{C}^n$ is isotropic with respect to some Kähler structure on $\mathbb{C}^n$, then it is rationally convex [DS]. This result was extended in 1999 by Guedj to the context of complex projective manifolds :

Theorem 1 (Guedj [Gu]). *Let (X, ω, J) be a closed Kähler manifold, such that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ is integral. Then any smooth compact isotropic submanifold $\mathcal{L} \subset X$ (possibly with boundary) is rationally convex, i.e. there exist complex hypersurfaces in X passing through any given point in the complement of $\mathcal{L}$ and avoiding $\mathcal{L}$.*

Because the concept of isotropic submanifold originates in symplectic geometry, it is natural to seek an analogue of this result for symplectic manifolds. Although the lack of an integrable almost-complex structure prevents the existence of holomorphic hypersurfaces in a general symplectic manifold, a suitable analogue may be found in Donaldson's construction of approximately holomorphic symplectic hypersurfaces.

Let (X, ω) be a closed compact symplectic manifold of real dimension $2n$. Unless otherwise stated, we will always assume that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ is integral ; this does not restrict the diffeomorphism type of X in any way. A compatible almost-complex structure J on X and the corresponding Riemannian metric g are also fixed.

Let L be a complex line bundle on X with first Chern class $c_1(L) = \frac{1}{2\pi}[\omega]$, endowed with a Hermitian structure and a Hermitian connection ∇^L whose curvature 2-form is $-i\omega$. It was shown by Donaldson in [D1] that, when the integer k is large enough, the line bundles $L^{\otimes k}$ admit many approximately J-holomorphic sections, some of which possess remarkable transversality properties ensuring that their zero sets are smooth symplectic submanifolds in X. Many interesting constructions in symplectic topology have recently been obtained by using the same techniques (see e.g. [A2], [D2] and [S]).

Let us recall the following definitions. The almost-complex structure J and the Hermitian connection on $L^{\otimes k}$ induced by that on L yield ∂ and $\bar\partial$ operators on $L^{\otimes k}$. Since the connection on $L^{\otimes k}$ has curvature $-ik\omega$, we introduce the rescaled metric $g_k = k\,g$ on X, in order to be able to consider uniform bounds for covariant derivatives of sections of $L^{\otimes k}$. As a consequence of this rescaling, the diameter of X is multiplied by $k^{1/2}$, and all derivatives of order p are divided by $k^{p/2}$.

Definition 1. *Let $(s_k)_{k\gg 0}$ be a sequence of sections of $L^{\otimes k}$ over X. The sections s_k are said to be* asymptotically holomorphic *if there exists a constant $C > 0$ such that, for all k and at every point of X, $|s_k| + |\nabla s_k| + |\nabla\nabla s_k| \leq C$ and $|\bar\partial s_k| + |\nabla\bar\partial s_k| \leq Ck^{-1/2}$, where the norms of the derivatives are evaluated with respect to the metrics $g_k = k\,g$.*

The sections s_k are said to be uniformly transverse to 0 *if there exists a constant $\eta > 0$ (independent of k) such that the sections s_k are* η-transverse to 0*, i.e. such that, for any k and at any point $x \in X$ where $|s_k(x)| < \eta$, the covariant derivative $\nabla s_k(x) : T_xX \to L_x^{\otimes k}$ is surjective and satisfies the bound $|\nabla s_k(x)|_{g_k} > \eta$.*

With these definitions, Donaldson's construction amounts to showing the existence of a sequence of sections s_k of $L^{\otimes k}$ which are at the same time asymptotically holomorphic and uniformly transverse to 0 [D1]. It then follows easily from these properties that, for large enough k, the zero sets W_k of s_k are smooth symplectic hypersurfaces in X.

Let $\mathcal{L}$ be a compact isotropic submanifold in X, not necessarily connected : we wish to show that one can get the symplectic hypersurfaces W_k to lie in $X - \mathcal{L}$. The fundamental reason why it is reasonable to expect such a result is that, since ω vanishes over $\mathcal{L}$, the line bundle $L_{|\mathcal{L}}$ comes equipped with a flat connection. However $L^{\otimes k}$ admits non-vanishing sections over $\mathcal{L}$ only when its restriction to $\mathcal{L}$ is topologically trivial ; if $\mathcal{L}$ is not simply connected, this can restrict the admissible values of the parameter k. For example, if $X = \mathbb{CP}^2$ and $\mathcal{L} = \mathbb{RP}^2$, an easy calculation in homology with $\mathbb{Z}/2$ coefficients shows that any symplectic submanifold of odd degree must intersect $\mathcal{L}$. Our main result is the following :

Theorem 2. *Let $\mathcal{L}$ be a compact isotropic submanifold in X, and let N be the order of the torsion part of $H_1(\mathcal{L}, \mathbb{Z})$. Then, for all large enough values of k, there exist asymptotically holomorphic sections s_k of $L^{\otimes k}$ over X whose zero sets W_k are smooth symplectic submanifolds, disjoint from $\mathcal{L}$ whenever k is a multiple of N. Moreover, W_k can be assumed to pass through any given point $x_0 \in X - \mathcal{L}$.*

This result is mildly surprising when one considers the results obtained in [D1] and [A1] indicating that, when k increases, the submanifolds W_k tend to fill all of X. There is no contradiction, though, as the distance by which the submanifolds W_k given by Theorem 2 stay away from $\mathcal{L}$ actually decreases like $k^{-1/2}$.

Remark 1. (*a*) *Theorem 2 remains valid when $\mathcal{L}$ has non-empty boundary ; see* [M] *for details.*

(*b*) *When X is a Kähler manifold, one can perform the construction in such a way that the sections s_k are holomorphic. The submanifolds W_k are then complex hypersurfaces ; this provides a new proof of Guedj's rational convexity result.*

(*c*) *When the cohomology class $\frac{1}{2\pi}[\omega]$ is no longer assumed to be integral, the line bundle L is no longer defined, but it is still possible to obtain symplectic hypersurfaces in X which avoid the submanifold $\mathcal{L}$ and pass through any given point in $X - \mathcal{L}$.*

Additional motivation for these results can be found in the work of Biran [B], where the notion of *Lagrange skeleton* of a symplectic manifold of Kähler type with respect to a hypersurface of Donaldson type is defined. As will be explained in §3, Theorem 2 can be interpreted in this context as a flexibility result for Lagrange skeleta in large degrees.

More importantly, it was observed by Seidel and Viterbo that Theorem 2 implies that if $\mathcal{L}$ is Lagrangian then its homology class is a primitive element of $H_n(X - W_k)$ (see §3) ; this remark might lead to obstructions to the existence of certain Lagrangian embeddings.

Note. Different proofs of Theorem 2 were obtained independently by the three authors ; the curious reader is referred to [M] and [Ga] for various alternate arguments and generalizations.

The authors wish to thank Claude Viterbo, Paul Seidel and Paul Biran for motivating discussions and for suggesting applications of Theorem 2. The authors are respectively thankful to Ivan Smith, Julien Duval, Bruno Sévennec and Emmanuel Giroux for discussions and advice.

2. Proof of Theorem 2

We first define the notion of concentrated sections of $L^{\otimes k}$:

Definition 2. *Asymptotically holomorphic sections s_k of $L^{\otimes k}$ are said to be* concentrated *over a subset $N \subset X$ if there exist positive constants λ, c and C (independent of k) such that for all $y \in N$, $|s_k(y)| \geq c$, and, for all $y \in X$, $|s_k(y)| \leq C\exp(-\lambda\, d(y,N)^2)$, where $d(.,.)$ is the distance induced by g_k. When the subset N consists of a single point $x \in X$, we say that the sections s_k are concentrated at x.*

With this terminology, recall the following result (Proposition 11 of [D1]) :

Lemma 1 (Donaldson)**.** *For all large enough k the line bundles $L^{\otimes k}$ admit asymptotically holomorphic sections $\sigma_{k,x}$ concentrated at any given point $x \in X$.*

As the properties of the sections $\sigma_{k,x}$ play an important role in the argument, let us recall briefly their construction.

Remember that, at any point $x \in X$, it is possible to find a local approximately holomorphic Darboux coordinate chart, i.e. a local symplectomorphism $\psi : (X, x, \omega) \to (\mathbb{C}^n, 0, \omega_0)$ such that, with respect to J and the standard complex structure of $\mathbb{C}^n$, $\bar\partial\psi(x) = 0$ and $|\nabla\bar\partial\psi|_g$ is bounded uniformly by a constant C. The compactness of X implies that the size of the neighborhood over which ψ is defined and the value of the constant C can be assumed not to depend on the chosen point x.

In our case, we will moreover require that, whenever the point x belongs to the given isotropic submanifold $\mathcal{L}$, the coordinate map ψ locally sends $\mathcal{L}$ to a linear subspace in $\mathbb{C}^n$ (obviously isotropic). The existence of Darboux coordinate charts with this property is a very classical result of Weinstein ([W], see also [McS]) ; it is an immediate observation that the coordinate map can still be chosen to satisfy $\bar\partial\psi(x) = 0$, and the compactness of $\mathcal{L}$ implies the existence of uniform estimates on $|\nabla\bar\partial\psi|$ and on the size of the coordinate chart.

In a Darboux coordinate chart, a suitable unitary gauge transformation leads to a local trivialization of $L^{\otimes k}$ in which the connection 1-form is given by $\frac{k}{4}\sum(z_j d\bar z_j - \bar z_j dz_j)$. The local section defined by $f_k(z) = \exp(-k|z|^2/4)$ is then holomorphic over a neighborhood of 0 in $\mathbb{C}^n$. Pulling back f_k via the coordinate chart ψ, one obtains sections $\hat\sigma_{k,x}$ of $L^{\otimes k}$ over a neighborhood of x in X, and it easily follows from the estimates on $\bar\partial\psi$ that these sections are asymptotically holomorphic.

Finally, multiplying $\hat\sigma_{k,x}$ by a smooth cut-off function vanishing at distance $k^{-1/6}$ from x yields the desired asymptotically holomorphic sections $\sigma_{k,x}$, easily shown to be concentrated at the point x (see [D1]).

Recall from [D1] (see also [A1]) that asymptotically holomorphic sections with uniform transversality estimates are constructed by an iterative process, where one starts with any given asymptotically holomorphic sections s_k of $L^{\otimes k}$ (e.g. $s_k = 0$) and perturbs them over small open subsets of X in order to achieve transversality over those subsets ; successive smaller and smaller perturbations are performed in such a way that the transversality property gained at each step is preserved by all subsequent perturbations, until transversality holds over all of X. In particular, given any constant $C > 0$ it is possible to ensure that the constructed sections $\tilde s_k$ differ from the given sections s_k by less than C in C^1 norm (i.e., at every point of X we have $|\tilde s_k - s_k| + |\nabla\tilde s_k - \nabla s_k|_{g_k} \le C$) [A1].

Therefore, in order to prove Theorem 2 (without requiring yet the submanifolds to pass through a given point of $X - \mathcal{L}$), it is sufficient to construct asymptotically holomorphic sections $\sigma_{k,\mathcal{L}}$ of $L^{\otimes k}$, concentrated over $\mathcal{L}$ for k ranging over all large enough multiples of $N = |\mathrm{Tor}\, H_1(\mathcal{L},\mathbb{Z})|$. By definition these sections satisfy a uniform lower bound over $\mathcal{L}$ by some constant $c > 0$, and perturbing them by less than $c/2$ we get (for large enough k) uniformly transverse sections which do not vanish over $\mathcal{L}$. Our next ingredient is the following observation :

Lemma 2. *Given any compact isotropic submanifold $\mathcal{L} \subset X$, there exists a constant $C_\mathcal{L} > 0$ such that, whenever k is a multiple of $N = |\mathrm{Tor}\, H_1(\mathcal{L},\mathbb{Z})|$, the restriction of $L^{\otimes k}$ to $\mathcal{L}$ admits a section τ_k such that $|\tau_k(x)| = 1$ and $|\nabla\tau_k(x)|_g \le C_\mathcal{L}$, i.e. $|\nabla\tau_k(x)|_{g_k} \le C_\mathcal{L}\, k^{-1/2}$, at every point $x \in \mathcal{L}$.*

Proof. Since $\mathcal{L}$ is isotropic, the restriction to $\mathcal{L}$ of the connection ∇^L on L is flat ; therefore the first Chern class $c_1(L_{|\mathcal{L}})$, although not necessarily trivial, belongs to the kernel of the natural map $\iota : H^2(\mathcal{L},\mathbb{Z}) \to H^2(\mathcal{L},\mathbb{R})$. By the universal coefficients theorem (see e.g. [BT], page 194), $\mathrm{Ker}(\iota) = \mathrm{Tor}\, H^2(\mathcal{L},\mathbb{Z}) \simeq \mathrm{Tor}\, H_1(\mathcal{L},\mathbb{Z})$. It follows that the order of $c_1(L_{|\mathcal{L}})$ divides N, so that the complex line bundle $L^{\otimes k}_{|\mathcal{L}}$ has zero first Chern class and hence is topologically trivial whenever k is a multiple of N.

Fix a trivialization of $L^{\otimes k}$ over $\mathcal{L}$, and consider the 1-form $\alpha_k \in \Omega^1(\mathcal{L}, i\mathbb{R})$ representing the connection on $L^{\otimes k}$ induced by ∇^L. We work with the metric on $\mathcal{L}$ induced by g, and observe that a suitable choice of trivialization of $L^{\otimes k}$ ensures that the 1-form α_k and its derivatives satisfy uniform bounds which depend only on the geometry of $\mathcal{L}$ and not on k.

Indeed, it is well-known that the moduli space of flat unitary connections on the trivial complex line bundle over $\mathcal{L}$ up to $U(1)$ gauge transformations is compact and isomorphic to $H^1(\mathcal{L},\mathbb{R})/H^1(\mathcal{L},\mathbb{Z})$. Therefore, a well-chosen gauge transformation makes it possible to obtain uniform bounds on the 1-form α_k and its derivatives, independently of k. More precisely, a first gauge transformation in the identity component can be used to make the closed 1-form α_k harmonic, while the flexibility coming from the connected components of the gauge group makes it possible to ensure that α_k lies in a fixed bounded subset of $H^1(\mathcal{L},\mathbb{R})$.

Let τ_k be the section of $L^{\otimes k}$ over $\mathcal{L}$ which identifies with the constant function 1 in the chosen trivialization : clearly, $|\tau_k| = 1$ at every point of $\mathcal{L}$ and the derivatives of τ_k are bounded by uniform constants independently of k with respect to the metric g. □

Remark. The bounds satisfied by α_k and $\nabla\tau_k$ depend on the minimum g-length $\delta(\mathcal{L})$ of a homotopically non-trivial loop in $\mathcal{L}$; in fact $C_{\mathcal{L}}$ must be at least of the order of $\delta(\mathcal{L})^{-1}$. This is one of the reasons why the submanifold $\mathcal{L}$ cannot be allowed to vary with k, another one being that we need to control the size of the balls centered at points of $\mathcal{L}$ which can be trivialized by Weinstein's theorem.

Throughout the remainder of this section we assume that k is a multiple of N. For each such k, let P_k be a finite set of points of $\mathcal{L}$ such that the balls of g_k-radius 1 centered at the points of P_k cover $\mathcal{L}$ and any two points of P_k are at g_k-distance at least $\frac{2}{3}$ from each other. Such a set can be constructed by covering $\mathcal{L}$ by finitely many balls of g_k-radius $\frac{1}{3}$ and iteratedly removing the points that are too close to each other (see also [D1]).

Define the sections

$$\sigma_{k,\mathcal{L}} = \sum_{p\in P_k} \frac{\tau_k(p)}{\sigma_{k,p}(p)}\,\sigma_{k,p}$$

of $L^{\otimes k}$ over X. The sections $\sigma_{k,\mathcal{L}}$ are linear combinations of the asymptotically holomorphic sections $\sigma_{k,p}$, with coefficients unitary complex numbers (recall that $|\tau_k(p)| = |\sigma_{k,p}(p)| = 1$). Therefore, because any two points of P_k are mutually g_k-distant of at least $\frac{2}{3}$ and because the sections $\sigma_{k,p}$ are concentrated at points, a standard argument ([D1],[S]) shows that the sections $\sigma_{k,\mathcal{L}}$ are uniformly bounded and asymptotically holomorphic.

We now show that the sections $\sigma_{k,\mathcal{L}}$ are concentrated over $\mathcal{L}$. The decay properties of $\sigma_{k,\mathcal{L}}$ away from $\mathcal{L}$ follow from the following lemma :

Lemma 3. *Let $P_k \subset X$ be a finite set of points whose mutual g_k-distance is bounded from below by a constant $\delta > 0$. Let $(\alpha_{k,p})_{p\in P_k}$ be a family of complex numbers such that $|\alpha_{k,p}| \le 1$ $\forall p \in P_k$, and let $s_k = \sum_{p\in P_k} \alpha_{k,p}\sigma_{k,p}$. Then there exist constants C_δ and λ_δ, independent of k and P_k, such that $|s_k(x)| \le C_\delta \exp(-\lambda_\delta d_{g_k}(x,P_k)^2)$ at every point of X.*

Proof. Because $\sigma_{k,p}$ is supported in $B_g(p, 2k^{-1/6})$, we can restrict ourselves to only considering points in a fixed ball around the given point $x \in X$; since the g_k-distance between any two points of P_k is greater than δ, this implies that the number of points $p \in P_k$ lying within a given fixed g_k-distance ρ of x is bounded by $Q(\rho)$, where Q is a polynomial depending only on δ. Therefore, using the existence of a bound $|\sigma_{k,p}(x)| \le C' \exp(-\lambda' d(x,p)^2)$ for $\sigma_{k,p}$ and ordering the points of P_k

according to their distance from x, we get the desired bound on $|s_k(x)|$ by summing over concentric slices. ☐

We immediately conclude that $|\sigma_{k,\mathcal{L}}(x)| \leq C_{2/3} \exp(-\lambda_{2/3} d_{g_k}(x,\mathcal{L})^2)$. It remains to be shown that the norm of $\sigma_{k,\mathcal{L}}$ at a point of $\mathcal{L}$ admits a uniform lower bound. For this, we first prove the following result :

Lemma 4. *If k is large enough, and if p and x are two points of $\mathcal{L}$ such that $d_{g_k}(p,x) \leq k^{1/10}$, then $\sigma_{k,p}(x) \neq 0$ and*

$$\left| \arg\left(\frac{\sigma_{k,p}(x)}{\tau_k(x)}\right) - \arg\left(\frac{\sigma_{k,p}(p)}{\tau_k(p)}\right) \right| \leq \frac{\pi}{4}.$$

Proof. Since the g-distance between x and p is less than $k^{-2/5}$, the cut-off function used to define $\sigma_{k,p}$ is equal to 1 at x, and therefore $\sigma_{k,p}(x) \neq 0$.

We work in the same local coordinate chart ψ and local trivialization of $L^{\otimes k}$ that were used to define $\sigma_{k,p}$; we write $\psi(x) = u$, and consider the radial path $\gamma(t) = \psi^{-1}(tu)$ from p to x. Recall that the connection on $L^{\otimes k}$ is expressed as $d + A_k = d + \frac{k}{4}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$, while $\sigma_{k,p}$ is locally given by the function $\exp(-\frac{k}{4}|z|^2)$. Therefore one easily checks that

$$(1) \qquad \int_0^1 \left(\frac{\nabla\sigma_{k,p}}{\sigma_{k,p}}\right)_{\gamma(t)} \cdot \gamma'(t)\, dt = \int_0^1 d(-\frac{k}{4}|z|^2)_{(tu)} \cdot u\, dt = -\frac{k}{4}|u|^2 \in \mathbb{R}.$$

Recall that by construction we require that ψ locally maps $\mathcal{L}$ to a linear subspace of $\mathbb{C}^n$. Therefore the radial path γ is contained in $\mathcal{L}$, and we can use the bound on $\nabla\tau_k$ given by Lemma 2 to obtain that

$$(2) \qquad \Big|\int_0^1 \left(\frac{\nabla\tau_k}{\tau_k}\right)_{\gamma(t)} \cdot \gamma'(t)\, dt\Big| \leq \int_0^1 |(\nabla\tau_k)_{\gamma(t)}| \cdot |\gamma'(t)|\, dt = O(k^{-2/5}).$$

Therefore,

$$\arg\left(\frac{\sigma_{k,p}(x)}{\tau_k(x)}\right) - \arg\left(\frac{\sigma_{k,p}(p)}{\tau_k(p)}\right) = \mathrm{Im}\left[\int_0^1 \left(\frac{\nabla\sigma_{k,p}}{\sigma_{k,p}} - \frac{\nabla\tau_k}{\tau_k}\right)_{\gamma(t)} \cdot \gamma'(t)\, dt\right]$$

is bounded by a constant times $k^{-2/5}$, which gives the result. ☐

Lemma 4 implies the existence of a uniform lower bound on $\sigma_{k,\mathcal{L}}$ at any point of $\mathcal{L}$. Indeed, consider a point $x \in \mathcal{L}$, and let p be the point of P_k closest to x. By construction $d_{g_k}(x,p) \leq 1$, and therefore there exists a constant $c > 0$ (independent of x, p and k) such that $|\sigma_{k,p}(x)| \geq c$. By Lemma 4 we know that the contributions of the various points $q \in P_k$ whose g_k-distance to x is less than $k^{1/10}$ cannot cancel each other, and we have

$$\left| \sum_{\substack{q \in P_k \\ d(x,q) \leq k^{1/10}}} \frac{\tau_k(q)}{\sigma_{k,q}(q)}\, \sigma_{k,q}(x) \right| \geq |\sigma_{k,p}(x)| \geq c.$$

On the other hand, Lemma 3 implies that the contribution of the remaining points of P_k decreases exponentially with k. Therefore, when k is large enough we get that $|\sigma_{k,\mathcal{L}}(x)| \geq c/2$ at any point x of $\mathcal{L}$; in fact, the argument also implies that $\sup_{x\in\mathcal{L}} |\arg(\sigma_{k,\mathcal{L}}(x)/\tau_k(x))|$ becomes arbitrarily small for large k.

We conclude that the asymptotically holomorphic sections $\sigma_{k,\mathcal{L}}$ are concentrated over $\mathcal{L}$, which ends the argument : perturbing $\sigma_{k,\mathcal{L}}$ by less than $c/4$ we obtain asymptotically holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$ satisfying a uniform transversality property, and by construction their zero sets are (asymptotically holomorphic) symplectic submanifolds which do not intersect $\mathcal{L}$.

The final step to complete the proof of Theorem 2 is to show that these asymptotically holomorphic hypersurfaces can be made to pass through a given point $x_0 \in X - \mathcal{L}$. Considering the sections $u_{k,x_0} = k^{1/2} z_1 \sigma_{k,x_0}$, where z_1 is a local approximately holomorphic coordinate function at x_0, the idea is to work with $\sigma_{k,\mathcal{L}} + u_{k,x_0}$ instead of $\sigma_{k,\mathcal{L}}$. Indeed, observing that for large k the support of u_{k,x_0} is disjoint from $\mathcal{L}$, a small perturbation of $\sigma_{k,\mathcal{L}} + u_{k,x_0}$ yields asymptotically holomorphic hypersurfaces W_k avoiding $\mathcal{L}$ and passing through a point x within unit g_k-distance of x_0. It is then possible to find a Hamiltonian diffeomorphism ϕ preserving $\mathcal{L}$, mapping x to x_0, and sufficiently close to the identity in order to ensure the asymptotic holomorphicity of $\phi(W_k)$. □

Remark. When $\mathcal{L}$ is Lagrangian, Theorem 2 can also be proved by arguing along the following lines. By Weinstein's Lagrangian neighborhood theorem, a neighborhood V of $\mathcal{L}$ in X is symplectomorphic to a neighborhood of the zero section in $T^*\mathcal{L}$ with its standard symplectic structure $dp \wedge dq$; the fibers of $\pi : T^*\mathcal{L} \to \mathcal{L}$ can be chosen g-orthogonal to $\mathcal{L}$ at every point of $\mathcal{L}$. Consider the trivialization of $L^{\otimes k}$ over $\mathcal{L}$ given by the section τ_k of Lemma 2, and extend it over V in such a way that the connection 1-form is given by $\beta_k = \pi^*\alpha_k - ik\,p\,dq$, where α_k is the same 1-form on $\mathcal{L}$ as in Lemma 2. It can then be checked that the sections of $L^{\otimes k}$ over V defined by $s_k = \exp(-\frac{1}{2}k|p|_g^2)$ (where $|\cdot|_g$ is the metric induced by $g_{|\mathcal{L}}$ on the fibers of $T^*\mathcal{L}$) are asymptotically holomorphic ; multiplying s_k by a suitable cut-off function we obtain asymptotically holomorphic sections concentrated over $\mathcal{L}$, from where Theorem 2 is easily obtained.

3. Remarks and applications

3.1. **The Kähler case.** We consider the case where (X, ω, J) is a Kähler manifold, and show how the construction can be performed in the holomorphic category (Remark 1 (b)) using the ideas of Donaldson (see pp. 696–700 of [D1]). The first observation is that near any point $x \in X$ there exists a local holomorphic section of L which, in the same local trivialization of L as in the proof of Lemma 1, is given by a function f such that $f(z) = 1 - \frac{1}{4}|z|^2 + O(|z|^3)$ and $df(z) = -\frac{1}{4}\sum_j (z_j d\bar{z}_j + \bar{z}_j dz_j) + O(|z|^2)$; see the proof of Lemma 36 of [D1].

Multiplying $f(z)^k$ by a smooth cut-off function at distance $k^{-1/6}$ from x yields asymptotically holomorphic sections $\sigma_{k,x}$ of $L^{\otimes k}$, concentrated at x as in Lemma 1 ; moreover, as observed by Donaldson in [D1], there exist holomorphic sections $\tilde{\sigma}_{k,x}$ of $L^{\otimes k}$ such that $\sup |\tilde{\sigma}_{k,x} - \sigma_{k,x}| \le C \exp(-ak^{1/3})$, with a and C positive constants (independent of k and x).

We now proceed as in §2.1, using the new sections $\sigma_{k,x}$ instead of those obtained in Lemma 1. The argument remains the same, the only difference being in the proof of Lemma 4 where the l.h.s. of (1) becomes equal to

$$\int_0^1 \frac{(d + A_k) f(z)^k_{(tu)}}{f(tu)^k} \cdot u\, dt = \int_0^1 k \Big(\frac{df}{f}\Big)_{(tu)} \cdot u\, dt = -\frac{k}{4}|u|^2 + O(k|u|^3).$$

Since $|u|$ is at most of the order of $k^{-2/5}$ the imaginary part of this quantity is bounded by $O(k^{-1/5})$, which is enough to prove Lemma 4 and hence construct $\sigma_{k,\mathcal{L}}$ as in §2.1.

Replacing $\sigma_{k,x}$ by $\tilde{\sigma}_{k,x}$ in the definition of $\sigma_{k,\mathcal{L}}$, we obtain holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$ which differ from $\sigma_{k,\mathcal{L}}$ by at most $C\exp(-ak^{1/3})\,\mathrm{card}(P_k)$ and therefore also satisfy a uniform lower bound over $\mathcal{L}$. It is then possible to conclude as usual, by adding a linear combination of the sections $\tilde{\sigma}_{k,x}$ to $\tilde{\sigma}_{k,\mathcal{L}}$ in order to achieve uniform transversality.

Alternately, given a point $x_0 \in X - \mathcal{L}$, one can add a multiple of $\tilde{\sigma}_{k,x_0}$ to $\tilde{\sigma}_{k,\mathcal{L}}$ in order to obtain holomorphic sections $\tilde{\sigma}_{k,\mathcal{L},x_0}$ which vanish at x_0 while remaining bounded away from zero over $\mathcal{L}$. In terms of the projective embeddings $i : X \to \mathbb{P}H^0(L^{\otimes k})^*$, these sections correspond to hyperplanes passing through $i(x_0)$ while avoiding $i(\mathcal{L})$. A small generic perturbation yields a hyperplane passing through $i(x_0)$ which intersects $i(X)$ transversely and still avoids $i(\mathcal{L})$; this gives smooth complex hypersurfaces passing through x_0 and avoiding $\mathcal{L}$, giving a new proof of Guedj's result.

3.2. **The non-integral case.** In this section we no longer assume that the cohomology class $\frac{1}{2\pi}[\omega]$ is integral, as in Remark 1 (c). As in [D1] the idea is to perturb the symplectic form ω into a symplectic form ω' such that $\frac{1}{2\pi}[\omega']$ is proportional to an integral class, and work with a multiple of ω'. It is however necessary to ensure that $\mathcal{L}$ remains isotropic.

Because $\frac{1}{2\pi}[\omega]$ lies in the kernel of the restriction map from $H^2(X,\mathbb{R})$ to $H^2(\mathcal{L},\mathbb{R})$, it is the image of a class $\alpha \in H^2(X,\mathcal{L};\mathbb{R})$. Moreover, $H^2(X,\mathcal{L};\mathbb{Q})$ contains elements lying arbitrarily close to α in $H^2(X,\mathcal{L};\mathbb{R})$. Therefore, by adding to ω an arbitrarily small closed 2-form vanishing over $\mathcal{L}$, we obtain a symplectic form ω' such that $\frac{1}{2\pi}[\omega']$ is the image of a class in $H^2(X,\mathcal{L};\mathbb{Q})$ and hence belongs to $H^2(X,\mathbb{Q})$. By construction, ω' satisfies up to multiplication by a constant factor the required integrality condition, and $\mathcal{L}$ is ω'-isotropic.

The symplectic form ω' admits a compatible almost-complex structure J', C^0-close to J ; since $\omega(v, J'v) > 0\ \forall v \in TX$, any J'-complex subspace is ω-symplectic. So, if a sequence of submanifolds $W_k \subset X$ is asymptotically J'-holomorphic, then W_k is a symplectic submanifold of (X,ω) for large enough k. One then concludes by applying Theorem 2 to (X,ω',J').

3.3. **Uniqueness up to isotopy.** It was shown in [A1] that the symplectic submanifolds constructed by Donaldson in [D1] are, for each large enough value of k, canonical up to symplectic isotopy, independently of the almost-complex structure J. One may ask whether in our case the submanifolds W_k are canonical up to a symplectic isotopy of X preserving $\mathcal{L}$; such a uniqueness property does not hold in general, because the homotopy class of the non-vanishing section s_k of $L^{\otimes k}$ over $\mathcal{L}$ plays a determining role.

Let γ be a non-contractible loop in $\mathcal{L}$ bounding a disc D in X : the homotopy class of the non-vanishing section $(s_k)_{|\gamma}$ over γ determines the number of zeroes of s_k over D, i.e. the linking number of W_k with γ, which can be modified by choosing different trivializations of $L^{\otimes k}$ over $\mathcal{L}$. Still, when $\mathcal{L}$ is simply connected the homotopy classes of the nowhere vanishing sections $(s_k)_{|\mathcal{L}}$ are uniquely determined.

Even though it seems reasonable to expect that the isotopy class of asymptotically holomorphic hypersurfaces in $X - \mathcal{L}$ should only depend on the homotopy class of $(s_k)_{|\mathcal{L}}$, our techniques do not allow us to prove so strong a statement ; we are only able to prove that the submanifolds constructed in §2 (using either the given proof or the alternate argument sketched at the end) are canonical up to symplectic isotopy in $X - \mathcal{L}$. For this, we use the control on the complex argument of $(s_k)_{|\mathcal{L}}$ given by the construction : it follows directly from Lemma 4 and the subsequent discussion that for large k the argument of s_k/τ_k remains small at every point of $\mathcal{L}$.

Proposition 1. *Let τ_k^0 and τ_k^1 be sections of $L^{\otimes k}$ over $\mathcal{L}$ belonging to the same homotopy class and such that $|\tau_k^i| \equiv 1$ and $|\nabla \tau_k^i|_g = O(1)$. Let s_k^0 and s_k^1 be asymptotically holomorphic sections of $L^{\otimes k}$ over X, uniformly transverse to 0, uniformly bounded from below over $\mathcal{L}$, and such that the bound $|\arg(s_k^i/\tau_k^i)| \leq \frac{\pi}{3}$ holds at every point of $\mathcal{L}$. Then for large enough k their zero sets W_k^0 and W_k^1 differ by a symplectic isotopy preserving $\mathcal{L}$.*

Proof. We use the same one-parameter argument as in [A1] in order to construct for large k a one-parameter family of asymptotically holomorphic sections s_k^t, bounded from below on $\mathcal{L}$, interpolating between s_k^0 and s_k^1. First, choosing a trivialization of $L^{\otimes k}$ over $\mathcal{L}$ to express τ_k^i in the form $\exp(\phi_k^i)$ for $i \in \{0, 1\}$, we define sections τ_k^t of $L^{\otimes k}_{|\mathcal{L}}$ for $t \in [0, 1]$ by $\tau_k^t = \exp((1-t)\phi_k^0 + t\phi_k^1)$. Observing that $|\tau_k^t| \equiv 1$ and $|\nabla \tau_k^t|_g = O(1)$ for all t, we can define sections $\sigma_{k,\mathcal{L}}^t = \sum_{p \in P_k} (\tau_k^t(p)/\sigma_{k,p}(p))\, \sigma_{k,p}$ of $L^{\otimes k}$ over X which are asymptotically holomorphic and concentrated over $\mathcal{L}$.

Define s_k^t to be equal to $(1-3t)s_k^0 + 3t\sigma_{k,\mathcal{L}}^0$ for $t \in [0, \frac{1}{3}]$, to $\sigma_{k,\mathcal{L}}^{3t-1}$ for $t \in [\frac{1}{3}, \frac{2}{3}]$ and to $(3-3t)\sigma_{k,\mathcal{L}}^1 + (3t-2)s_k^1$ for $t \in [\frac{2}{3}, 1]$. All these sections are asymptotically holomorphic ; observing that for $i \in \{0, 1\}$ the arguments of s_k^i and $\sigma_{k,\mathcal{L}}^i$ both remain within $\frac{\pi}{3}$ of that of τ_k^i at every point of $\mathcal{L}$, they also satisfy a uniform lower bound by some constant $c > 0$ at every point of $\mathcal{L}$.

Let $\gamma > 0$ be the uniform transversality estimate satisfied by s_k^i for $i \in \{0, 1\}$. Applying the main theorem of [A1], we obtain, provided that k is large enough, uniformly transverse sections $\tilde{s}_k^t$ of $L^{\otimes k}$ depending continuously on t and differing from s_k^t by at most $\frac{1}{2}\inf(c, \gamma)$ in C^1 norm ; slightly modifying this 1-parameter family near its extremities we can safely assume that $\tilde{s}_k^0 = s_k^0$ and $\tilde{s}_k^1 = s_k^1$ (see Corollary 2 in [A1]). The zero sets of $\tilde{s}_k^t$ are then symplectic hypersurfaces $W_k^t \subset X - \mathcal{L}$ realizing a smooth isotopy between W_k^0 and W_k^1. The argument in §4.2 of [A1] then shows that this smooth isotopy can be turned into a symplectic isotopy preserving $\mathcal{L}$ (observe that all the quantities appearing in the argument can be chosen to vanish over a neighborhood of $\mathcal{L}$). □

A final remark about the homotopy class of the sections we construct in the non simply connected case : the homotopy class of $(s_k)_{|\mathcal{L}}$ as given by our construction is in fact related to the evaluation of ω on elements of $\pi_2(X, \mathcal{L})$. More precisely, given a loop $\gamma \subset \mathcal{L}$ bounding a disc D in X, the trivialization of $L^{\otimes k}$ over γ which minimizes the norm of the connection 1-form differs from the one which extends over D by an amount of twisting approximately equal to $\frac{1}{2\pi}\int_D k\omega$; therefore, in the construction of W_k we obtain a linking number differing from this amount by at most a bounded quantity.

3.4. **Behavior of concentrated sections along normal slices.** For any point $x \in \mathcal{L}$, let N_x be the image by the exponential map of the metric g of a small disc in the normal space to $\mathcal{L}$ at x. Let $\sigma_{k,\mathcal{L}}$ be the asymptotically holomorphic sections concentrated over $\mathcal{L}$ constructed in §2. The following Lemma will be useful for applications.

Lemma 5. *There exist constants $\delta > 0$ and $\gamma > 0$, independent of k, such that the restriction of $|\sigma_{k,\mathcal{L}}|^2$ to the intersection of N_x with $B_{g_k}(x,\delta)$ is strictly concave, with second derivatives bounded from above by $-\gamma$ w.r.t. g_k, and reaches its maximum at a point within g_k-distance $o(1)$ from x. The set of all these maxima is a smooth submanifold $\mathcal{L}'_k$, C^0-converging towards $\mathcal{L}$ as k increases. Moreover, when X is Kähler the same properties remain true for the holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$ constructed in §3.1.*

Proof. Fix a value of k and a point $p \in P_k$ such that $d_{g_k}(x,p) \le k^{1/10}$, and work in the approximately holomorphic Darboux coordinate chart used to define $\sigma_{k,p}$; recalling that $\mathcal{L}$ is locally mapped to a linear subspace, let N'_x be the affine subspace through x orthogonal to $\mathcal{L}$ in these coordinates. Since x lies at g-distance less than $k^{-2/5}$ from p where the coordinate map is an isometry, N_x and N'_x are very close to each other (their angle at x is at most $O(k^{-2/5})$). Moreover, the restriction to N'_x of the function $f(z) = \exp(-\frac{1}{4}|z|^2)$ is strictly concave (with a uniform upper bound on its second derivatives) and admits a maximum at x ; therefore, $f_{|N_x}$ is also strictly concave and admits a maximum within g-distance $O(k^{-4/5})$ from x. Since $\sigma_{k,p}$ coincides with f^k near x, the same property holds for $|\sigma_{k,p}|^2$, except that the upper bound on second derivatives depends on $d_{g_k}(p,x)$ and only holds over a ball of fixed g_k-radius around x.

Next, recall from the proof of Lemma 4 that the contributions to $\sigma_{k,\mathcal{L}}$ coming from the various points of P_k lying within g_k-distance $k^{1/10}$ from x do not cancel each other at x, and more precisely their complex arguments at x differ from each other by at most $O(k^{-2/5})$. Of course this no longer remains true as soon as one moves away from $\mathcal{L}$; still, by a computation similar to the proof of Lemma 4 we can obtain control on the manner in which the complex arguments of the various contributions to $\sigma_{k,\mathcal{L}}$ differ from each other at a point close to x.

More precisely, consider a geodesic arc γ joining x to a nearby point y in N_x, and let p be a point of P_k within g_k-distance $k^{1/10}$. Then

$$\mathrm{Im} \int_0^1 \left(\frac{\nabla \sigma_{k,p}}{\sigma_{k,p}} \right)_{\gamma(t)} \cdot \gamma'(t)\, dt = \int_0^1 -\tfrac{ik}{4} \sum z_j d\bar{z}_j - \bar{z}_j dz_j \cdot \gamma'(t)\, dt$$

is equal to $-\frac{k}{2}\omega_0(x-p, y-x) + O(k\, d_g(x,p)^2 d_g(x,y))$, where ω_0 is the standard symplectic form on $\mathbb{C}^n$ and the error term comes from the non-linearity of N_x in the Darboux coordinate chart. In particular, if p, p' and y are at bounded g_k-distance from x then the difference of complex arguments between the contributions of $\sigma_{k,p}$ and $\sigma_{k,p'}$ to $\sigma_{k,\mathcal{L}}(y)$ is given by $\phi_{p,p'}(y) = \frac{k}{2}\omega_0(p-p', y-x) + O(k^{-2/5})$, where the first term is bounded by a fixed constant times $d_{g_k}(y,x)$.

Fix a large constant $D > 0$ (independent of k and x), and let us first restrict ourselves to the sum $\sigma_{k,\mathcal{L},x,D}$ of the contributions of the points of P_k within g_k-distance D from x. It follows from the above remarks that there exists a constant $\delta(D) > 0$ (of the order of D^{-1}) such that $|\sigma_{k,\mathcal{L},x,D}|^2$ is a strictly concave function at every point of $N_x \cap B_{g_k}(x, \delta(D))$, with a uniform upper bound (independent of k, D and x) on its second derivatives. Indeed,

$$|\sigma_{k,\mathcal{L},x,D}(y)|^2 = \sum_p |\sigma_{k,p}(y)|^2 + \sum_{p \neq p'} |\sigma_{k,p}(y)|\,|\sigma_{k,p'}(y)|\,\cos\phi_{p,p'}(y).$$

When $d_{g_k}(y,x)$ is not too large, $\cos\phi_{p,p'}$ has second derivatives bounded from above by $o(1)$ (by the above expression of $\phi_{p,p'}$ and the corresponding bounds on its first and second derivatives) ; therefore, using the lower bounds on $|\sigma_{k,p}|$, $|\sigma_{k,p'}|$ and $\cos\phi_{p,p'}$, the upper bounds on their second derivatives and the estimates on their first derivatives near x, we obtain that all the terms in the sum are strictly concave functions, thus yielding the desired concavity property for $|\sigma_{k,\mathcal{L},x,D}|^2$.

Moreover, since the total contribution of the remaining points of P_k to the section $\sigma_{k,\mathcal{L}}$ decreases exponentially fast as a function of D, it cannot affect the concavity property provided that D is chosen large enough.

The contributions of the points within distance $k^{1/10}$ from x reach their maxima over N_x within g-distance $O(k^{-4/5})$ from x and their arguments at x differ by $O(k^{-2/5})$, while the remaining terms decrease exponentially fast with k. Therefore, the value of $|\sigma_{k,\mathcal{L}}(x)|^2$ is sufficiently close to the maximal possible one in order to guarantee that the maximum of $|\sigma_{k,\mathcal{L}}|^2$ over N_x is reached within g_k-distance $o(1)$ from x.

Finally, the smoothness of the set $\mathcal{L}'_k$ of all maxima is an immediate consequence of the smoothness of $\sigma_{k,\mathcal{L}}$ and of the uniform concavity property.

In the Kähler case, recall from §3.1 that the sections $\sigma_{k,p}$ are now constructed using the local holomorphic section $f(z) = 1 - \frac{1}{4}|z|^2 + O(|z|^3)$, for which the maximum over N'_x is reached not necessarily at x but at an arbitrary point within g-distance $O(k^{-4/5})$ from x ; however this does not affect the properties of $|\sigma_{k,p}|^2_{|N_x}$ that we have used. Similarly, the fact that f is no longer real-valued affects the complex arguments of the various contributions to $\sigma_{k,\mathcal{L}}$, both at a point $x \in \mathcal{L}$ (bound by $O(k^{-1/5})$ instead of $O(k^{-2/5})$ in Lemma 4, see §3.1) and outside $\mathcal{L}$ (but it turns out that these extra contributions do not affect the estimates) ; still, the argument remains valid without modification. Finally, since the holomorphic section $\tilde{\sigma}_{k,\mathcal{L}}$ differs from $\sigma_{k,\mathcal{L}}$ by an amount decreasing exponentially fast with k, it enjoys the same concavity and maximum properties as $\sigma_{k,\mathcal{L}}$, so that the conclusion remains valid in this case as well. □

Remark. The assertions of Lemma 5 are also trivially satisfied by the concentrated sections obtained in the alternate proof of Theorem 2 outlined at the end of §2.

3.5. **Relations with Lagrange skeleta.** Let X be a compact Kähler manifold, let s be a holomorphic section of $L^{\otimes k}$, transverse to 0, and consider the smooth hypersurface $W = s^{-1}(0)$. It is a result of Biran [B] that the section s determines a splitting $X = B \sqcup \Delta$, where B is a "standard" symplectic disc bundle over W and Δ is an isotropic CW-complex called the *Lagrange skeleton* of (X, W). The skeleton Δ is obtained as the union of the ascending varieties of all the critical points of the plurisubharmonic function $\log|s|^2$; it is well-known that these critical points are all of index at least n. Combined with standard results in Lagrangian intersection theory, this result provides powerful restrictions on Lagrangian embeddings. For example, any simply connected embedded Lagrangian submanifold in X must intersect either W or Δ (otherwise it could be disjointed from itself by a Hamiltonian flow in $B - W$).

Biran's result is generally expected to remain valid in the more general case of a symplectic manifold and a symplectic hypersurface "of Donaldson type". However, to be on the safe side we will assume throughout this section that X is Kähler, considering only the construction of §3.1.

Proposition 2. *Let $\mathcal{L}$ be a compact isotropic submanifold of X. Then for large k there exist holomorphic sections s_k of $L^{\otimes k}$, transverse to 0 and non-vanishing over $\mathcal{L}$, such that $\mathcal{L}$ is contained in arbitrarily small neighborhoods of the Lagrange skeleta Δ_k corresponding to their zero sets W_k.*

Proof. We use the notations of §3.1, and consider the local behavior near $\mathcal{L}$ of the transverse sections s_k constructed as small perturbations of the concentrated holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$. By Lemma 5 we know that the restriction of $|\tilde{\sigma}_{k,\mathcal{L}}|^2$ to each normal slice N_x is locally concave and reaches its maximum close to $\mathcal{L}$. Therefore, choosing the transverse sections s_k close enough to $\tilde{\sigma}_{k,\mathcal{L}}$ we conclude that the restriction of $h_k = \log |s_k|^2$ to N_x admits a unique local maximum at g_k-distance less than $\frac{1}{2}\delta$ from x ; as in Lemma 5, the set of these local maxima is a smooth submanifold $\mathcal{L}''_k$ in X, obtained from $\mathcal{L}$ by an arbitrarily small deformation.

Observe that, by construction, every critical point of $h_{k|\mathcal{L}''_k}$ is also a critical point of h_k, with index increased by $\operatorname{codim} \mathcal{L}$. Moreover, although the union Λ_k of the ascending varieties of these critical points is not exactly $\mathcal{L}''_k$, one expects it to be a small deformation of $\mathcal{L}$ as well. More precisely, observe that the gradient of h_k is directed inwards at every point of the boundary of the δ-tubular neighborhood $T_\delta(\mathcal{L})$ of $\mathcal{L}$ (w.r.t. g_k). This implies, first, that every point of Λ_k lies at g_k-distance less than δ from $\mathcal{L}$, since all ascending trajectories remain in $T_\delta(\mathcal{L})$. Conversely, consider the disc $D_x = N_x \cap B_{g_k}(x, \delta)$ and its image by the downward gradient flow of h_k : since no trajectory can re-enter $T_\delta(\mathcal{L})$, the algebraic intersection number of the disc with $\mathcal{L}''_k$ constantly remains equal to 1, which implies that $D_x \cap \Lambda_k$ is non-empty. In particular $\mathcal{L}$ is contained in the δ-neighborhood of Λ_k, which is itself contained in the Lagrange skeleton. ☐

3.6. **Obstructions to Lagrangian embeddings.** In this section, we no longer assume that X is Kähler, but we assume that $\mathcal{L}$ is Lagrangian (i.e., $\dim \mathcal{L} = n$). It was suggested to us by Seidel, Viterbo and Biran that Theorem 2 might provide obstructions to the existence of certain Lagrangian embeddings by arguing along the following lines.

Consider the asymptotically holomorphic sections s_k of $L^{\otimes k}$, bounded from below over $\mathcal{L}$ and uniformly transverse to 0, given by Theorem 2, and their zero sets W_k. It follows from Lemma 5 that, if the sections constructed in §2 are chosen sufficiently close to the concentrated sections $\sigma_{k,\mathcal{L}}$, their norms reach local maxima over the transverse slices N_x along smooth submanifolds $\mathcal{L}''_k$ obtained by slightly deforming $\mathcal{L}$. Moreover, after an arbitrarily small perturbation we can assume that $h_k = \log |s_k|^2$ is a generic Morse function over $X - W_k$, without affecting the other properties.

Consider a point $x \in \mathcal{L}''_k$ where the restriction of h_k to $\mathcal{L}''_k$ reaches a local minimum : it is a critical point of index n of h_k. However the sections s_k are asymptotically holomorphic and uniformly transverse to 0, so it follows from a result of Donaldson [D1] that the critical points h_k are all of index at least n. Therefore, the genericity condition on h_k implies that the stable manifold Δ_x is a topological disc in $X - W_k$, with boundary mapped to W_k, and intersecting $\mathcal{L}''_k$ transversely at x. Observe that Δ_x is the image by the downward gradient flow of h_k of the small disc $\Delta_x \cap T_\delta(\mathcal{L})$, where $T_\delta(\mathcal{L})$ is the δ-tubular neighborhood of $\mathcal{L}$. However, the downward gradient flow is pointing outwards at every point of the boundary of $T_\delta(\mathcal{L})$, so that x is the only intersection between $\mathcal{L}''_k$ and Δ_x, and the intersection pairing between these two cycles evaluates to 1. This implies that the homology class $[\mathcal{L}''_k] \in H_n(X - W_k)$ is a primitive element. Since $\mathcal{L}''_k$ is isotopic to $\mathcal{L}$, we obtain the following

Proposition 3. *The element* $[\mathcal{L}] \in H_n(X - W_k)$ *is primitive.*

Moreover, when $\mathcal{L}$ is not connected we can apply the same argument to the minima of h_k over each component individually, obtaining that the fundamental classes of the various components of $\mathcal{L}$ are linearly independent primitive classes in $H_n(X - W_k)$.

When X is a complex projective manifold, working with the holomorphic sections of §3.1 and assuming moreover that $\mathcal{L}$ is simply connected, it is an interesting question to ask whether the smooth complex hypersurfaces W_k are always isotopic in $X - \mathcal{L}$ to hypersurfaces H_k arbitrarily close to a given hyperplane section H of X avoiding $\mathcal{L}$. A positive answer would imply that $[\mathcal{L}]$ is primitive in $H_n(X - H)$ as well, providing a new proof of a theorem of Gromov.

However, even though no problem with homotopy classes of sections over $\mathcal{L}$ is to be feared in the simply connected case, the isotopy result of §3.3 does not apply in this context, as we have no control over the complex argument of the holomorphic section of $L^{\otimes k}$ defining H_k. Whether a refinement of Proposition 1 can handle this case or not remains an open question.

References

[A1] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[A2] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$, Invent. Math. **139** (2000), 551–602.

[B] P. Biran, *Lagrangian barriers and symplectic embeddings*, preprint.

[BT] R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.

[D1] S.K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. **44** (1996), 666–705.

[D2] S.K. Donaldson, *Lefschetz pencils on symplectic manifolds*, preprint.

[Du] J. Duval, *Convexité rationnelle des tores lagrangiens de* $\mathbb{C}^2$, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 943–948.

[DS] J. Duval, N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. **79** (1995), 487–513.

[Ga] D. Gayet, Ph.D. thesis, Université Toulouse III, in preparation.

[Gu] V. Guedj, *Approximation of currents on complex manifolds*, Math. Ann. **313** (1999), 437–474.

[McS] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, 1995.

[M] J.-P. Mohsen, Ph.D. thesis, ENS Lyon, in preparation.
[S] J.-C. Sikorav, *Construction de sous-variétés symplectiques (d'après S. K. Donaldson et D. Auroux)*, Séminaire Bourbaki. Vol. 1997/98. Astérisque **252** (1998), 231–253.
[W] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Advances in Math. **6** (1971), 329–346.

Department of Mathematics, M.I.T., Cambridge MA 02139-4307, USA
E-mail address: auroux@math.mit.edu

Laboratoire Emile Picard, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4, France
E-mail address: gayet@picard.ups-tlse.fr

UMPA, ENS-Lyon, UMR 5669, 46 allée d'Italie, 69364 Lyon Cedex 7, France
E-mail address: jpmohsen@umpa.ens-lyon.fr

A REMARK ABOUT DONALDSON'S CONSTRUCTION OF SYMPLECTIC SUBMANIFOLDS

D. AUROUX

Abstract. We describe a simplification of Donaldson's arguments for the construction of symplectic hypersurfaces [4] or Lefschetz pencils [5] that makes it possible to avoid any reference to Yomdin's work on the complexity of real algebraic sets.

1. Introduction

Donaldson's construction of symplectic submanifolds [4] is unquestionably one of the major results obtained in the past ten years in symplectic topology. What sets it apart from many of the results obtained during the same period is that it appeals neither to Seiberg-Witten theory, nor to pseudo-holomorphic curves; in fact, most of Donaldson's argument is a remarkable succession of elementary observations, combined in a particularly clever way. One ingredient of the proof that does not qualify as elementary, though, is an effective version of Sard's theorem for approximately holomorphic complex-valued functions over a ball in $\mathbb{C}^n$ (Theorem 20 in [4]). The proof of this result, which occupies a significant portion of Donaldson's paper (§4 and §5 of [4]), appeals to very subtle considerations about the complexity of real algebraic sets, following ideas of Yomdin [6].

Methods similar to those in [4] were subsequently used to perform various other constructions, leading in particular to Donaldson's result that symplectic manifolds carry structures of symplectic Lefschetz pencils [5], or to the result that symplectic 4-manifolds can be realized as branched coverings of $\mathbb{CP}^2$ [2]. It was observed in [3] that, whereas Donaldson's construction of submanifolds can be thought of in terms of an estimated transversality result for sections of line bundles, the subsequent constructions can be interpreted in terms of estimated transversality with respect to stratifications in jet bundles.

As remarked at the end of §4 in [3], the transversality of the r-jet of a section to a given submanifold in the bundle of r-jets is equivalent to the non-intersection of the $(r + 1)$-jet of the section with a certain (possibly singular) submanifold of greater codimension in the bundle of $(r + 1)$-jets. This is of particular interest because the effective Sard theorem for approximately holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}^m$ admits a conceptually much easier proof in the case where $m > n$ [2]. In the case of the construction of symplectic submanifolds, the formalism of jet bundles can be completely eliminated from the presentation; the purpose of this note is to present the resulting simplified argument for Donaldson's result (§§2–3). We also observe (see §4) that a similar simplification is possible for the higher-rank local result required for the construction of symplectic Lefschetz pencils [5].

2. Overview of Donaldson's argument

We first review Donaldson's construction of symplectic submanifolds [4], using the terminology and notations of [2]. Let (X^{2n},ω) be a compact symplectic manifold, and assume that the cohomology class $\frac{1}{2\pi}[\omega]$ is integral. Endow X with an ω-compatible almost-complex structure J and the corresponding Riemannian metric $g=\omega(\cdot,J\cdot)$. Consider a Hermitian line bundle L over X such that $c_1(L)=\frac{1}{2\pi}[\omega]$, equipped with a Hermitian connection ∇ having curvature $-i\omega$. The almost-complex structure J induces a splitting of the connection into $\nabla=\partial+\bar\partial$. We are interested in approximately holomorphic sections of the line bundles $L^{\otimes k}$ $(k\gg 0)$ satisfying a certain estimated transversality property: indeed, if we can find a section s such that $|\bar\partial s|\ll|\partial s|$ at every point where s vanishes, then the zero set of s is automatically a smooth symplectic submanifold in X (cf. e.g. Proposition 3 of [4]). The philosophical justification of the construction is that, as the twisting parameter k increases, one starts probing the geometry of X at very small scales, where the effects due to the non-integrability of J become negligible. This phenomenon is due to the curvature $-ik\omega$ of the connection on $L^{\otimes k}$, and leads us to work with a rescaled metric $g_k=k\,g$ (the metric induced by J and $k\omega$).

Let $(s_k)_{k\gg 0}$ be a sequence of sections of Hermitian vector bundles E_k equipped with Hermitian connections over X. We make the following definitions:

Definition 1. *The sections s_k are* asymptotically holomorphic *if there exist constants $(C_p)_{p\in\mathbb{N}}$ such that, for all k and at every point of X, $|s_k|\le C_0$, $|\nabla^p s_k|_{g_k}\le C_p$ and $|\nabla^{p-1}\bar\partial s_k|_{g_k}\le C_p k^{-1/2}$ for all $p\ge 1$.*

Definition 2. *The sections s_k are* uniformly transverse to 0 *if there exists a constant $\eta>0$ independent of k such that the sections s_k are* η-transverse to 0*, i.e. if at any point $x\in X$ where $|s_k(x)|<\eta$, the linear map $\nabla s_k(x):T_xX\to(E_k)_x$ is surjective and has a right inverse of norm less than η^{-1} w.r.t. the metric g_k.*

When $\mathrm{rank}(E_k)>n$, uniform transversality means that $|s_k(x)|\ge\eta$ at every point of X; on the other hand, when E_k is a line bundle and the sections s_k are asymptotically holomorphic, uniform transversality can be rephrased as a uniform lower bound on $|\partial s_k|$ at all points where $|s_k|<\eta$ (which by the above observation is enough to ensure the symplecticity of $s_k^{-1}(0)$ for large k). With this terminology, Donaldson's result can be reformulated as follows (cf. Theorem 5 of [4]):

Theorem 1. *For large values of k, the line bundles $L^{\otimes k}$ admit sections s_k that are asymptotically holomorphic and uniformly transverse to* 0*.*

The proof of Theorem 1 starts with a couple of preliminary lemmas about the existence of approximately holomorphic rescaled Darboux coordinates on X and of large families of well-concentrated asymptotically holomorphic sections of $L^{\otimes k}$.

Lemma 1. *There exists a constant $c>0$ such that near any point $x\in X$, for any integer k, there exist local complex Darboux coordinates $z_k=(z_k^1,\dots,z_k^n):(X,x)\to(\mathbb{C}^n,0)$ for the symplectic structure $k\omega$, such that the following estimates hold uniformly in x and k at every point of the ball $B_{g_k}(x,c\sqrt{k})$: $|z_k(y)|=O(\mathrm{dist}_{g_k}(x,y))$, $|\bar\partial z_k(y)|_{g_k}=O(k^{-1/2}\mathrm{dist}_{g_k}(x,y))$, $|\nabla^r\bar\partial z_k|_{g_k}=O(k^{-1/2})$, $|\nabla^r z_k|_{g_k}=O(1)$ $\forall r\ge 1$; and denoting by $\psi_k:(\mathbb{C}^n,0)\to(X,x)$ the inverse map, the estimates $|\bar\partial\psi_k(z)|_{g_k}=O(k^{-1/2}|z|)$, $|\nabla^r\bar\partial\psi_k|_{g_k}=O(k^{-1/2})$ and $|\nabla^r\psi_k|_{g_k}=O(1)$ hold $\forall r\ge 1$ at every point* *of the ball $B_{\mathbb{C}^n}(0, c\sqrt{k})$, where $\bar\partial\psi_k$ is defined with respect to the almost-complex structure J on X and the standard complex structure on $\mathbb{C}^2$.*

Lemma 1 is identical to Lemma 3 of [2], or to the discussion on pp. 674–675 of [4] if one keeps track carefully of the available estimates; the idea is simply to start with usual Darboux coordinates for ω, compose them with a linear transformation to ensure holomorphicity at the origin, and then rescale them by a factor of $\sqrt{k}$.

Definition 3. *A section s of E_k has* Gaussian decay *in C^r norm away from a point $x \in X$ if there exist a polynomial P and a constant $\lambda > 0$ such that for all $y \in X$, $|s(y)|$, $|\nabla s(y)|_{g_k}$, ..., $|\nabla^r s(y)|_{g_k}$ are all bounded by $P(d(x,y))\exp(-\lambda\, d(x,y)^2)$, where $d(.,.)$ is the distance induced by g_k. The decay properties of a family of sections are said to be* uniform *if P and λ can be chosen independently of k and of the point x at which decay occurs for a given section.*

Lemma 2. *Given any point $x \in X$, for all large enough k, there exist asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x}$ of $L^{\otimes k}$ over X, such that $|s^{\mathrm{ref}}_{k,x}| \geq c_0$ at every point of the ball of g_k-radius 1 centered at x, for some universal constant $c_0 > 0$, and such that the sections $s^{\mathrm{ref}}_{k,x}$ have uniform Gaussian decay away from x in C^2 norm.*

Lemma 2 is essentially Proposition 11 of [4]. Considering a local trivialization of $L^{\otimes k}$ where the connection 1-form is $\frac{1}{4}\sum(z_k^j d\bar z_k^j - \bar z_k^j dz_k^j)$, the sections $s^{\mathrm{ref}}_{k,x}$ are constructed by multiplication of the function $\exp(-|z_k|^2/4)$ by a suitable cut-off function at distance $k^{1/6}$ from the origin.

The central ingredient is the following result about the near-critical sets of approximately holomorphic functions (used in the special case $m = 1$):

Proposition 1. *Let f be a function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}^m$. Let δ be a constant with $0 < \delta < \frac{1}{4}$, and let $\eta = \delta\log(\delta^{-1})^{-p}$ where p is a fixed integer depending only on n and m. Assume that f satisfies the bounds $|f|_{C^0(B^+)} \leq 1$ and $|\bar\partial f|_{C^1(B^+)} \leq \eta$. Then there exists $w \in \mathbb{C}^m$ with $|w| \leq \delta$ such that $f - w$ is η-transverse to 0 over the interior ball B of radius 1.*

The case $m = 1$ is Theorem 20 of [4]; the comparatively much easier case $m > n$ is Proposition 2 of [2]; the general case is proved in [5]. In all cases the proof begins with an approximation of f first by a holomorphic function (using general elliptic theory), then by a polynomial g of degree $O(\log(\eta^{-1}))$ (by truncating the power series expansion at the origin). The proof in the case $m = 1$ then appeals to a rather sophisticated result on the complexity of real algebraic sets to control the size of the set of points where ∂g is small (the near-critical points) [4]. Meanwhile, in the case $m > n$, since we only have to find w such that $|f - w| \geq \eta$ at every point of B, it is sufficient to observe that the image of the polynomial map g is contained in a complex algebraic hypersurface H in $\mathbb{C}^m$; the result then follows from a standard result about the volume of a tubular neighborhood of H, which can be estimated using an explicit bound on the degree of H [2].

Given asymptotically holomorphic sections s_k of $L^{\otimes k}$ and a point $x \in X$, one can apply Proposition 1 to the complex-valued functions $f_k = s_k/s^{\mathrm{ref}}_{k,x}$ (defined over a neighborhood of x) in order to find constants w_k such that the functions $f_k - w_k$ are uniformly transverse to 0 near x; multiplying by $s^{\mathrm{ref}}_{k,x}$, it follows that the sections $s_k - w_k s^{\mathrm{ref}}_{k,x}$ are uniformly transverse to 0 near x. Therefore, we have:

Proposition 2. *There exist constants c, c', p, $\delta_0 > 0$ such that, given a real number $\delta \in (0, \delta_0)$, a sequence of asymptotically holomorphic sections s_k of $L^{\otimes k}$ and a point $x \in X$, for large enough k there exist asymptotically holomorphic sections $\tau_{k,x}$ of $L^{\otimes k}$ with the following properties:* (a) $|\tau_{k,x}|_{C^1, g_k} < \delta$, (b) *the sections $\frac{1}{\delta}\tau_{k,x}$ have uniform Gaussian decay away from x in C^1 norm, and* (c) *the sections $s_k + \tau_{k,x}$ are η-transverse to 0 at every point of the ball $B_{g_k}(x, c)$, with $\eta = c'\delta\log(\delta^{-1})^{-p}$.*

This result lets us achieve estimated transversality over a small ball in X by adding to s_k a small well-concentrated perturbation. Uniform transversality over the entire manifold X is achieved by proceeding iteratively, adding successive perturbations to the sections in order to obtain transversality properties over larger and larger subsets of X. The key observation is that estimated transversality is an open property (preserved under C^1-small perturbations). Since the transversality estimate decreases after each perturbation, it is important to obtain global uniform transversality after a number of steps that remains bounded independently of k; this is made possible by the uniform decay properties of the perturbations, using a beautiful observation of Donaldson. The reader is referred to §3 of [4] or to Proposition 3 of [2] for details.

3. The simplified argument

Keeping the same general strategy, the proof of Theorem 1 can be simplified by appealing to a result weaker than Proposition 1, namely the following statement:

Proposition 3. *Let f be a function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}$. Let δ be a constant with $0 < \delta < \frac{1}{4}$, and let $\eta = \delta\log(\delta^{-1})^{-p'}$ where p' is a fixed integer depending only on n. Assume that f satisfies the bounds $|f|_{C^1(B^+)} \leq 1$ and $|\bar\partial f|_{C^2(B^+)} \leq \eta$. Then there exists $w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1}$ with $|w| \leq \delta$ such that the function $f - w_0 - \sum w_i z_i$ is η-transverse to 0 over the interior ball B of radius 1.*

Proof. Let $g = (g_0, \dots, g_n) : B^+ \to \mathbb{C}^{n+1}$ be the function defined by $g_i = \partial f/\partial z_i$ for $1 \leq i \leq n$ and $g_0 = f - \sum_{i=1}^n z_i g_i$. The bounds on f immediately imply that $|g|_{C^0(B^+)} \leq C_n$ and $|\bar\partial g|_{C^1(B^+)} \leq C_n\eta$, for some constant C_n depending only on the dimension. We can safely choose the constant p' appearing in the definition of η to be larger than the constant p appearing in Proposition 1. Therefore we can apply Proposition 1 in its easy version ($m = n+1$) to the function g, after scaling down by the constant factor C_n. This gives us a constant $w = (w_0, \dots, w_n) \in \mathbb{C}^{n+1}$, bounded by δ, and such that $|g - w| \geq \alpha$ at every point of B, where $\alpha = \delta\log((\delta/C_n)^{-1})^{-p}$.

Define $\tilde f = f - w_0 - \sum w_i z_i$ and $\tilde g = g - w$, and observe that $\partial\tilde f/\partial z_i = \tilde g_i$ for $1 \leq i \leq n$ and $\tilde f - \sum_{i=1}^n z_i \tilde g_i = \tilde g_0$. Let $z \in B$ be a point where $|\partial\tilde f| < \frac{1}{4}\alpha$. Since $\partial\tilde f/\partial z_i = \tilde g_i$, and since $|\tilde g(z)| \geq \alpha$ by construction, we have the inequality $|\tilde g_0(z)| > \frac{3}{4}\alpha$. However, $|\tilde f(z) - \tilde g_0(z)| = |\sum z_i \tilde g_i(z)| \leq |z|\,|\partial\tilde f(z)| < \frac{1}{4}\alpha$ (recall that z belongs to the unit ball). Therefore $|\tilde f(z)| > \frac{1}{2}\alpha$.

Conversely, at any point $z \in B$ where $|\tilde f| \leq \frac{1}{2}\alpha$ we must have $|\partial\tilde f(z)| \geq \frac{1}{4}\alpha$. However, because of the bound on $\bar\partial\tilde f = \bar\partial f$, if we assume that $\eta < \frac{1}{8}\alpha$ then this inequality implies that $\nabla\tilde f(z)$ is surjective and admits a right inverse of norm at most $(\frac{1}{8}\alpha)^{-1}$. Hence we conclude from the previous discussion that $\tilde f$ is $\frac{1}{8}\alpha$-transverse to 0 over B. Finally, we observe that, because $\delta < \frac{1}{4}$, if the constant p' is chosen large enough then $\eta = \delta \log(\delta^{-1})^{-p'} < \frac{1}{8}\alpha = \frac{1}{8}\delta \log((\delta/C_n)^{-1})^{-p}$, so that $\tilde{f}$ is η-transverse to 0 over B. □

Although it is weaker, Proposition 3 is in practice interchangeable with the case $m = 1$ of Proposition 1, in particular for the purpose of proving Proposition 2.

Proof of Proposition 2. We use the same argument as Donaldson [4]: we work in approximately holomorphic Darboux coordinates on a neighborhood of the given point x, using Lemma 1. Using the sections $s^{\mathrm{ref}}_{k,x}$ given by Lemma 2 to define local trivializations of $L^{\otimes k}$, the sections s_k can be identified with complex-valued functions $f_k = s_k/s^{\mathrm{ref}}_{k,x}$. The estimates on s_k and $s^{\mathrm{ref}}_{k,x}$ imply that the functions f_k are approximately holomorphic near the origin (in particular $|\bar\partial f_k|_{C^2} = O(k^{-1/2})$); after a suitable rescaling of the coordinates and of the functions by uniform constant factors, we can assume additionally that $|f_k|_{C^1} \leq 1$ near the origin, and that the estimates hold over a neighborhood of the origin that contains the ball B^+. Therefore, the assumptions of Proposition 3 are satisfied provided that k is sufficiently large to ensure that $k^{-1/2} \ll \eta$.

By Proposition 3, we can find $w_k = (w_{k,0}, \dots, w_{k,n}) \in \mathbb{C}^{n+1}$, with $|w_k| \leq \delta$, such that $\tilde{f}_k = f_k - w_{k,0} - \sum w_{k,i} z_i$ is γ-transverse to 0 over the unit ball, where $\gamma = \delta \log(\delta^{-1})^{-p'}$. Define $\tau_{k,x} = -w_{k,0} s^{\mathrm{ref}}_{k,x} - \sum w_{k,i} z^i_k s^{\mathrm{ref}}_{k,x}$. The estimates on z^i_k from Lemma 1 and on $s^{\mathrm{ref}}_{k,x}$ from Lemma 2 imply that the sections $z_{k,i} s^{\mathrm{ref}}_{k,x}$ of $L^{\otimes k}$ are asymptotically holomorphic and have uniform Gaussian decay away from x. Therefore, it is easy to check that the sections $\frac{1}{\delta}\tau_{k,x}$ are asymptotically holomorphic and have uniform Gaussian decay. Moreover, because there exist uniform bounds on $s^{\mathrm{ref}}_{k,x}$ and $z^i_k s^{\mathrm{ref}}_{k,x}$, one easily checks that $|\tau_{k,x}|_{C^1, g_k}$ is bounded by some constant multiple of δ; decreasing the required bound on $|w_k|$, we can assume that the constant is equal to 1, to the expense of inserting a constant factor in the above expression for γ. Finally, observing that $s_k + \tau_{k,x} = \tilde{f}_k s^{\mathrm{ref}}_{k,x}$ over a neighborhood of x, it is straightforward to check that the γ-transversality to 0 of $\tilde{f}_k$ and the lower bound satisfied by $s^{\mathrm{ref}}_{k,x}$ imply a uniform transversality property of the desired form for the section $s_k + \tau_{k,x}$. □

Remark 1. Proposition 3 also admits a version for one-parameter families of functions: given functions $f_t : B^+ \to \mathbb{C}$ depending continuously on a parameter $t \in [0, 1]$ and satisfying the assumptions of Proposition 3 for all values of t, we can find constants $w_t \in \mathbb{C}^{n+1}$, depending continuously on t, such that the conclusion holds for all values of t. This is because the auxiliary functions $g_t : B^+ \to \mathbb{C}^{n+1}$ introduced in the proof also depend continuously on t, which allows us to appeal to the one-parameter version of Proposition 1 (cf. e.g. Proposition 2 of [2]). We can therefore simplify the argument proving the asymptotic uniqueness of the constructed submanifolds [1] in the same manner as the construction itself.

Remark 2. The idea behind the modified argument can be interpreted as follows in terms of 1-jets of sections: let $\mathcal{J}^1 L^{\otimes k} = L^{\otimes k} \oplus (T^*X^{1,0} \otimes L^{\otimes k})$, and define the 1-jet of a section $s_k \in \Gamma(L^{\otimes k})$ as $j^1 s_k = (s_k, \partial s_k) \in \Gamma(\mathcal{J}^1 L^{\otimes k})$. The jet bundles carry natural Hermitian metrics (induced by those on $L^{\otimes k}$ and the metrics g_k on the cotangent bundle), and natural Hermitian connections for which the 1-jets of asymptotically holomorphic sections of $L^{\otimes k}$ are asymptotically holomorphic sections of $\mathcal{J}^1 L^{\otimes k}$. It is worth noting that the natural connection on $\mathcal{J}^1 L^{\otimes k}$ is *not* the connection ∇ induced by the connection on $L^{\otimes k}$ and the Levi-Civita connection, because $\bar\partial^\nabla(s_k,\partial s_k)=(\bar\partial s_k,\bar\partial\partial s_k)$ differs from $(\bar\partial s_k,-\partial\bar\partial s_k)$ (which is bounded by $O(k^{-1/2})$) by the curvature term $-ik\omega s_k$. Therefore, we must instead work with the Hermitian connection $\tilde\nabla$ characterized by the formula $\bar\partial^{\tilde\nabla}(\sigma^0,\sigma^1)=\bar\partial^\nabla(\sigma^0,\sigma^1)+(0,ik\omega\sigma^0)$, where ω is viewed as a $(0,1)$-form with values in $T^*X^{1,0}$.

Observe that the 1-jets $j^1\sigma_{k,x,0},\dots,j^1\sigma_{k,x,n}$, where $\sigma_{k,x,0}=s^{\mathrm{ref}}_{k,x}$ and $\sigma_{k,x,i}=z_k^i s^{\mathrm{ref}}_{k,x}$ for $1\le i\le n$, are asymptotically holomorphic sections of $\mathcal{J}^1L^{\otimes k}$, with uniform Gaussian decay away from x, which form a local frame of the jet bundle over a neighborhood of x. Therefore, given asymptotically holomorphic sections s_k and a point $x\in X$, there exist local complex-valued functions $g_{k,0},\dots,g_{k,n}$ such that $j^1s_k=\sum g_{k,i}\,j^1\sigma_{k,x,i}$. Moreover, remark that a section of $L^{\otimes k}$ is uniformly transverse to 0 if and only if its 1-jet satisfies a uniform lower bound. Therefore, our argument actually amounts to a local perturbation of j^1s_k, using the given local frame $\{j^1\sigma_{k,x,i}\}$, in order to bound it away from 0; because the rank of the jet bundle is $n+1>n$, the easy version of Proposition 1 is sufficient for that purpose. The curious reader is referred to [3] for a more detailed discussion of estimated transversality using the formalism of jet bundles.

4. The higher-rank local result

We now formulate and prove an analogue of Proposition 3 for functions with values in $\mathbb{C}^m$ ($m\le n$); as in the case $m=1$, the statement differs from Proposition 1 by allowing the extra freedom of affine perturbations rather than restricting oneself to constants.

Proposition 4. *Let f be a function defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$ with values in $\mathbb{C}^m$, $m\le n$. Let δ be a constant with $0<\delta<\frac14$, and let $\eta=\delta\log(\delta^{-1})^{-p'}$ where p' is a fixed integer depending only on m and n. Assume that f satisfies the bounds $|f|_{C^0(B^+)}\le 1$ and $|\bar\partial f|_{C^1(B^+)}\le\eta$. Then there exists $w=(w_0,w_1,\dots,w_n)\in\mathbb{C}^{m(n+1)}$ (each w_i is an element of $\mathbb{C}^m$) with $|w|\le\delta$ such that the function $f-w_0-\sum w_iz_i$ is η-transverse to 0 over the interior ball B of radius 1.*

Moreover, given a one-parameter family of functions $f_t:B^+\to\mathbb{C}$ depending continuously on a parameter $t\in[0,1]$ and satisfying the above assumptions for all t, we can find constants $w_t\in\mathbb{C}^{m(n+1)}$, depending continuously on t, such that the conclusion holds for all values of t.

This statement is essentially interchangeable with Proposition 1 for all practical applications, and in particular the case $m=n$ allows us to simplify noticeably the argument for Donaldson's construction of symplectic Lefschetz pencils [5]. Indeed, the main problem to be solved is the following: given pairs of asymptotically holomorphic sections (s_k^0,s_k^1) of $L^{\otimes k}$, defining $\mathbb{CP}^1$-valued maps $f_k=[s_k^0:s_k^1]$ away from the base loci, one must perturb them so that the differentials ∂f_k (which are sections of rank n vector bundles) become uniformly transverse to 0. This ensures the non-degeneracy of the singular points of the pencil. The manner in which the problem reduces to the $m=n$ case of Proposition 1 is explained in detail in [5], and the reduction to Proposition 4 is essentially identical except that the resulting perturbations of (s_k^0,s_k^1) are products of $s^{\mathrm{ref}}_{k,x}$ by quadratic (rather than linear) polynomials.

Proof. Although for technical reasons we cannot use directly the case $m > n$ of Proposition 1, the argument presents many similarities with §2.3 of [2]; we accordingly skip the details whenever the two arguments parallel each other in an obvious manner. As in the case of Proposition 1, we first use the bounds on f to find an approximation by a polynomial $h : \mathbb{C}^n \to \mathbb{C}^m$ of degree $d = O(\log(\eta^{-1}))$ such that $|h - f|_{C^1(B)} \leq c\,\eta$ for some constant c (see Lemmas 27 and 28 of [4]). Observe that, if we can perturb h by less than δ to make it $(c+1)\eta$-transverse to 0 over B, then because transversality is an open property the desired result on f will follow immediately. So we are reduced to the case of a polynomial function $h = (h^1, \dots, h^m)$ of degree $d = O(\log(\eta^{-1}))$.

If $w = (w_0, \dots, w_n)$ is a vector in $\mathbb{C}^{m(n+1)}$, denote by $(w_i^j)_{1\leq j\leq m}$ the components of w_i, and let $\vec{w} = (w_1, \dots, w_n) \in \mathbb{C}^{m\times n}$. The set of choices to be avoided for w is

$$S = \{w \in \mathbb{C}^{m(n+1)},\ \exists z \in B \text{ s.t. } h(z) - w_0 - \textstyle\sum w_i z_i = 0,\ \bigwedge^m(\partial h(z) - \vec{w}) = 0\}.$$

Indeed, observe that $h - w_0 - \sum w_i z_i$ is transverse to 0 over B (without any estimate) if and only if $w \notin S$. We now define a polynomial function $g : \mathbb{C}^{N-1} \to \mathbb{C}^N$, where $N = m(n+1)$, which parametrizes a dense subset of S. Given elements $z = (z_i)_{1\leq i\leq n} \in \mathbb{C}^n$, $\theta = (\theta_i^j)_{1\leq i\leq n,\ 1\leq j\leq m-1} \in \mathbb{C}^{(m-1)n}$ and $\lambda = (\lambda_j)_{1\leq j\leq m-1} \in \mathbb{C}^{m-1}$, we define $g(z,\theta,\lambda) \in \mathbb{C}^{m(n+1)}$ by the formulas

$$\begin{cases} g_i^j(z,\theta,\lambda) = \dfrac{\partial h^j}{\partial z_i}(z) + \theta_i^j & \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m-1,\\ g_i^m(z,\theta,\lambda) = \dfrac{\partial h^m}{\partial z_i}(z) + \displaystyle\sum_{j=1}^{m-1} \lambda_j \theta_i^j & \text{for } 1 \leq i \leq n,\\ g_0^j(z,\theta,\lambda) = h^j(z) - \displaystyle\sum_{i=1}^{n} g_i^j(z,\theta,\lambda) z_i & \text{for } 1 \leq j \leq m. \end{cases}$$

One easily checks that the image by g of the subset $\{(z,\theta,\lambda) \in \mathbb{C}^{N-1},\ z \in B\}$ is contained in S, in which it is a dense subset. Observe that g is a polynomial map with the same degree d as h (provided that $d \geq 2$). Therefore, the image $g(\mathbb{C}^{N-1})$ is contained in an algebraic surface $H \subset \mathbb{C}^N$, of degree at most $D = N\,d^{N-1}$. Indeed, denoting by E the space of polynomials of degree at most D in N variables and by E' the space of polynomials of degree at most dD in $N-1$ variables, we have $\dim E = \binom{D+N}{N} > \binom{dD+N-1}{N-1} = \dim E'$, so that the map from E to E' defined by $P \mapsto P \circ g$ cannot be injective, and a non-zero element of its kernel provides an equation for the hypersurface H (see §2.3 of [2] for details).

Since $g(B \times \mathbb{C}^{(m-1)n} \times \mathbb{C}^{m-1})$ is dense in S, we conclude that $S \subset H$. From this point on, the argument is very similar to §2.3 of [2], to which the reader is referred for details. Standard results on complex algebraic hypersurfaces, essentially amounting to the well-known monotonicity formula, allow us to bound the size of S and of its tubular neighborhoods (cf. e.g. Lemma 4 of [2]). In particular, denoting by $\bar{B}$ the ball of radius δ centered at the origin in $\mathbb{C}^N$ and by V_0 the volume of the unit ball in dimension $2N-2$, we have $vol_{2N-2}(H \cap \bar{B}) \leq DV_0\delta^{2N-2}$, while given any point $x \in H$ we have $vol_{2N-2}(H \cap B(x,\eta)) \geq V_0\eta^{2N-2}$. Therefore, choosing a suitable covering of $\bar{B}$ by balls of radius η, one can show that $H \cap \bar{B}$ is contained in the union of $M = C\,D\,\delta^{2N-2}\eta^{-(2N-2)}$ balls of radius η, where C is a constant depending only on N. As a consequence, the neighborhood $Z = \{w \in \mathbb{C}^N,\ |w| \leq \delta,\ \mathrm{dist}(w,S) \leq (3c+3)\eta\}$ is contained in the union of M balls of radius $(3c+4)\eta$.

A simple comparison of the volumes implies that, if the constant p' is chosen suitably large, then the volume of Z is much smaller than that of the ball $\bar{B}$, and therefore $\bar{B} - Z$ is not empty, i.e. $\bar{B}$ contains an element w which lies at distance more than $(3c+3)\eta$ from S. Moreover, using a standard isoperimetric inequality we can show that $\bar{B} - Z$ contains a unique large connected component; it follows that, in the case where the data depends continuously on a parameter $t \in [0,1]$, the subset $\bigsqcup \{t\} \times (\bar{B} - Z_t) \subset [0,1] \times \bar{B}$ contains a preferred large connected component, in which we can choose elements w_t depending continuously on t.

To complete the proof of Proposition 4, we only need to show that, if $w \in \bar{B}$ lies at distance more than $(3c+3)\eta$ from S, then $\tilde{f} = f - w_0 - \sum w_i z_i$ is η-transverse to 0 over B. In fact, it is sufficient to show that $\tilde{h} = h - w_0 - \sum w_i z_i$ is $(c+1)\eta$-transverse to 0 over B, because $|\tilde{h} - \tilde{f}|_{C^1(B)} = |h - f|_{C^1(B)} \leq c\eta$ and transversality is an open property. We conclude using the following lemma:

Lemma 3. *If w lies at distance more than 3α from S for some constant $\alpha > 0$, then $\tilde{h} = h - w_0 - \sum w_i z_i$ is α-transverse to 0 over B.*

To prove Lemma 3, we first provide an alternative definition of α-transversality:

Lemma 4. *Let $L : E \to F$ be a linear map between Hermitian complex vector spaces, and choose a constant $\alpha > 0$. The two following properties are equivalent:*
(i) L is surjective and has a right inverse $R : F \to E$ of norm at most α^{-1},
(ii) for every unit vector v in F, the component $\langle v, L\rangle = v^* L$ of L along v is a linear form on E such that $|v^* L| \geq \alpha$.

Proof. If (i) holds, then given any unit vector $v \in F$, the vector $u = Rv$ is such that $|u| \leq \alpha^{-1}$ and $\langle v, Lu\rangle = |v|^2 = 1$. Therefore the linear form $\langle v, L\rangle$ has norm at least α, and (ii) holds.

Conversely, assume (ii) holds. Then for any $v \in F$ we have $|v^* L| \geq \alpha |v|$, i.e. $v^* L L^* v \geq \alpha^2 |v|^2$. Therefore, the Hermitian endomorphism LL^* of F is positive definite and has eigenvalues $\geq \alpha^2$. It follows that it admits an inverse $U = (LL^*)^{-1}$ of operator norm at most α^{-2}. We have $LL^*U = \mathrm{Id}$, and $|L^*Uv|^2 = \langle v, ULL^*Uv\rangle = \langle v, Uv\rangle \leq \alpha^{-2}|v|^2$, so that $R = L^*U$ is a right inverse of norm at most α^{-1}. □

Proof of Lemma 3. Assume that $\tilde{h}$ is not α-transverse to 0 over B: using the definition and Lemma 4, there exists a point $z \in B$ and a unit vector $v \in \mathbb{C}^m$ such that $|\tilde{h}(z)| < \alpha$ and $|\langle v, \partial\tilde{h}(z)\rangle| < \alpha$. Let $u = (u_0, u_1, \dots, u_n) \in \mathbb{C}^{m(n+1)}$ be such that $u_i = \langle v, \partial\tilde{h}/\partial z_i\rangle\, v$ and $u_0 = \tilde{h}(z) - \sum z_i u_i$. We clearly have $|(u_1, \dots, u_n)| < \alpha$, and $|u_0| < 2\alpha$, so that $|u| < 3\alpha$. On the other hand, if we consider the function $\hat{h} = h - (w_0 + u_0) - \sum (w_i + u_i) z_i$, then by construction $\hat{h}(z) = 0$ and $\langle v, \partial\hat{h}(z)\rangle = 0$. Therefore $w + u \in S$, and so w is within distance 3α of S. □

References

[1] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[2] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$, Invent. Math. **139** (2000), 551–602.

[3] D. Auroux, *Estimated transversality in symplectic geometry and projective maps*, in "Symplectic Geometry and Mirror Symmetry", Proc. 4th KIAS International Conference, Seoul (2000), World Scientific, 2001, 1–30 (math.SG/0010052).

[4] S. K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Diff. Geom. **44** (1996), 666–705.
[5] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Diff. Geom. **53** (1999), 205–236.
[6] Y. Yomdin, *The geometry of critical and near-critical values of differentiable mappings*, Math. Annalen **264** (1983), 495–515.

FIBER SUMS OF GENUS 2 LEFSCHETZ FIBRATIONS

D. AUROUX

Abstract. Using the recent results of Siebert and Tian about the holomorphicity of genus 2 Lefschetz fibrations with irreducible singular fibers, we show that any genus 2 Lefschetz fibration becomes holomorphic after fiber sum with a holomorphic fibration.

1. Introduction

Symplectic Lefschetz fibrations have been the focus of a lot of attention since the proof by Donaldson that, after blow-ups, every compact symplectic manifold admits such structures [3]. Genus 2 Lefschetz fibrations, where the first non-trivial topological phenomena arise, have been particularly studied. Most importantly, it has recently been shown by Siebert and Tian that every genus 2 Lefschetz fibration without reducible fibers and with "transitive monodromy" is holomorphic [9]. The statement becomes false if reducible singular fibers are allowed, as evidenced by the construction by Ozbagci and Stipsicz [7] of genus 2 Lefschetz fibrations with non-complex total space (similar examples have also been constructed by Ivan Smith; the reader is also referred to the work of Amorós et al [1] and Korkmaz [5] for related constructions).

It has been conjectured by Siebert and Tian that any genus 2 Lefschetz fibration should become holomorphic after fiber sum with sufficiently many copies of the rational genus 2 Lefschetz fibration with 20 irreducible singular fibers. The purpose of this paper is to prove this conjecture by providing a classification of genus 2 Lefschetz fibrations up to stabilization by such fiber sums. The result is the following (see §2 and Definition 4 for notations):

Theorem 1. *Let F be any factorization of the identity element as a product of positive Dehn twists in the mapping class group* Map_2*. Then there exist integers* $\epsilon \in \{0,1\}$*,* $k \geq 0$ *and* $m \geq 0$ *such that, for any large enough integer* n*, the factorization* $F \cdot (W_0)^n$ *is Hurwitz equivalent to* $(W_0)^{n+k} \cdot (W_1)^\epsilon \cdot (W_2)^m$*.*

Corollary 2. *Let* $f : X \to S^2$ *be a genus* 2 *Lefschetz fibration. Then the fiber sum of* f *with sufficiently many copies of the rational genus* 2 *Lefschetz fibration with* 20 *irreducible singular fibers is isomorphic to a holomorphic fibration.*

2. Mapping class group factorizations

Recall that a Lefschetz fibration $f : X \to S^2$ is a fibration admitting only isolated singularities, all lying in distinct fibers of f, and near which a local model for f in orientation-preserving complex coordinates is given by $(z_1, z_2) \mapsto z_1^2 + z_2^2$. We will only consider the case $\dim X = 4$, where the smooth fibers are compact oriented surfaces (of genus $g = 2$ in our case), and the singular fibers present nodal singularities obtained by collapsing a simple closed loop (the *vanishing cycle*) in the smooth fiber. The monodromy of the fibration around a singular fiber is given by a positive Dehn twist along the vanishing cycle.

Denoting by $q_1, \dots, q_r \in S^2$ the images of the singular fibers and choosing a reference point in S^2, we can characterize the fibration f by its *monodromy* $\psi : \pi_1(S^2 - \{q_1, \dots, q_r\}) \to \mathrm{Map}_g$, where $\mathrm{Map}_g = \pi_0\mathrm{Diff}^+(\Sigma_g)$ is the mapping class group of an oriented genus g surface. It is a classical result (cf. [4]) that the monodromy morphism ψ is uniquely determined up to conjugation by an element of Map_g and a braid acting on $\pi_1(S^2 - \{q_i\})$, and that it determines the isomorphism class of the Lefschetz fibration f.

While all positive Dehn twists along non-separating curves are mutually conjugate in Map_g, there are different types of twists along separating curves, according to the genus of each component delimited by the curve. When $g = 2$, only two cases can occur: either the curve splits the surface into two genus 1 components, or it is homotopically trivial and the corresponding singular fiber contains a sphere component of square -1. The latter case can always be avoided by blowing down the total space of the fibration; if the blown-down fibration can be shown to be holomorphic, then by performing the converse blow-up procedure we conclude that the original fibration was also holomorphic. Therefore, in all the following we can assume that our Lefschetz fibrations are *relatively minimal*, i.e. have no homotopically trivial vanishing cycles.

The monodromy of a Lefschetz fibration can be encoded in a *mapping class group factorization* by choosing an ordered system of generating loops $\gamma_1, \dots, \gamma_r$ for $\pi_1(S^2 - \{q_i\})$, such that each loop γ_i encircles only one of the points q_i and $\prod \gamma_i$ is homotopically trivial. The monodromy of the fibration along each of the loops γ_i is a Dehn twist τ_i; we can then describe the fibration in terms of the relation $\tau_1 \cdot \ldots \cdot \tau_r = 1$ in Map_2. The choice of the loops γ_i (and therefore of the twists τ_i) is of course not unique, but any two choices differ by a sequence of *Hurwitz moves* exchanging consecutive factors: $\tau_i \cdot \tau_{i+1} \to (\tau_{i+1})_{\tau_i^{-1}} \cdot \tau_i$ or $\tau_i \cdot \tau_{i+1} \to \tau_{i+1} \cdot (\tau_i)_{\tau_{i+1}}$, where we use the notation $(\tau)_\phi = \phi^{-1}\tau\phi$, i.e. if τ is a Dehn twist along a loop δ then $(\tau)_\phi$ is the Dehn twist along the loop $\phi(\delta)$.

Definition 3. *A* factorization $F = \tau_1 \cdot \ldots \cdot \tau_r$ *in* Map_g *is an ordered tuple of positive Dehn twists. We say that two factorizations are* Hurwitz equivalent ($F \sim F'$) *if they can be obtained from each other by a sequence of Hurwitz moves.*

It is well-known that a Lefschetz fibration is characterized by a factorization of the identity element in Map_g, uniquely determined up to Hurwitz equivalence and simultaneous conjugation of all factors by a same element of Map_g.

Let ζ_i ($1 \le i \le 5$) and σ be the Dehn twists represented in Figure 1. It is well-known (cf. e.g. [2], Theorem 4.8) that Map_2 admits the following presentation:

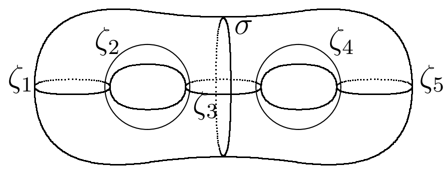


FIGURE 1

– generators: $\zeta_1, \dots, \zeta_5$.
– relations: $\zeta_i\zeta_j = \zeta_j\zeta_i$ if $|i-j| \geq 2$; $\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1}$;
$(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5)^6 = 1$; $I = \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1$ is central; $I^2 = 1$.

It is easy to check that σ can be expressed in terms of the generators $\zeta_1, \dots, \zeta_5$ as $\sigma = (\zeta_1\zeta_2)^6 = (\zeta_4\zeta_5)^6 = (\zeta_1\zeta_2)^3(\zeta_4\zeta_5)^3 I$.

We can fix a hyperelliptic structure on the genus 2 surface Σ, i.e. a double covering map $\Sigma \to S^2$ (with 6 branch points), in such a way that $\zeta_1, \dots, \zeta_5$ become the lifts of standard half-twists exchanging consecutive branch points in S^2. The element I then corresponds to the hyperelliptic involution (i.e. the non-trivial automorphism of the double covering). The fact that I is central means that every diffeomorphism of Σ is compatible with the hyperelliptic structure, up to isotopy. In fact, Map_2 is closely related to the braid group $B_6(S^2)$ acting on the branch points of the double covering. The group $B_6(S^2)$ admits the following presentation (cf. Theorem 1.1 of [2]):

– generators: $x_1, \dots, x_5$ (half-twists exchanging two consecutive points).
– relations: $x_i x_j = x_j x_i$ if $|i-j| \geq 2$; $x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$;
$x_1 x_2 x_3 x_4 x_5^2 x_4 x_3 x_2 x_1 = 1$.

Consider a S^2-bundle $\pi : P \to S^2$, and a smooth curve $B \subset P$ intersecting a generic fiber in 6 points, everywhere transverse to the fibers of π except for isolated nondegenerate complex tangencies. The curve B can be characterized by its *braid monodromy*, or equivalently by a factorization in the braid group $B_6(S^2)$, with each factor a positive half-twist, defined by considering the motion of the 6 intersection points of B with the fiber of π upon moving around the image of a tangency point. As before, this factorization is only defined up to Hurwitz equivalence and simultaneous conjugation (see also [6] for the case of plane curves).

There exists a lifting morphism from $B_6(S^2)$ to $\mathrm{Map}_2/\langle I \rangle$, defined by $x_i \mapsto \zeta_i$. Given a half-twist in $B_6(S^2)$, exactly one of its two possible lifts to Map_2 is a Dehn twist about a non-separating curve. This allows us to lift the braid factorization associated to the curve $B \subset P$ to a mapping class group factorization; the product of the resulting factors is equal to 1 if the homology class represented by B is divisible by two, and to I otherwise. In the first case, we can construct a genus 2 Lefschetz fibration by considering the double covering of P branched along B, and its monodromy is exactly the lift of the braid monodromy of the curve B. This construction always yields Lefschetz fibrations without reducible singular fibers; however, if we additionally allow some blow-up and blow-down operations (on P and its double covering respectively), then we can also handle the case of reducible singular fibers (see §3 below and [8]). It is worth mentioning that Siebert and Tian have shown the converse result: given any genus 2 Lefschetz fibration, it can be realized as a double covering of a S^2-bundle over S^2 (with additional blow-up and blow-down operations in the case of reducible singular fibers) [8].

3. Holomorphic genus 2 fibrations

We are interested in the properties of certain specific factorizations in Map_2.

Definition 4. *Let* $W_0 = (T)^2$, $W_1 = (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^6$, *and* $W_2 = \sigma \cdot (\zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_1 \cdot \zeta_2 \cdot \zeta_3)^2 \cdot (T)$, *where* $T = \zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1$.

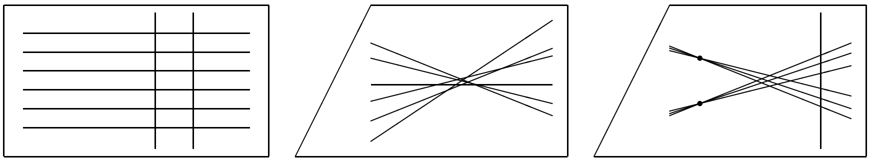


FIGURE 2

In this definition the notation $(\cdots)^n$ means that the sequence of Dehn twists is repeated n times. It is fairly easy to check that W_0, W_1 and W_2 are all factorizations of the identity element of Map_2 as a product of 20, 30, and 29 positive Dehn twists respectively (for W_0 and W_1 this follows immediately from the presentation of Map_2; see below for W_2).

Lemma 5. *The factorization W_0 describes the genus 2 Lefschetz fibration f_0 on the rational surface obtained as a double covering of $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a smooth algebraic curve B_0 of bidegree $(6, 2)$. The factorization W_1 corresponds to the genus 2 Lefschetz fibration f_1 on the blown-up K3 surface obtained as a double covering of $\mathbb{F}_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ branched along a smooth algebraic curve B_1 in the linear system $|6L|$, where L is a line in $\mathbb{CP}^2$ avoiding the blown-up point.*

Proof. B_0 can be degenerated into a singular curve D_0 consisting of 6 sections and 2 fibers intersecting in 12 nodes (see Figure 2). We can recover B_0 from D_0 by first smoothing the intersections of the first section with the two fibers, giving us a component of bidegree $(1, 2)$, and then smoothing the remaining 10 nodes, each of which produces two vertical tangencies. The braid factorization corresponding to B_0 can therefore be expressed as $((x_1)^2 \cdot (x_2)^2_{x_1} \cdot (x_3)^2_{x_2x_1} \cdot (x_4)^2_{x_3x_2x_1} \cdot (x_5)^2_{x_4x_3x_2x_1})^2$, or equivalently after suitable Hurwitz moves, $(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_5 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_1)^2$. Lifting this braid factorization to the mapping class group, we obtain W_0 as claimed. Alternately, it is easy to check that the braid factorization for a smooth curve of bidegree $(6, 1)$ is $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_5 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_1$; we can then conclude by observing that B_0 is the fiber sum of two such curves.

In the case of the curve B_1, by definition the braid monodromy is exactly that of a smooth plane curve of degree 6 as defined by Moishezon in [6]; it can be computed e.g. by degenerating B_1 to a union of 6 lines in generic position (D_1 in Figure 2), and is known to be given by the factorization $(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)^6$. Lifting to Map_2, we obtain that the monodromy factorization for the corresponding double branched covering is exactly W_1. □

Lemma 6. *Let $\tau \in \mathrm{Map}_2$ be a Dehn twist, and let F be a factorization of a central element (1 or I) of Map_2. If $F \sim \tau \cdot F'$ for some F', then the factorization $(F)_\tau$ obtained from F by simultaneous conjugation of all factors by τ is Hurwitz equivalent to F.*

Proof. We have: $(F)_\tau \sim \tau \cdot (F')_\tau \sim F' \cdot \tau \sim (\tau)_{F'} \cdot F' = \tau \cdot F' \sim F$. The first and last steps follow from the assumption; the second step corresponds to moving τ to the right across all the factors of $(F')_\tau$, while in the third step all the factors of F' are moved to the right across τ. Also observe that $(\tau)_{F'} = \tau$ because the product of all factors in F' commutes with τ. □

Lemma 7. *The factorizations T, W_0 and W_1 are* fully invariant*, i.e. for any element $\gamma \in \mathrm{Map}_2$ we have $(T)_\gamma \sim T$, $(W_0)_\gamma \sim W_0$, and $(W_1)_\gamma \sim W_1$.*

Proof. It is obviously sufficient to prove that $(T)_{\zeta_i} \sim T$ and $(W_1)_{\zeta_i} \sim W_1$ for all $1 \le i \le 5$. By moving the first ζ_i factor in T or W_1 to the left, we obtain a Hurwitz equivalent factorization of the form $\zeta_i \cdot \ldots$; therefore the result follows immediately from Lemma 6. □

A direct consequence of Lemma 7 is that all fiber sums of the holomorphic fibrations f_0 and f_1 (with monodromies W_0 and W_1) are *untwisted*. More precisely, when two Lefschetz fibrations with monodromy factorizations F and F' are glued to each other along a fiber, the resulting fibration normally depends on the isotopy class ϕ of a diffeomorphism between the two fibers to be identified, and its monodromy is given by a factorization of the form $(F) \cdot (F')_\phi$. However, when the building blocks are made of copies of f_0 and f_1, Lemma 7 implies that the result of the fiber sum operation is independent of the chosen identification diffeomorphisms; e.g., we can always take $\phi = 1$.

Lemma 8. $(W_1)^2 \sim (W_0)^3$.

Proof. Let $\rho = \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_1\zeta_2\zeta_3\zeta_4\zeta_1\zeta_2\zeta_3\zeta_1\zeta_2\zeta_1$ be the reflection of the genus 2 surface Σ about its central axis. It follows from Lemma 7 that $(W_1)^2 \sim W_1 \cdot (W_1)_\rho = (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^6 \cdot (\zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1)^6$. The central part of this factorization is exactly T; after moving it to the right, we obtain the new identity $(W_1)^2 \sim (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^5 \cdot (\zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1)^5 \cdot (T)_{(\zeta_5\zeta_4\zeta_3\zeta_2\zeta_1)^5}$. Repeating the same operation four more times, we get $(W_1)^2 \sim \prod_{j=0}^{5} (T)_{(\zeta_5\zeta_4\zeta_3\zeta_2\zeta_1)^j}$. Using the full invariance property of T (Lemma 7), it follows that $(W_1)^2 \sim (T)^6 = (W_0)^3$.

A more geometric argument is as follows: $(W_1)^2$ is the monodromy factorization of the fiber sum $f_1 \# f_1$ of two copies of f_1, which is a double covering of the fiber sum of two copies of $(\mathbb{F}_1, B_1)$. Therefore $f_1 \# f_1$ is a double covering of $(\mathbb{F}_2, B')$ where $\mathbb{F}_2 = \mathbb{P}(O \oplus O(2))$ is the second Hirzebruch surface and B' is a smooth algebraic curve in the linear system $|6S|$, where S is a section of $\mathbb{F}_2$ ($S \cdot S = 2$). On the other hand $(W_0)^3$ is the monodromy factorization of the fiber sum $f_0 \# f_0 \# f_0$, which is a double covering of the fiber sum of three copies of $(\mathbb{F}_0, B_0)$, i.e. a double covering of $(\mathbb{F}_0, B'')$ where B'' is a smooth algebraic curve of bidegree $(6, 6)$. The conclusion follows from the fact that $(\mathbb{F}_2, B')$ and $(\mathbb{F}_0, B'')$ are deformation equivalent. □

We can now reformulate the holomorphicity result obtained by Siebert and Tian [9] in terms of mapping class group factorizations. Say that a factorization is *transitive* if the images of the factors under the morphism $\mathrm{Map}_2 \to S_6$ mapping ζ_i to the transposition $(i, i+1)$ generate the entire symmetric group S_6.

Theorem 9 (Siebert-Tian [9]). *Any transitive factorization of the identity element as a product of positive Dehn twists along non-separating curves in Map_2 is Hurwitz equivalent to a factorization of the form $(W_0)^k \cdot (W_1)^\epsilon$ for some integers $k \ge 0$ and $\epsilon \in \{0, 1\}$.*

What Siebert and Tian have shown is in fact that any such factorization is the monodromy of a holomorphic Lefschetz fibration, which can be realized as a double covering of a ruled surface branched along a smooth connected holomorphic curve intersecting the generic fiber in 6 points. However, we can always assume that the ruled surface is either $\mathbb{F}_0$ or $\mathbb{F}_1$ (either by the topological classification of ruled surfaces or using Lemma 8). In the first case, the branch curve has bidegree $(6, 2k)$ for some integer k, and the corresponding monodromy is $(W_0)^k$, while in the second case the branch curve realizes the homology class $6[L] + 2k[F]$ for some integer k (here F is a fiber of $\mathbb{F}_1$), and the corresponding monodromy is $(W_0)^k \cdot W_1$.

We now look at examples of genus 2 fibrations with reducible singular fibers.

Definition 10. *Let $B_2 \subset \mathbb{F}_1$ be an algebraic curve in the linear system $|6L + F|$, presenting two triple points in the same fiber F_0. Let P_2 be the surface obtained by blowing up $\mathbb{F}_1$ at the two triple points of B_2, and denote by $\hat{B}_2$ and $\hat{F}_0$ the proper transforms of B_2 and F_0 in P_2. Consider the double covering $\pi : \hat{X}_2 \to P_2$ branched along $\hat{B}_2 \cup \hat{F}_0$, and let X_2 be the surface obtained by blowing down the -1-curve $\pi^{-1}(\hat{F}_0)$ in $\hat{X}_2$.*

Let us check that this construction is well-defined. The easiest way to construct the curve B_2 is to start with a curve C of degree 7 in $\mathbb{CP}^2$ with two triple points p_1 and p_2. If we choose C generically, we can assume that the three branches of C through p_i intersect each other transversely and are transverse to the line L_0 through p_1 and p_2. Therefore the line L_0 intersects C transversely in another point p, and by blowing up $\mathbb{CP}^2$ at p we obtain the desired curve B_2 (see also below). Next, we blow up the two triple points p_1 and p_2, which turns B_2 into a smooth curve $\hat{B}_2$, disjoint from $\hat{F}_0$. Denoting by E_1 and E_2 the exceptional divisors of the two blow-ups, we have $[\hat{B}_2] = 6[L] + [F] - 3[E_1] - 3[E_2]$ and $[\hat{F}_0] = [F] - [E_1] - [E_2]$, so that $[\hat{B}_2] + [\hat{F}_0] = 6[L] + 2[F] - 4[E_1] - 4[E_2]$ is divisible by 2; therefore the double covering $\pi : \hat{X}_2 \to P_2$ is well-defined.

The complex surface $\hat{X}_2$ is equipped with a natural holomorphic genus 2 fibration $\hat{f}_2$, obtained by composing $\pi : \hat{X}_2 \to P_2$ with the natural projections to $\mathbb{F}_1$ and then to S^2. The fiber of $\hat{f}_2$ corresponding to $F_0 \subset \mathbb{F}_1$ consists of three components: two elliptic curves of square -2 obtained as double coverings of the exceptional curves E_1 and E_2 in P_2, with 4 branch points in each case (three on $\hat{B}_2$ and one on $\hat{F}_0$), and a rational curve of square -1, the preimage of $\hat{F}_0$. After blowing down the rational component, we obtain on X_2 a holomorphic genus 2 fibration f_2, with one reducible fiber consisting of two elliptic components. It is easy to check that near this singular point f_2 presents the local model expected of a Lefschetz fibration, and that the vanishing cycle for this fiber is the loop obtained by lifting any simple closed loop that separates the two triple points of B_2 inside the fiber F_0 of $\mathbb{F}_1$.

Lemma 11. *The complex surface X_2 carries a natural holomorphic genus 2 Lefschetz fibration, with monodromy factorization W_2.*

Proof. We need to calculate the braid monodromy factorization associated to the curve $B_2 \subset \mathbb{F}_1$. For this purpose, observe that B_2 can be degenerated to a union D_2 of 6 lines in two groups of three, L_1, L_2, L_3 and L_4, L_5, L_6, and a fiber F, with two triple points and 15 nodes (cf. Figure 2). The monodromy around the fiber containing the two triple points is given by the braid $\delta = (x_1x_2)^3(x_4x_5)^3$. The 9 nodes corresponding to the mutual intersections of the two groups of three lines give rise to 18 vertical tangencies in B_2, and the corresponding factorization is

$$x_3^2 \cdot (x_4)^2_{x_3} \cdot (x_5)^2_{x_4x_3} \cdot [x_2^2 \cdot (x_3)^2_{x_2} \cdot (x_4)^2_{x_3x_2}]_{(x_5x_4x_3)} \cdot [x_1^2 \cdot (x_2)^2_{x_1} \cdot (x_3)^2_{x_2x_1}]_{(x_4x_3x_2x_5x_4x_3)}.$$

After suitable Hurwitz moves, this expression can be rewritten as

$x_3 \cdot x_4 \cdot (x_5)^2 \cdot x_4 \cdot x_3 \cdot [x_2 \cdot x_3 \cdot (x_4)^2 \cdot x_3 \cdot x_2]_{(x_5x_4x_3)} \cdot [x_1 \cdot x_2 \cdot (x_3)^2 \cdot x_2 \cdot x_1]_{(x_4x_3x_2x_5x_4x_3)}$,

or equivalently as $x_3 \cdot x_4 \cdot x_5 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_3 \cdot x_2 \cdot x_1 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_5 \cdot x_4 \cdot x_3$, which is in turn equivalent to $(x_3 \cdot x_4 \cdot x_5 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_1 \cdot x_2 \cdot x_3)^2$. Finally, the six intersections of the lines $L_1, \dots, L_6$ with the fiber F give rise to 10 vertical tangencies, for which the same argument as for Lemma 5 gives the monodromy factorization $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_5 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_1$. We conclude by lifting the monodromy of B_2 to the mapping class group, observing that the contribution δ of the fiber containing the triple points lifts to the Dehn twist σ. □

Theorem 12. *Fix integers $m \geq 0$, $\epsilon \in \{0,1\}$ and $k \geq \frac{3}{2}m+1$. Then the Hirzebruch surface $\mathbb{F}_{m+\epsilon} = \mathbb{P}(O \oplus O(m+\epsilon))$ contains a complex curve $B_{k,\epsilon,m}$ in the linear system $|6S + (m+2k)F|$ (where S is a section of square $(m+\epsilon)$ and F is a fiber), having $2m$ triple points lying in m distinct fibers of $\mathbb{F}_{m+\epsilon}$ as its only singularities.*

Moreover, after blowing up $\mathbb{F}_{m+\epsilon}$ at the $2m$ triple points, passing to a double covering, and blowing down m rational -1-curves, we obtain a complex surface and a holomorphic genus 2 fibration $f_{k,\epsilon,m} : X_{k,\epsilon,m} \to S^2$ with monodromy factorization $(W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^m$.

Proof. We first construct the curve $B_{k,\epsilon,m}$ by perturbation of a singular configuration $D_{k,\epsilon,m}$ consisting of 6 sections of $\mathbb{F}_{m+\epsilon}$ together with $m+2k$ fibers. Since the case of smooth curves is a classical result, we can assume that $m \geq 1$. Also observe that, since the intersection number of $B_{k,\epsilon,m}$ with a fiber is equal to 6, the $2m$ triple points must come in pairs lying in the same fiber: $p_{2i-1}, p_{2i} \in F_i$, $1 \leq i \leq m$.

Let u_0 and u_1 be generic sections of the line bundle $O_{\mathbb{CP}^1}(m+\epsilon)$, without common zeroes. Define six sections $S_{\alpha,\beta}$ ($\alpha \in \{0,1\}$, $\beta \in \{0,1,2\}$) of $\mathbb{F}_{m+\epsilon}$ as the projectivizations of the sections $(1, (-1)^\alpha u_0 + c_{\alpha\beta} u_1)$ of $O_{\mathbb{CP}^1} \oplus O_{\mathbb{CP}^1}(m+\epsilon)$, where $c_{\alpha\beta}$ are small generic complex numbers. The three sections $S_{0,\beta}$ intersect each other in $(m+\epsilon)$ triple points, in the fibers $F_1, \dots, F_{m+\epsilon}$ above the points of $\mathbb{CP}^1$ where u_1 vanishes, and similarly for the three sections $S_{1,\beta}$; generic choices of parameters ensure that all the other intersections between these six sections are transverse and lie in different fibers of $\mathbb{F}_{m+\epsilon}$. We define $D_{k,\epsilon,m}$ to be the singular configuration consisting of the six sections $S_{\alpha,\beta}$ together with $m+2k$ generic fibers of $\mathbb{F}_{m+\epsilon}$ intersecting the sections in six distinct points.

Let $s \in H^0(\mathbb{F}_{m+\epsilon}, O(mF))$ be the product of the sections of $O(F)$ defining the fibers $F_1, \dots, F_m$ containing $2m$ of the triple points of $D_{k,\epsilon,m}$. Let s' be a generic section of the line bundle $L = O(D_{k,\epsilon,m} - 4mF) = O(6S + (2k-3m)F)$ over $\mathbb{F}_{m+\epsilon}$. Because $k \geq \frac{3}{2}m+1$, the linear system $|L|$ is base point free, and so we can assume that s' does not vanish at any of the double or triple points of $D_{k,\epsilon,m}$. Finally, let s_0 be the section defining $D_{k,\epsilon,m}$.

We consider the section $s_\lambda = s_0 + \lambda s^4 s' \in H^0(\mathbb{F}_{m+\epsilon}, O(6S + (2k+m)F))$, where $\lambda \neq 0$ is a generic small complex number. Because the perturbation $s^4 s'$ vanishes at order 4 at each of the triple points $p_1, \dots, p_{2m}$ of $D_{k,\epsilon,m}$ in the fibers $F_1, \dots, F_m$, it is easy to check that all the curves $D(\lambda) = s_\lambda^{-1}(0)$ present triple points at $p_1, \dots, p_{2m}$. On the other hand, since $s^4 s'$ does not vanish at any of the other singular points of $D_{k,\epsilon,m}$, for generic λ the curve $D(\lambda)$ presents no other singularities than the triple points $p_1, \dots, p_{2m}$ (this follows e.g. from Bertini's theorem); this gives us the curve $B_{k,\epsilon,m}$ with the desired properties. Moreover, generic choices of the parameters ensure that the vertical tangencies of $B_{k,\epsilon,m}$ all lie in distinct fibers of $\mathbb{F}_{m+\epsilon}$; in that case, the double covering construction will give rise to a Lefschetz fibration.

The braid monodromy of the curve $B_{k,\epsilon,m}$ can be computed using the existence of a degeneration to the singular configuration $D_{k,\epsilon,m}$ (taking $\lambda \to 0$ in the above construction); a calculation similar to the proofs of Lemma 5 and Lemma 11 yields that it consists of k copies of the braid factorization of the curve B_0, ϵ copies of the braid factorization of B_1, and m copies of the braid factorization of B_2. Another way to see this is to observe that the surface $\mathbb{F}_{m+\epsilon}$ admits a decomposition into a fiber sum of k copies of $\mathbb{F}_0$ and $m+\epsilon$ copies of $\mathbb{F}_1$, in such a way that the singular configuration $D_{k,\epsilon,m}$ naturally decomposes into k copies of D_0, ϵ copies of a degenerate version of D_1 presenting some triple points, and m copies of D_2. After a suitable smoothing, we obtain that the pair $(\mathbb{F}_{m+\epsilon}, B_{k,\epsilon,m})$ splits as the untwisted fiber sum $k\,(\mathbb{F}_0, B_0)\#\epsilon\,(\mathbb{F}_1, B_1)\#m\,(\mathbb{F}_1, B_2)$.

By the same process as in the construction of the surface X_2, we can blow up the $2m$ triple points of $B_{k,\epsilon,m}$, take a double covering branched along the proper transforms of $B_{k,\epsilon,m}$ and of the fibers through the triple points, and blow down m rational components, to obtain a complex surface $X_{k,\epsilon,m}$ equipped with a holomorphic genus 2 Lefschetz fibration $f_{k,\epsilon,m} : X_{k,\epsilon,m} \to S^2$. Because of the structure of $B_{k,\epsilon,m}$, it is easy to see that $f_{k,\epsilon,m}$ splits into an untwisted fiber sum of k copies of f_0, ϵ copies of f_1, and m copies of f_2. Therefore, its monodromy is described by the factorization $(W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^m$. □

4. Proof of the main result

In order to prove Theorem 1, we will use the following lemma which allows us to trade one reducible singular fiber against a collection of irreducible singular fibers:

Lemma 13. $(\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (T) \cdot (W_2) \sim \sigma \cdot (W_0) \cdot (W_1)$.

Proof. Let $\Phi = \zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_1 \cdot \zeta_2 \cdot \zeta_3$, and observe that $(\zeta_i)_\Phi = \zeta_{6-i}$ for $i \in \{1,2,4,5\}$. Therefore, we have

$$(\zeta_4 \cdot \zeta_5)^3 \cdot (\Phi)^2 \sim (\Phi) \cdot \zeta_1 \cdot \zeta_2 \cdot \zeta_1 \cdot (\Phi) \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_5 \sim (\Phi) \cdot (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^3.$$

Moreover, $(\zeta_1 \cdot \zeta_2)^3 \cdot (\Phi) \sim \zeta_1 \cdot \zeta_2 \cdot \zeta_1 \cdot (\Phi) \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_5 \sim (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^3$. It follows that $(\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (\Phi)^2 \sim W_1$. Recalling that $(\zeta_1\zeta_2)^3(\zeta_4\zeta_5)^3 = \sigma I$ and $W_2 = \sigma \cdot (\Phi)^2 \cdot (T)$, and using the invariance property of T (Lemma 7), we have

$$(\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (T) \cdot (W_2) \sim \sigma \cdot (T) \cdot (\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (\Phi)^2 \cdot (T) \sim \sigma \cdot (T) \cdot (W_1) \cdot (T),$$

which is Hurwitz equivalent to $\sigma \cdot (T)^2 \cdot (W_1) = \sigma \cdot (W_0) \cdot (W_1)$. □

Proof of Theorem 1. We argue by induction on the number m of reducible singular fibers. If there are no separating Dehn twists, then after summing with at least one copy of W_0 to ensure transitivity, we obtain a transitive factorization of the identity element into Dehn twists along non-separating curves, which by Theorem 9 is of the expected form.

Assume that Theorem 1 holds for all factorizations with $m-1$ separating Dehn twists, and consider a factorization F with m separating Dehn twists. By Hurwitz moves we can bring one of the separating Dehn twists to the right-most position in F and assume that $F = (F') \cdot \tilde{\sigma}$, where $\tilde{\sigma}$ is a Dehn twist about a loop separating two genus 1 components. Clearly, there exists an element $\phi \in \mathrm{Map}_2$ such that $\tilde{\sigma} = (\sigma)_{\phi^{-1}}$. Using the relation $I^2 = 1$, we can express each ζ_i^{-1} as a product of the generators $\zeta_1, \dots, \zeta_5$, and therefore ϕ can be expressed as a positive word involving only the generators $\zeta_1, \dots, \zeta_5$ (and not their inverses).

Starting with the factorization $\tilde{\sigma} \cdot (W_0)^n$, we can selectively move σ to the right across the various factors W_0, sometimes conjugating the factors of W_0 and sometimes conjugating $\tilde{\sigma}$. If we choose the factors by which we conjugate $\tilde{\sigma}$ according to the expression of ϕ in terms of $\zeta_1, \dots, \zeta_5$, and if n is sufficiently large, we obtain that $\tilde{\sigma} \cdot (W_0)^n \sim (F'') \cdot \sigma$, for some factorization F'' involving only non-separating Dehn twists. Therefore, using Lemma 8 and Lemma 13 we have

$$F \cdot (W_0)^{n+4} \sim F' \cdot \tilde{\sigma} \cdot (W_0)^n \cdot W_0 \cdot (W_1)^2 \sim F' \cdot F'' \cdot \sigma \cdot W_0 \cdot (W_1)^2 \sim \tilde{F} \cdot W_2,$$

where $\tilde{F} = F' \cdot F'' \cdot (\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot T \cdot W_1$. Next we observe that $\tilde{F}$ is a factorization of the identity element with $m-1$ separating Dehn twists, therefore by assumption there exist integers $\tilde{n}, k, \epsilon$ such that $\tilde{F} \cdot (W_0)^{\tilde{n}} \sim (W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^{m-1}$. It follows that $F \cdot (W_0)^{n+\tilde{n}+4} \sim \tilde{F} \cdot W_2 \cdot (W_0)^{\tilde{n}} \sim \tilde{F} \cdot (W_0)^{\tilde{n}} \cdot W_2 \sim (W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^m$. This concludes the proof, since it is clear that the splitting remains valid after adding extra copies of W_0. $\square$

Proof of Corollary 2. First of all, as observed at the beginning of §2 we can assume that f is relatively minimal, i.e. all vanishing cycles are homotopically non-trivial. Let F be a monodromy factorization corresponding to f, and observe that by Theorem 1 we have a splitting of the form $F \cdot (W_0)^n \sim (W_0)^{n+k} \cdot (W_1)^\epsilon \cdot (W_2)^m$. If n is chosen large enough then $n+k \geq \frac{3}{2}m+1$, and so by Theorem 12 the right-hand side is the monodromy of the holomorphic fibration $f_{n+k,\epsilon,m}$, while the left-hand side corresponds to the fiber sum $f \#n\, f_0$. $\square$

Remark. Pending a suitable extension of the result of Siebert and Tian to higher genus hyperelliptic Lefschetz fibrations with transitive monodromy and irreducible singular fibers, the techniques described here can be generalized to higher genus hyperelliptic fibrations almost without modification. The main difference is the existence of different types of reducible fibers, classified by the genera h and $g-h$ of the two components; this makes it necessary to replace W_2 with a larger collection of building blocks, obtained e.g. from complex curves in $\mathbb{F}_1$ that intersect the generic fiber in $2g+2$ points and present two multiple points with multiplicities $2h+1$ and $2(g-h)+1$ in the same fiber.

References

[1] J. Amorós, F. Bogomolov, L. Katzarkov, T. Pantev, *Symplectic Lefschetz fibrations with arbitrary fundamental groups (with an appendix by I. Smith)*, J. Diff. Geom. **54** (2000), 489–545.

[2] J. Birman, *Braids, links and mapping class groups*, Princeton Univ. Press, 1974.

[3] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Diff. Geom. **53** (1999), 205–236.

[4] A. Kas, *On the handlebody decomposition associated to a Lefschetz fibration*, Pacific J. Math. **89** (1980), 89–104.

[5] M. Korkmaz, *Noncomplex smooth 4-manifolds with Lefschetz fibrations*, Internat. Math. Res. Notices **2001**, 115–128.

[6] B. Moishezon, *Stable branch curves and braid monodromies*, Algebraic Geometry (Chicago, 1980), Lecture Notes in Math. **862**, Springer, 1981, 107–192.

[7] B. Ozbagci, A. Stipsicz, *Noncomplex smooth 4-manifolds with genus 2 Lefschetz fibrations*, Proc. Amer. Math. Soc. **128** (2000), 3125–3128.

[8] B. Siebert, G. Tian, *On hyperelliptic symplectic Lefschetz fibrations of four manifolds*, Comm. Contemp. Math. **1** (1999), 255–280.
[9] B. Siebert, G. Tian, *On the holomorphicity of genus two Lefschetz fibrations*, preprint.

FUNDAMENTAL GROUPS OF COMPLEMENTS OF PLANE CURVES AND SYMPLECTIC INVARIANTS

D. AUROUX, S.K. DONALDSON, L. KATZARKOV, AND M. YOTOV

Abstract. Introducing the notion of stabilized fundamental group for the complement of a branch curve in $\mathbb{CP}^2$, we define effectively computable invariants of symplectic 4-manifolds that generalize those previously introduced by Moishezon and Teicher for complex projective surfaces. Moreover, we study the structure of these invariants and formulate conjectures supported by calculations on new examples.

1. Introduction

Using approximately holomorphic techniques first introduced in [5], it was shown in [2] (see also [1]) that compact symplectic 4-manifolds with integral symplectic class can be realized as branched covers of $\mathbb{CP}^2$ and can be investigated using the braid group techniques developed by Moishezon and subsequently by Moishezon and Teicher for the study of complex surfaces (see e.g. [13]):

Theorem 1.1 ([2]). *Let (X, ω) be a compact symplectic 4-manifold, and let L be a line bundle with $c_1(L) = \frac{1}{2\pi}[\omega]$. Then there exist branched covering maps $f_k : X \to \mathbb{CP}^2$ defined by approximately holomorphic sections of $L^{\otimes k}$ for all large enough values of k; the corresponding branch curves $D_k \subset \mathbb{CP}^2$ admit only nodes (both orientations) and complex cusps as singularities, and give rise to well-defined braid monodromy invariants. Moreover, up to admissible creations and cancellations of pairs of nodes in the branch curve, for large k the topology of f_k is a symplectic invariant.*

This makes it possible to associate to (X, ω) a sequence of invariants (indexed by $k \gg 0$) consisting of two objects: the braid monodromy characterizing the branch curve D_k, and the *geometric monodromy representation* $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_n$ ($n = \deg f_k$) characterizing the n-fold covering of $\mathbb{CP}^2 - D_k$ induced by f_k [2]. These invariants are extremely powerful (from them one can recover (X, ω) up to symplectomorphism) but too complicated to handle in practical cases.

In the study of complex surfaces, Moishezon and Teicher have shown that the fundamental group $\pi_1(\mathbb{CP}^2 - D)$ (or, restricting to an affine subset, $\pi_1(\mathbb{C}^2 - D)$) can be computed explicitly in some simple examples; generally speaking, this group has been expected to provide a valuable invariant for distinguishing diffeomorphism types of complex surfaces of general type. However, in the symplectic case, it is affected by creations and cancellations of pairs of nodes and cannot be used immediately as an invariant.

We will introduce in §2 a certain quotient G_k (resp. $\bar{G}_k$) of $\pi_1(\mathbb{C}^2 - D_k)$ (resp. $\pi_1(\mathbb{CP}^2 - D_k)$), the *stabilized fundamental group*, which remains invariant under creations and cancellations of pairs of nodes. As an immediate corollary of the construction and of Theorem 1.1, we obtain the following

Theorem 1.2. *For large enough k, the stabilized groups $G_k = G_k(X,\omega)$ (resp. $\bar{G}_k(X,\omega)$) and their reduced subgroups $G_k^0 = G_k^0(X,\omega)$ are symplectic invariants of the manifold (X,ω).*

These invariants can be computed explicitly in various examples, some due to Moishezon, Teicher and Robb, others new; these examples will be presented in §4, and a brief overview of the techniques involved in the computations is given in §6 and §7. The new examples include double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along arbitrary complex curves (Theorem 4.6 and §7); similar methods should apply to other double covers as well, thus providing results for both types of so-called Horikawa surfaces. The calculations described in §7, which rely on various innovative tools in addition to a suitable reformulation of the methods developed by Moishezon and Teicher, go well beyond the scope of results accessible using only the previously known techniques, and may present interest of their own for applications in algebraic geometry.

The available data suggest several conjectures about the structure of the stabilized fundamental groups.

First of all, it appears that in most examples the stabilization operation does not actually affect the fundamental group. The only known exceptions are given by "small" linear systems with insufficient ampleness properties, where the stabilization is a quotient by a non-trivial subgroup (see §4). Therefore we have the following

Conjecture 1.3. *Assume that (X,ω) is a complex surface, and let D_k be the branch curve of a generic projection to $\mathbb{CP}^2$ of the projective embedding of X given by the linear system $|kL|$. Then, provided that k is large enough, the stabilization operation is trivial, i.e. $G_k(X,\omega) \simeq \pi_1(\mathbb{C}^2 - D_k)$ and $\bar{G}_k(X,\omega) \simeq \pi_1(\mathbb{CP}^2 - D_k)$.*

An important class of fundamental groups for which the conjecture holds will be described in §3.

Moreover, the structure of the stabilized fundamental groups seems to be remarkably simple, at least when the manifold X is simply connected; in all known examples they are extensions of a symmetric group by a solvable group, while there exist plane curves with much more complicated complements [4, 6]. In fact these groups seem to be largely determined by intersection pairing data in $H_2(X,\mathbb{Z})$. More precisely, the following result will be proved in §5:

Definition 1.4. *Let Λ_k be the image of the map $\lambda_k : H_2(X,\mathbb{Z}) \to \mathbb{Z}^2$ defined by $\lambda_k(\alpha) = (\alpha \cdot L_k, \alpha \cdot R_k)$, where $L_k = k\,c_1(L)$ and $R_k = c_1(K_X) + 3L_k$ are the classes in $H^2(X,\mathbb{Z})$ Poincaré dual to a hyperplane section and to the ramification curve respectively.*

Theorem 1.5. *If the symplectic manifold X is simply connected, then there exists a natural surjective homomorphism $\phi_k : \mathrm{Ab}\, G_k^0(X,\omega) \to (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_{n_k} \simeq (\mathbb{Z}^2/\Lambda_k)^{n_k-1}$, where $n_k = \deg f_k = L_k \cdot L_k$, and $\mathcal{R}_{n_k}$ is the reduced regular representation of S_{n_k} (isomorphic to $\mathbb{Z}^{n_k-1}$).*

The map ϕ_k is (G_k, S_{n_k})-equivariant, in the sense that $\phi_k(g^{-1}\gamma g) = \theta_k(g) \cdot \phi_k(\gamma)$ for any elements $g \in G_k(X,\omega)$ and $\gamma \in \mathrm{Ab}\, G_k^0(X,\omega)$ (cf. also Lemma 5.2).

In the examples discussed in §4, the group G_k^0 is always close to being abelian, and ϕ_k is always an isomorphism. It seems likely that the injectivity of ϕ_k can be proved using techniques similar to those described in §6–7. Therefore, it makes sense to formulate the following

Conjecture 1.6. *If the symplectic manifold X is simply connected and k is large enough, then* $\mathrm{Ab}\, G_k^0(X,\omega) \simeq (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_{n_k}$*, and the commutator subgroup* $[G_k^0, G_k^0]$ *is a quotient of* $(\mathbb{Z}_2)^2$.

Conjectures 1.3 and 1.6 provide an almost complete tentative description of the structure of fundamental groups of branch curve complements in high degrees. In relation with the property $(*)$ introduced in §3, they also provide a framework to explain various observations and conjectures made in [14] and [12].

The obtained results seem to indicate that fundamental groups of branch curve complements cannot be used as invariants to symplectically distinguish homeomorphic manifolds. This is in sharp contrast with the braid monodromy data, which completely determines the symplectomorphism type of (X,ω) [2] ; how to introduce effectively computable invariants retaining more of the information contained in the braid monodromy remains an open question.

2. Braid monodromy and stabilized fundamental groups

Let D_k be the branch curve of a covering map $f_k : X \to \mathbb{CP}^2$ as in Theorem 1.1. Braid monodromy invariants are defined by considering a generic projection $\pi : \mathbb{CP}^2 - \{\mathrm{pt}\} \to \mathbb{CP}^1$: the pole of the projection lies away from D_k, and a generic fiber of π intersects D_k in $d = \deg D_k$ distinct points, the only exceptions being fibers through cusps or nodes of D_k, or fibers that are tangent to D_k at one of its smooth points ("vertical tangencies"). Moreover we can assume that the special points (cusps, nodes and vertical tangencies) of D_k all lie in different fibers of π.

By restricting ourselves to an affine subset $\mathbb{C}^2 \subset \mathbb{CP}^2$, choosing a base point and trivializing the fibration π, we can view the monodromy of $\pi_{|D_k}$ as a group homomorphism from $\pi_1(\mathbb{C} - \{q_i\})$ (where q_i are the images by π of the special points of D_k) to the braid group B_d. More precisely, the monodromy around a vertical tangency is a half-twist (a braid that exchanges two of the d intersection points of the fiber with D_k by rotating them around each other counterclockwise along a certain path); the monodromy around a positive (resp. negative) node is the square (resp. the inverse of the square) of a half-twist; the monodromy around a cusp is the cube of a half-twist [13, 2].

It is sometimes convenient to choose an ordered system of generating loops for $\pi_1(\mathbb{C} - \{q_i\})$ (one loop going around each q_i), and to express the monodromy as a *braid factorization*, i.e. a decomposition of the central braid Δ^2 (the monodromy around the fiber at infinity, due to the non-triviality of the fibration π over $\mathbb{CP}^1$) into the product of the monodromies along the chosen generating loops. However, this braid factorization is only well-defined up to simultaneous conjugation of all factors (i.e., a change in the choice of the identification of the fibers with $\mathbb{R}^2$) and *Hurwitz equivalence* (i.e., a rearrangement of the factors due to a different choice of the system of generating loops).

The braid monodromy determines in a very explicit manner the fundamental groups $\pi_1(\mathbb{C}^2 - D_k)$ and $\pi_1(\mathbb{CP}^2 - D_k)$. Indeed, consider a generic fiber $\ell \simeq \mathbb{C} \subset \mathbb{CP}^2$ of the projection π (e.g. the fiber containing the base point), intersecting D_k in d distinct points. The free group $\pi_1(\ell - (\ell \cap D_k)) = F_d$ is generated by a system of d loops going around the various points in $\ell \cap D_k$. The inclusion map $i : \ell - (\ell \cap D_k) \to \mathbb{C}^2 - D_k$ induces a surjective homomorphism $i_* : F_d \to \pi_1(\mathbb{C}^2 - D_k)$.

Definition 2.1. *The images of the standard generators of the free group F_d and their conjugates are called geometric generators of $\pi_1(\mathbb{C}^2 - D_k)$; the set of all geometric generators will be denoted by Γ_k.*

By the Zariski-Van Kampen theorem, $\pi_1(\mathbb{C}^2 - D_k)$ is realized as a quotient of F_d by relations corresponding to the various special points (vertical tangencies, nodes, cusps) of D_k; these relations express the fact that the action of the braid monodromy on F_d induces a trivial action on $\pi_1(\mathbb{C}^2 - D_k)$. To each factor in the braid factorization one can associate a pair of elements $\gamma_1, \gamma_2 \in \Gamma_k$ (small loops around the two portions of D_k that meet at the special point), well-determined up to simultaneous conjugation. The relation corresponding to a tangency is $\gamma_1 \sim \gamma_2$; for a node (of either orientation) it is $[\gamma_1, \gamma_2] \sim 1$; for a cusp it becomes $\gamma_1\gamma_2\gamma_1 \sim \gamma_2\gamma_1\gamma_2$. Taking into account all the special points of D_k (i.e. considering the entire braid monodromy), we obtain a presentation of $\pi_1(\mathbb{C}^2 - D_k)$. Moreover, $\pi_1(\mathbb{CP}^2 - D_k)$ is obtained from $\pi_1(\mathbb{C}^2 - D_k)$ just by adding the extra relation $g_1 \dots g_d \sim 1$, where g_i are the images of the standard generators of F_d under the inclusion.

It follows from this discussion that the creation or cancellation of a pair of nodes in D_k may affect $\pi_1(\mathbb{C}^2 - D_k)$ and $\pi_1(\mathbb{CP}^2 - D_k)$ by adding or removing commutation relations between geometric generators. Although it is reasonable to expect that negative nodes can always be cancelled in the branch curves given by Theorem 1.1, the currently available techniques are insufficient to prove such a statement. Instead, a more promising approach is to compensate for these changes in the fundamental groups by considering certain quotients where one stabilizes the group by adding commutation relations between geometric generators. The resulting group is in some sense more natural than $\pi_1(\mathbb{C}^2 - D_k)$ from the symplectic point of view, and as a side benefit it is often easier to compute (see §7). Moreover, it also turns out that, in many cases, no information is lost in the stabilization process (see §3).

In order to define the stabilized group G_k, first observe that, because the branching index of f_k above a smooth point of D_k is always 2, the geometric monodromy representation morphism $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_n$ describing the topology of the covering above $\mathbb{CP}^2 - D_k$ maps all geometric generators to transpositions in S_n. As seen above, to each nodal point of D_k one can associate geometric generators $\gamma_1, \gamma_2 \in \Gamma_k$, one for each of the two intersecting portions of D_k, so that the corresponding relation in $\pi_1(\mathbb{C}^2 - D_k)$ is $[\gamma_1, \gamma_2] \sim 1$. Since the branching occurs in disjoint sheets of the cover, the two transpositions $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are necessarily disjoint (i.e. they are distinct and commute). Therefore, adding or removing pairs of nodes amounts to adding or removing relations given by commutators of geometric generators associated to disjoint transpositions.

Definition 2.2. *Let K_k (resp. $\bar{K}_k$) be the normal subgroup of $\pi_1(\mathbb{C}^2 - D_k)$ (resp. $\pi_1(\mathbb{CP}^2 - D_k)$) generated by all commutators $[\gamma_1, \gamma_2]$ where $\gamma_1, \gamma_2 \in \Gamma_k$ are such that $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are disjoint transpositions. The* stabilized fundamental group *is defined as $G_k = \pi_1(\mathbb{C}^2 - D_k)/K_k$, resp. $\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)/\bar{K}_k$.*

Certain natural subgroups of G_k and $\bar{G}_k$ will play an important role in the following sections. Define the *linking number* homomorphism $\delta_k : \pi_1(\mathbb{C}^2 - D_k) \to \mathbb{Z}$ by $\delta_k(\gamma) = 1$ for every $\gamma \in \Gamma_k$; similarly one can define $\bar{\delta}_k : \pi_1(\mathbb{CP}^2 - D_k) \to \mathbb{Z}_d$.

When D_k is irreducible (which is the general case), these can also be thought of as abelianization maps from the fundamental groups to the homology groups $H_1(\mathbb{C}^2 - D_k, \mathbb{Z}) \simeq \mathbb{Z}$ and $H_1(\mathbb{CP}^2 - D_k, \mathbb{Z}) \simeq \mathbb{Z}_d$.

Lemma 2.3. $\operatorname{Ker}\delta_k \simeq \operatorname{Ker}\bar{\delta}_k$.

Proof. Since $\pi_1(\mathbb{CP}^2 - D_k) = \pi_1(\mathbb{C}^2 - D_k)/\langle g_1 \dots g_d\rangle$ and $\delta_k(g_1 \dots g_d) = d$, it is sufficient to show that the product $g_1 \dots g_d$ belongs to the center of $\pi_1(\mathbb{C}^2 - D_k)$. Observe that the relation in $\pi_1(\mathbb{C}^2 - D_k)$ coming from a special point of D_k can be rewritten in the form $g \sim b_* g\ \forall g \in F_d$, where $b \in B_d$ is the braid monodromy around the given special point, acting on F_d. In particular, if we consider the braid monodromy as a factorization $\Delta^2 = \prod b_i$, we obtain that $g \sim (\prod b_i)_* g = (\Delta^2)_* g$ for any element g. However the action of the braid Δ^2 on F_d is exactly conjugation by $g_1 \dots g_d$; we conclude that $g_1 \dots g_d$ commutes with any element of $\pi_1(\mathbb{C}^2 - D_k)$, hence the result. □

The homomorphisms δ_k and $\bar{\delta}_k$ are obviously surjective. Moreover, θ_k is also surjective, because of the connectedness of X: the subgroup $\operatorname{Im}\theta_k \subseteq S_n$ is generated by transpositions and acts transitively on $\{1, \dots, n\}$, so it is equal to S_n. However, the image of $\theta_k^+ = (\theta_k, \delta_k) : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ is the index 2 subgroup $\{(\sigma, i) : \operatorname{sgn}(\sigma) \equiv i \bmod 2\}$, and similarly for $\bar{\theta}_k^+ = (\theta_k, \bar{\delta}_k) : \pi_1(\mathbb{CP}^2 - D_k) \to S_n \times \mathbb{Z}_d$ (note that d is always even). Since $K_k \subseteq \operatorname{Ker}\theta_k^+$, we can make the following definition:

Definition 2.4. *Let* $H_k^0 = \operatorname{Ker}\theta_k^+ \simeq \operatorname{Ker}\bar{\theta}_k^+$. *The* reduced subgroup *of* G_k *is* $G_k^0 = H_k^0/K_k$. *We have the following exact sequences:*

$$1 \longrightarrow G_k^0 \longrightarrow G_k \longrightarrow S_n \times \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$
$$1 \longrightarrow G_k^0 \longrightarrow \bar{G}_k \longrightarrow S_n \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

Theorem 1.2 is now obvious from the definitions and from Theorem 1.1: since creating a pair of nodes amounts to adding a relation of the form $[\gamma_1, \gamma_2] \sim 1$ where $[\gamma_1, \gamma_2] \in K_k$ (resp. $\bar{K}_k$), by construction it does not affect the groups G_k, $\bar{G}_k$ and G_k^0, which are therefore symplectic invariants for k large enough.

3. $\tilde{B}_n$-groups and their stabilizations

Denote by B_n (resp. P_n, $P_{n,0}$) the braid group on n strings (resp. the subgroups of pure braids and pure braids of degree 0), and denote by $X_1, \dots, X_{n-1}$ the standard generators of B_n. Recall that X_i is a half-twist along a segment joining the points i and $i+1$, and that the relations among these generators are $[X_i, X_j] = 1$ if $|i-j| \geq 2$ and $X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1}$.

Let $\tilde{B}_n$ be the quotient of B_n by the commutator of half-twists along two paths intersecting transversely in one point: $\tilde{B}_n = B_n/[X_2, X_3^{-1}X_1^{-1}X_2X_1X_3]$. The maps $\sigma : B_n \to S_n$ (induced permutation) and $\delta : B_n \to \mathbb{Z}$ (degree) factor through $\tilde{B}_n$, so one can define the subgroups $\tilde{P}_n = \operatorname{Ker}\sigma$ and $\tilde{P}_{n,0} = \operatorname{Ker}(\sigma, \delta)$. The structure of $\tilde{B}_n$ and its subgroups is described in detail in §1 of [9]; unlike P_n and $P_{n,0}$ which are quite complicated, these groups are fairly easy to understand: $\tilde{P}_{n,0}$ is solvable, its commutator subgroup is $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] \simeq \mathbb{Z}_2$ and its abelianization is $\operatorname{Ab}(\tilde{P}_{n,0}) \simeq \mathbb{Z}^{n-1}$ (it can in fact be identified naturally with the reduced regular representation $\mathcal{R}_n$ of S_n). More precisely, we have:

Lemma 3.1 (Moishezon). *Let x_i be the image of X_i in $\tilde{B}_n$, and define $s_1 = x_1^2$, $\eta = [x_1^2, x_2^2]$, $u_i = [x_i^{-1}, x_{i+1}^2]$ for $1 \le i \le n-2$, and $u_{n-1} = [x_{n-2}^2, x_{n-1}]$. Then $\tilde{P}_{n,0}$ is generated by $u_1, \dots, u_{n-1}$, and $\tilde{P}_n$ is generated by $s_1, u_1, \dots, u_{n-1}$.*

The relations among these elements are $[u_i, u_j] = 1$ if $|i-j| \ge 2$, $[u_i, u_{i+1}] = \eta$, $[s_1, u_i] = 1$ if $i \ne 2$, and $[s_1, u_2] = \eta$. The element η is central in $\tilde{B}_n$, has order 2 (i.e. $\eta^2 = 1$), and generates the commutator subgroups $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] = [\tilde{P}_n, \tilde{P}_n] \simeq \mathbb{Z}_2$ (in particular, for any two adjacent half-twists x and y we have $[x^2, y^2] = \eta$). As a consequence, $\mathrm{Ab}(\tilde{P}_n) \simeq \mathbb{Z}^n$ and $\mathrm{Ab}(\tilde{P}_{n,0}) \simeq \mathbb{Z}^{n-1}$.

Moreover, the action of $\tilde{B}_n$ on $\tilde{P}_n$ by conjugation is given by the following formulas: $x_i^{-1} s_1 x_i = s_1$ if $i \ne 2$, $x_2^{-1} s_1 x_2 = s_1 u_2^{-1}$; $x_i^{-1} u_j x_i = u_j$ if $|i-j| \ge 2$, $x_i^{-1} u_j x_i = u_i u_j$ if $|i-j| = 1$, and $x_i^{-1} u_i x_i = u_i^{-1}\eta$.

Proof. Most of the statement is a mere reformulation of Definition 8 and Theorem 1 in §1.5 of [9]. The only difference is that we define u_i directly in terms of the generators of $\tilde{B}_n$, while Moishezon defines $u_1 = (x_2 x_1^2 x_2^{-1}) x_2^{-2} = x_1^{-1} x_2^2 x_1 x_2^{-2}$ and constructs the other u_i by conjugation. In fact, $u_i = x^2 y^{-2}$ whenever x and y are two adjacent half-twists having respectively i and $i+1$ among their end points and such that $xyx^{-1} = x_i$; our definition of u_i corresponds to the choice $x = x_i^{-1} x_{i+1} x_i$ and $y = x_{i+1}$ for $i \le n-2$, and $x = x_{n-2}$ and $y = x_{n-1} x_{n-2} x_{n-1}^{-1}$ for $i = n-1$. Also note that Moishezon's formula for $x_2^{-1} s_1 x_2$ is inconsistent, due to a mistake in equation (1.25) of [9]; the formula we give is corrected. □

Intuitively speaking, the reason why $\tilde{B}_n$ is a fairly small group is that, due to the extra commutation relations, very little is remembered about the path supporting a given half-twist, namely just its two endpoints and the total number of times that it circles around the $n-2$ other points. This can be readily checked on simple examples (e.g., half-twists exchanging the first two points along a path that encircles only one of the $n-2$ other points: since these differ by conjugation by half-twists along paths presenting a single transverse intersection, they represent the same element in $\tilde{B}_n$). More generally, we have the following fact:

Lemma 3.2. *The elements of $\tilde{B}_n$ corresponding to half-twists exchanging the first two points are exactly those of the form $x_1 u_1^k \eta^{k(k-1)/2}$ for some integer k.*

Proof. Any half-twist exchanging the first two points can be put in the form $\gamma x_1 \gamma^{-1}$, where $\gamma \in \tilde{P}_n$ can be expressed as $\gamma = s_1^\alpha u_1^{\beta_1} \cdots u_{n-1}^{\beta_{n-1}} \eta^\epsilon$. Using Lemma 3.1, we have $x_1^{-1} \gamma x_1 = s_1^\alpha (u_1^{-1}\eta)^{\beta_1} (u_1 u_2)^{\beta_2} u_3^{\beta_3} \cdots u_{n-1}^{\beta_{n-1}} \eta^\epsilon$. Since $(u_1 u_2)^{\beta_2} = \eta^{\beta_2(\beta_2-1)/2} u_1^{\beta_2} u_2^{\beta_2}$, we can rewrite this equality as $x_1^{-1} \gamma x_1 = u_1^{-2\beta_1} \eta^{\beta_1} u_1^{\beta_2} \eta^{\beta_2(\beta_2-1)/2} \gamma = u_1^k \eta^{k(k-1)/2} \gamma$, where $k = \beta_2 - 2\beta_1$. Multiplying by x_1 on the left and γ^{-1} on the right we obtain $\gamma x_1 \gamma^{-1} = x_1 u_1^k \eta^{k(k-1)/2}$. □

Lemma 3.3. *Let $x, y \in \tilde{B}_n$ be elements corresponding to half-twists along paths with mutually disjoint endpoints. Then $[x, y] = 1$.*

Proof. The result is trivial when the paths corresponding to x and y are disjoint or intersect only once. In general, after conjugation we can assume that $x = \gamma x_1 \gamma^{-1}$ for some $\gamma \in \tilde{P}_n$, and $y = x_3$. By Lemma 3.2, $x = x_1 u_1^k \eta^{k(k-1)/2}$ for some integer k. Since x_1, u_1 and η all commute with x_3, we conclude that $[x, y] = 1$ as desired. □

Lemma 3.4. *Let $x, y \in \tilde{B}_n$ be elements corresponding to half-twists along paths with one common endpoint. Then $xyx = yxy$.*

Proof. After conjugation we can assume that $x = x_1$ and $y = \gamma x_2 \gamma^{-1}$ for some $\gamma \in \tilde{P}_n$. By the classification of half-twists in $\tilde{B}_n$ (Lemma 3.2), there exists an integer k such that $y = x_2 u_2^k \eta^{k(k-1)/2} = x_2 (s_1 u_2^{-1})^{-k} s_1^k = s_1^{-k} x_2 s_1^k$. Therefore $xyx = x_1 s_1^{-k} x_2 s_1^k x_1 = s_1^{-k}(x_1 x_2 x_1) s_1^k = s_1^{-k}(x_2 x_1 x_2) s_1^k = yxy$. □

It must be noted that Lemmas 3.3 and 3.4 have also been obtained by Robb [12].

Lemma 3.5. *The group $\tilde{B}_n$ admits automorphisms ϵ_i such that $\epsilon_i(x_i) = x_i u_i$ and $\epsilon_i(x_j) = x_j$ for every $j \neq i$. Moreover, $\epsilon_i(u_i) = u_i \eta$ and $\epsilon_i(u_j) = u_j$ $\forall j \neq i$.*

Proof. By Lemmas 3.3 and 3.4, the half-twists $x_1, \ldots, x_{i-1}, (x_i u_i), x_{i+1}, \ldots, x_{n-1}$ satisfy exactly the same relations as the standard generators of $\tilde{B}_n$. So ϵ_i is a well-defined group homomorphism from $\tilde{B}_n$ to itself, and it is injective. The formulas for $\epsilon_i(u_i)$ and $\epsilon_i(u_j)$ are easily checked. The surjectivity of ϵ_i follows from the identity $\epsilon_i(x_i u_i^{-1} \eta) = x_i$. □

The following definition is motivated by the very particular structure of the fundamental groups of branch curve complements computed by Moishezon for generic projections of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ [9, 10], which seems to be a feature common to a much larger class of examples (see §4):

Definition 3.6. *Define $\tilde{B}_n^{(2)} = \{(x, y) \in \tilde{B}_n \times \tilde{B}_n,\ \sigma(x) = \sigma(y) \text{ and } \delta(x) = \delta(y)\}$. We say that the group $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$ if there exists an isomorphism ψ from $\pi_1(\mathbb{C}^2 - D_k)$ to a quotient of $\tilde{B}_n^{(2)}$ such that, for any geometric generator $\gamma \in \Gamma_k$, there exist two half-twists $x, y \in \tilde{B}_n$ such that $\sigma(x) = \sigma(y) = \theta_k(\gamma)$ and $\psi(\gamma) = (x, y)$.*

In other words, $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$ if there exists a surjective homomorphism from $\tilde{B}_n^{(2)}$ to $\pi_1(\mathbb{C}^2 - D_k)$ which maps pairs of half-twists to geometric generators, in a manner compatible with the S_n-valued homomorphisms σ and θ_k.

Remark 3.7. *If $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$, then the kernel of the homomorphism $\theta_k^+ : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ is a quotient of $\tilde{P}_{n,0} \times \tilde{P}_{n,0}$ and therefore a solvable group; in particular its commutator subgroup is a quotient of $(\mathbb{Z}_2)^2$, and its abelianization is a quotient of $\mathbb{Z}^2 \otimes \mathcal{R}_n \simeq (\mathbb{Z} \oplus \mathbb{Z})^{n-1}$.*

As an immediate consequence of Definition 3.6 and Lemma 3.3, we have:

Proposition 3.8. *If $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$, then the stabilization operation is trivial, i.e. $K_k = \{1\}$, $G_k = \pi_1(\mathbb{C}^2 - D_k)$, and $G_k^0 = \operatorname{Ker} \theta_k^+$.*

Proof. Let $\gamma, \gamma' \in \Gamma_k$ be such that $\theta_k(\gamma)$ and $\theta_k(\gamma')$ are disjoint transpositions. Consider the isomorphism ψ given by Definition 3.6: there exist half-twists $x, x', y, y' \in \tilde{B}_n$ such that $\psi(\gamma) = (x, y)$ and $\psi(\gamma') = (x', y')$. Since $\theta_k(\gamma) = \sigma(x) = \sigma(y)$ and $\theta_k(\gamma') = \sigma(x') = \sigma(y')$ are disjoint transpositions, x and x' have disjoint endpoints, and similarly for y and y'. Therefore, by Lemma 3.3 we have $[x, x'] = 1$ and $[y, y'] = 1$, so that $[\psi(\gamma), \psi(\gamma')] = 1$, and therefore $[\gamma, \gamma'] = 1$. We conclude that $K_k = \{1\}$, which ends the proof. □

Let $D_{p,q}$ be the branch curve of a generic polynomial map $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ of bidegree (p, q), $p, q \geq 2$. As will be shown in §4, it follows from the computations in [9] that $\pi_1(\mathbb{C}^2 - D_{p,q})$ satisfies property $(*)$. This property also holds for the complement of the branch curve of a generic polynomial map from $\mathbb{CP}^2$ to itself in degree ≥ 3, as follows from the calculations in [10] (see also [15]), and in various other examples as well (see §4). It is an interesting question to determine whether this remarkable structure of branch curve complements extends to generic high-degree projections of arbitrary algebraic surfaces; this would tie in nicely with a conjecture of Teicher about the virtual solvability of these fundamental groups [14], and would also imply Conjecture 1.3.

4. Examples

As follows from pp. 696–700 of [5], if the symplectic manifold X happens to be Kähler, then all approximately holomorphic constructions can actually be carried out using genuine holomorphic sections of $L^{\otimes k}$ over X, and as a consequence the $\mathbb{CP}^2$-valued maps given by Theorem 1.1 coincide up to isotopy with projective maps defined by generic holomorphic sections of $L^{\otimes k}$; therefore, in the case of complex projective surfaces all calculations can legitimately be performed within the framework of complex algebraic geometry.

The fundamental groups of complements of branch curves have already been computed for generic projections of various complex projective surfaces. In many cases, these computations only hold for specific linear systems, and do not apply to the high degree situation that we wish to consider.

Nevertheless, it is worth mentioning that, if $D \subset \mathbb{CP}^2$ is the branch curve of a generic linear projection of a hypersurface of degree n in $\mathbb{CP}^3$, then it has been shown by Moishezon that $\pi_1(\mathbb{C}^2 - D) \simeq B_n$ [7]. In fact, in this specific case there is a well-defined geometric monodromy representation morphism θ_B with values in the braid group B_n rather than in the symmetric group S_n as usual, because the n preimages of any point in $\mathbb{CP}^2 - D$ lie in a fiber of the projection $\mathbb{CP}^3 - \{\mathrm{pt}\} \to \mathbb{CP}^2$, which after trivialization over an affine subset can be identified with $\mathbb{C}$. Moishezon's computations then show that $\theta_B : \pi_1(\mathbb{C}^2 - D) \to B_n$ is an isomorphism. An attempt to quotient out B_n by commutators as in the definition of stabilized fundamental groups yields $\tilde{B}_n$: in this case the stabilization operation is non-trivial. However this situation is specific to the linear system $O(1)$, and one expects the fundamental groups of branch curve complements to behave differently when one instead considers projections given by sections of $O(k)$ for $k \gg 0$.

Moishezon's result about hypersurfaces in $\mathbb{CP}^3$ has been extended by Robb to the case of complete intersections (still considering only linear projections to $\mathbb{CP}^2$ rather than arbitrary linear systems) [12]. The result is that, if D is the branch curve for a complete intersection of degree n in $\mathbb{CP}^m$ ($m \geq 4$), then the group $\pi_1(\mathbb{C}^2 - D)$ is isomorphic to $\tilde{B}_n$. It is worth noting that, in this example, the stabilization operation is trivial. In fact, the groups $\pi_1(\mathbb{C}^2 - D)$ can be shown to have property $(*)$ (observe that $\tilde{B}_n$ is the quotient of $\tilde{B}_n^{(2)}$ by its subgroup $1 \times \tilde{P}_{n,0}$).

Conjecture 1.6 holds for $k = 1$ in these two families of examples: we have $\mathrm{Ab}\, G^0 \simeq \mathbb{Z}^{n-1}$ and $[G^0, G^0] \simeq \mathbb{Z}_2$ in both cases, while $\mathbb{Z}^2/\Lambda_1 \simeq \mathbb{Z}$ because the canonical class is proportional to the hyperplane class which is primitive.

More interestingly for our purposes, the calculations have also been carried out in the case of arbitrarily positive linear systems by Moishezon for two fundamental examples: $\mathbb{CP}^1 \times \mathbb{CP}^1$ [9], and $\mathbb{CP}^2$ [10] (unpublished, see also [15] for a summary).

Theorem 4.1 (Moishezon)**.** *Let $D_{p,q}$ be the branch curve of a generic polynomial map $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ of bidegree (p,q), $p, q \geq 2$. Then the group $\pi_1(\mathbb{C}^2 - D_{p,q})$*

satisfies property $(*)$*, and its subgroup* $H^0_{p,q} = \operatorname{Ker}\theta^+_{p,q}$ *has the following structure:* $\operatorname{Ab} H^0_{p,q}$ *is isomorphic to* $(\mathbb{Z}_2 \oplus Z_{p-q})^{n-1}$ *if* p *and* q *are even, and* $(\mathbb{Z}_{2(p-q)})^{n-1}$ *if* p *or* q *is odd* (*here* $n = 2pq$)*; the commutator subgroup* $[H^0_{p,q}, H^0_{p,q}]$ *is isomorphic to* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *when* p *and* q *are even, and* $\mathbb{Z}_2$ *if* p *or* q *is odd.*

In fact, Moishezon identifies $\pi_1(\mathbb{C}^2 - D_{p,q})$ with a quotient of the semi-direct product $\tilde{B}_n \ltimes \tilde{P}_{n,0}$, where $\tilde{B}_n$ acts from the right on $\tilde{P}_{n,0}$ by conjugation [9]. However it is easy to observe that the map $\kappa : \tilde{B}_n \ltimes \tilde{P}_{n,0} \to \tilde{B}^{(2)}_n$ defined by $\kappa(x,u) = (x, xu)$ is a group isomorphism (recall the group structure on $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ is given by $(x,u)(x',u') = (xx', x'^{-1}ux'u')$). The factor $\tilde{P}_{n,0}$ of the semi-direct product corresponds to the normal subgroup $1 \times \tilde{P}_{n,0}$ of $\tilde{B}^{(2)}_n$, while the factor $\tilde{B}_n$ corresponds to the diagonally embedded subgroup $\tilde{B}_n = \{(x,x)\} \subset \tilde{B}^{(2)}_n$.

Moreover, by carefully going over the various formulas identifying a set of geometric generators for $\pi_1(\mathbb{C}^2 - D_{p,q})$ with certain specific elements in $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ (Propositions 8 and 10 of [9]; cf. also §1.4, Definition 24 and Remarks 28–29 of [9]), or equivalently in $\tilde{B}^{(2)}_n$ after applying the isomorphism κ, it is relatively easy to check that each geometric generator corresponds to a pair of half-twists with the expected end points in $\tilde{B}^{(2)}_n$ (see also §6 for more details). Therefore, property $(*)$ and Conjecture 1.3 hold for these groups.

Conjecture 1.6 also holds for $\mathbb{CP}^1 \times \mathbb{CP}^1$. Indeed, $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$ is generated by classes α and β corresponding to the two factors; the hyperplane section class is $L = p\alpha + q\beta$, while the ramification curve is $R = 3L + K = (3p-2)\alpha + (3q-2)\beta$. Therefore, the subgroup $\Lambda_{p,q}$ of $\mathbb{Z}^2$ is generated by $(\alpha \cdot L, \alpha \cdot R) = (q, 3q-2)$ and $(\beta \cdot L, \beta \cdot R) = (p, 3p-2)$. An easy computation shows that the quotient $\mathbb{Z}^2/\Lambda_{p,q} = \mathbb{Z}^2/\langle (q, 3q-2), (p, 3p-2)\rangle \simeq \mathbb{Z}^2/\langle (q,2), (p,2)\rangle$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{p-q}$ when p and q are even, and to $\mathbb{Z}_{2(p-q)}$ otherwise.

It is worth noting that this nice description for $p, q \geq 2$ completely breaks down in the insufficiently ample case $p = 1$, where it follows from computations of Zariski [17] that $\pi_1(\mathbb{C}^2 - D_{1,q}) \simeq B_{2q}$. So both Conjecture 1.3 and Conjecture 1.6 require a sufficient amount of ampleness in order to hold ($p, q \geq 2$).

Theorem 4.2 (Moishezon)**.** *Let* D_k *be the branch curve of a generic polynomial map* $\mathbb{CP}^2 \to \mathbb{CP}^2$ *of degree* $k \geq 3$*. Then the group* $\pi_1(\mathbb{C}^2 - D_k)$ *satisfies property* $(*)$*, and its subgroup* $H^0_k = \operatorname{Ker}\theta^+_k$ *has the following structure:* $\operatorname{Ab} H^0_k$ *is isomorphic to* $(\mathbb{Z} \oplus \mathbb{Z}_3)^{n-1}$ *if* k *is a multiple of* 3*, and to* $\mathbb{Z}^{n-1}$ *otherwise* (*here* $n = k^2$)*; the commutator subgroup* $[H^0_k, H^0_k]$ *is trivial for* k *even and isomorphic to* $\mathbb{Z}_2$ *for* k *odd.*

In this case too, Moishezon in fact identifies $\pi_1(\mathbb{C}^2 - D_k)$ with a quotient of $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ [10] (see also [15]). Property $(*)$ and Conjecture 1.3 hold for $\mathbb{CP}^2$ when $k \geq 3$, but for $k = 2$ the group $\pi_1(\mathbb{C}^2 - D_2)$ is much larger.

Since $H_2(\mathbb{CP}^2, \mathbb{Z})$ is generated by the class of a line, Λ_k is the subgroup of $\mathbb{Z}^2$ generated by $(k, 3k-3)$, and $\mathbb{Z}^2/\Lambda_k$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$ when k is a multiple of 3 and to $\mathbb{Z}$ otherwise. Therefore Conjecture 1.6 holds for $\mathbb{CP}^2$ when $k \geq 3$.

Results for certain projections of Del Pezzo and K3 surfaces have also been announced by Robb in [12].

Theorem 4.3 (Robb)**.** *Let* X *be either a cubic hypersurface in* $\mathbb{CP}^3$ *or a* $(2,2)$ *complete intersection in* $\mathbb{CP}^4$*, and let* D_k *be the branch curve of a generic algebraic map* $X \to \mathbb{CP}^2$ *given by sections of* $O(kH)$*, where* H *is the hyperplane section* *and* $k \geq 2$. *Then the subgroup* $H^0_k = \operatorname{Ker}\theta^+_k$ *of* $\pi_1(\mathbb{C}^2 - D_k)$ *has abelianization* $\operatorname{Ab} H^0_k \simeq \mathbb{Z}^{n-1}$.

Theorem 4.4 (Robb)**.** *Let* X *be a K3 surface realized either as a degree* 4 *hypersurface in* $\mathbb{CP}^3$, *a* $(3,2)$ *complete intersection in* $\mathbb{CP}^4$ *or a* $(2,2,2)$ *complete intersection in* $\mathbb{CP}^5$, *and let* D_k *be the branch curve of a generic algebraic map* $X \to \mathbb{CP}^2$ *given by sections of* $O(kH)$, *where* H *is the hyperplane section and* $k \geq 2$. *Then the subgroup* $H^0_k = \operatorname{Ker}\theta^+_k$ *of* $\pi_1(\mathbb{C}^2 - D_k)$ *has abelianization* $\operatorname{Ab} H^0_k \simeq (\mathbb{Z} \oplus \mathbb{Z}_k)^{n-1}$.

Although to our knowledge no detailed proofs of Theorems 4.3 and 4.4 have appeared yet, it appears very likely from the sketch of argument given in [12] that property $(*)$ and Conjecture 1.3 will hold for these examples as well. In any case we can compare Robb's results with the answers predicted by Conjecture 1.6.

In the case of the Del Pezzo surfaces, the hyperplane class H is primitive, and $K = -H$ (so $R_k = (3k-1)H$), so that the subgroup $\Lambda_k \subset \mathbb{Z}^2$ is generated by $(k, 3k-1)$, and $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z}$, which is in agreement with Theorem 4.3. In the case of the K3 surfaces, the hyperplane class H is again primitive, but $K = 0$ and $R_k = 3kH$, so that Λ_k is now generated by $(k, 3k)$, and $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z} \oplus \mathbb{Z}_k$, in agreement with Theorem 4.4.

The following result for the Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}(O_{\mathbb{CP}^1} \oplus O_{\mathbb{CP}^1}(1))$ is new to our knowledge; however partial results about this surface have been obtained by Moishezon, Robb and Teicher [11, 16], and an ongoing project of Teicher and coworkers is expected to yield another proof of the same result.

Theorem 4.5. *Let* $D_{p,q}$ *be the branch curve of a generic algebraic map* $\mathbb{F}_1 \to \mathbb{CP}^2$ *given by three sections of the linear system* $O(pF + qE)$, *where* F *is the class of a fiber,* E *is the exceptional section, and* $p > q \geq 2$. *Then the group* $\pi_1(\mathbb{C}^2 - D_{p,q})$ *satisfies property* $(*)$, *and its subgroup* $H^0_{p,q} = \operatorname{Ker}\theta^+_{p,q}$ *has the following structure:* $\operatorname{Ab} H^0_{p,q} \simeq (\mathbb{Z}_{3q-2p})^{n-1}$, *where* $n = (2p-q)q$, *and the commutator subgroup* $[H^0_{p,q}, H^0_{p,q}]$ *is isomorphic to* $\mathbb{Z}_2$ *if* p *is odd and* q *even, and trivial in all other cases.*

The proof relies on the observation that $\mathbb{F}_1$ is the blow-up of $\mathbb{CP}^2$ at one point. Recalling the interpretation of a symplectic (or Kähler) blow-up as the collapsing of an embedded ball, it is easy to check that $\mathbb{F}_1$ can be degenerated to a union of planes in a manner similar to $\mathbb{CP}^2$, only with some components missing; most of the calculations performed by Moishezon in [10] for $\mathbb{CP}^2$ can then be re-used in this context, with the only changes occurring along the exceptional curve E. More details are given in §6.2.

As a consequence of property $(*)$, Conjecture 1.3 holds for this example. So does Conjecture 1.6: indeed, $H_2(\mathbb{F}_1, \mathbb{Z})$ is generated by F and E. Recalling that $F \cdot F = 0$, $F \cdot E = 1$, $E \cdot E = -1$, and letting $L_{p,q} = pF + qE$ and $R_{p,q} = 3L_{p,q} + K = (3p-3)F + (3q-2)E$, we obtain that $\Lambda_{p,q} \subset \mathbb{Z}^2$ is generated by $(F \cdot L_{p,q}, F \cdot R_{p,q}) = (q, 3q-2)$ and $(E \cdot L_{p,q}, E \cdot R_{p,q}) = (p-q, 3p-3q-1)$. Therefore $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z}^2/\langle (q, 3q-2), (p-q, 3p-3q-1)\rangle \simeq \mathbb{Z}_{3q-2p}$.

A much wider class of examples, including an infinite family of surfaces of general type, can be investigated if one brings approximately holomorphic techniques into the picture, although this makes it only possible to obtain results about the stabilized fundamental groups of branch curve complements (cf. §2) rather than the actual fundamental groups.

Theorem 4.6. *For given integers $a, b \geq 1$ and $p, q \geq 2$, let $X_{a,b}$ be the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a smooth algebraic curve of degree $(2a, 2b)$, and let $L_{p,q}$ be the linear system over $X_{a,b}$ defined as the pullback of $O_{\mathbb{P}^1\times\mathbb{P}^1}(p,q)$ via the double cover. Let $D_{p,q}$ be the branch curve of a generic approximately holomorphic perturbation of an algebraic map $X_{a,b} \to \mathbb{CP}^2$ given by three sections of $L_{p,q}$. Then the* stabilized *fundamental group $G_{p,q}(X_{a,b}) = \pi_1(\mathbb{C}^2 - D_{p,q})/K_{p,q}$ satisfies property $(*)$, and its reduced subgroup $G^0_{p,q}(X_{a,b}) = \operatorname{Ker}\theta^+_{p,q}/K_{p,q}$ has the following structure: $\operatorname{Ab} G^0_{p,q}(X_{a,b}) \simeq (\mathbb{Z}^2/\langle (p, a-2), (q, b-2)\rangle)^{n-1}$, where $n = 4pq$, and the commutator subgroup $[G^0_{p,q}(X_{a,b}), G^0_{p,q}(X_{a,b})]$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if a, b, p, q are all even, trivial if a or b is odd and $a + p$ or $b + q$ is odd, and isomorphic to $\mathbb{Z}_2$ in all other cases.*

More precisely, the setup that we consider starts with a holomorphic map from $X_{a,b}$ to $\mathbb{CP}^2$ that factors through the double cover $X_{a,b} \to \mathbb{CP}^1 \times \mathbb{CP}^1$. Such a map is of course not generic in any sense; however there is a natural explicit way to perturb it in the approximately holomorphic category (see §7), giving rise to the branch curves $D_{p,q}$ that we consider. The map can also be perturbed in the holomorphic category, which at least for p and q large enough yields a branch curve that is equivalent to $D_{p,q}$ up to creations and cancellations of pairs of nodes. So, on the level of stabilized groups, our result does give an answer that is relevant from both the symplectic and algebraic points of view. Moreover, it is expected that, at least for p and q large enough, the fundamental groups themselves (rather than their stabilized quotients) should satisfy property $(*)$.

Theorem 4.6 implies that Conjecture 1.6 holds for the manifolds $X_{a,b}$. Indeed, $X_{a,b}$ can also be described topologically as follows: in $\mathbb{CP}^1 \times \mathbb{CP}^1$ consider $2a$ curves of the form $\mathbb{CP}^1 \times \{\mathrm{pt}\}$ and $2b$ curves of the form $\{\mathrm{pt}\} \times \mathbb{CP}^1$, and blow up their $4ab$ intersection points to obtain a manifold $Y_{a,b}$ containing disjoint rational curves $C_1, \dots, C_{2a}$ (of square $-2b$) and $C'_1, \dots, C'_{2b}$ (of square $-2a$). Then $X_{a,b}$ is the double cover of $Y_{a,b}$ branched along $C_1 \cup \dots \cup C_{2a} \cup C'_1 \cup \dots \cup C'_{2b}$. Now, consider the preimages $\tilde{C}_i = \pi^{-1}(C_i)$ and $\tilde{C}'_i = \pi^{-1}(C'_i)$, and let $L_{p,q} = p\pi^*\alpha + q\pi^*\beta$ and $R_{p,q} = 3L_{p,q} + K_{X_{a,b}} = (3p + a - 2)\pi^*\alpha + (3q + b - 2)\pi^*\beta$, where α and β are the homology generators corresponding to the two factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$. We have $(\tilde{C}_i \cdot L_{p,q}, \tilde{C}_i \cdot R_{p,q}) = (q, 3q + b - 2)$ and $(\tilde{C}'_i \cdot L_{p,q}, \tilde{C}'_i \cdot R_{p,q}) = (p, 3p + a - 2)$. It is easily shown that these two elements of $\mathbb{Z}^2$ generate the subgroup $\Lambda_{p,q}$; therefore $\mathbb{Z}^2/\Lambda_{p,q} = \mathbb{Z}^2/\langle (q, 3q + b - 2), (p, 3p + a - 2)\rangle \simeq \mathbb{Z}^2/\langle (p, a - 2), (q, b - 2)\rangle$.

The techniques involved in the proof of Theorem 4.6, which will be discussed in §7, extend to double covers of other examples for which the answer is known, possibly including iterated double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$. One example of particular interest is that of double covers of Hirzebruch surfaces branched along disconnected curves, for which we make the following conjecture:

Conjecture 4.7. *Given integers $m, a \geq 1$, let $X_{2m,a}$ be the double cover of the Hirzebruch surface $\mathbb{F}_{2m}$ branched along the union of the exceptional section Δ_∞ and a smooth algebraic curve in the homology class $(2a - 1)[\Delta_0]$ (where Δ_0 is the zero section, of square $2m$). Given integers $p, q \geq 2$ such that $p > 2mq$, let $L_{p,q}$ be the linear system over $X_{2m,a}$ defined as the pullback of $O_{\mathbb{F}_{2m}}(pF + q\Delta_\infty)$ via the double cover. Let $D_{p,q}$ be the branch curve of a generic approximately holomorphic perturbation of an algebraic map $X_{2m,a} \to \mathbb{CP}^2$ given by three sections of $L_{p,q}$. Then the* *reduced stabilized fundamental group* $G^0_{p,q}(X_{2m,a}) = \operatorname{Ker}\theta^+_{p,q}/K_{p,q}$ *has abelianization* $\operatorname{Ab} G^0_{p,q}(X_{2m,a}) \simeq (\mathbb{Z}^2/\langle (p-2mq, m-2), (2q, 2a-4)\rangle)^{n-1}$.

5. Stabilized fundamental groups and homological data

Consider a compact symplectic 4-manifold X such that $H_1(X,\mathbb{Z}) = 0$ and a branched covering map $f_k : X \to \mathbb{CP}^2$ determined by three sections of $L^{\otimes k}$, with branch curve $D_k \subset \mathbb{CP}^2$ and geometric monodromy representation morphism $\theta_k : \pi_1(\mathbb{C}^2 - D_k) \to S_n$. The purpose of this section is to construct a natural morphism $\psi_k : \operatorname{Ker}\theta_k \to (\mathbb{Z}^2/\Lambda_k) \otimes \bar{\mathcal{R}}_n \simeq (\mathbb{Z}^2/\Lambda_k)^n$ (where $\bar{\mathcal{R}}_n \simeq \mathbb{Z}^n$ is the regular representation of S_n) and use its properties to prove Theorem 1.5.

Fix a base point p_0 in $\mathbb{C}^2 - D_k$, and let $p_1, \dots, p_n$ be its preimages by f_k. Let $\gamma \in \pi_1(\mathbb{C}^2 - D_k)$ be a loop in the complement of D_k such that $\theta_k(\gamma) = \mathrm{Id}$. Since the monodromy of the branched cover f_k along γ is trivial, $f_k^{-1}(\gamma)$ is the union of n disjoint closed loops in X. Denote by γ_i the lift of γ that starts at the point p_i. Since $H_1(X,\mathbb{Z}) = 0$, there exists a surface (or rather a 2-chain) $S_i \subset X$ such that $\partial S_i = \gamma_i$. Since $\gamma \subset \mathbb{C}^2 - D_k$, the loop γ_i intersects neither the ramification curve R_k nor the preimage L_k of the line at infinity in $\mathbb{CP}^2$. Therefore, there exist well-defined algebraic intersection numbers $\lambda_i = S_i \cdot L_k$ and $\rho_i = S_i \cdot R_k \in \mathbb{Z}$. However, there are various possible choices for the surface S_i, and the relative cycle $[S_i]$ is only well-defined up to an element of $H_2(X,\mathbb{Z})$. Therefore, the pair $(\lambda_i, \rho_i) \in \mathbb{Z}^2$ is only defined up to an element of the subgroup Λ_k.

Definition 5.1. *With the above notations, we denote by* $\psi_k : \operatorname{Ker}\theta_k \to (\mathbb{Z}^2/\Lambda_k)^n$ *the morphism defined by* $\psi_k(\gamma) = ((S_i \cdot L_k, S_i \cdot R_k))_{1\le i\le n}$.

In fact, there is no canonical ordering of the preimages of p_0, and ψ_k more naturally takes values in $(\mathbb{Z}^2/\Lambda_k) \otimes \bar{\mathcal{R}}_n$, as evidenced by Lemma 5.2 below.

Definition 5.1 can naturally be extended to the case $H_1(X,\mathbb{Z}) \neq 0$ by instead considering the morphism $\tilde{\psi}_k : \operatorname{Ker}\theta_k \to H_1(X - L_k - R_k, \mathbb{Z})^n$ which maps a loop γ to the homology classes of its lifts γ_i in $X - L_k - R_k$. However, the properties to be expected of this morphism in general are not entirely clear, due to the lack of available non-simply connected examples (even though the techniques in §6–7 could probably be applied to the 4-manifold $\Sigma \times \mathbb{CP}^1$ for any Riemann surface Σ).

We now investigate the various properties of ψ_k.

Lemma 5.2. *For every* $\gamma \in \operatorname{Ker}\theta_k$ *and* $g \in \pi_1(\mathbb{C}^2 - D_k)$, $\psi_k(g^{-1}\gamma g) = \theta_k(g)\cdot\psi_k(\gamma)$, *where* S_n *acts on* $(\mathbb{Z}^2/\Lambda_k)^n$ *by permuting the factors (i.e.,* ψ_k *is equivariant).*

Proof. Denoting by σ the permutation $\theta_k(g)$, observe that the lifts of $g^{-1}\gamma g$ are freely homotopic to those of γ, and more precisely that the lift of $g^{-1}\gamma g$ through $p_{\sigma(i)}$ is freely homotopic to the lift of γ through p_i. Therefore, the $\sigma(i)$-th component of $\psi_k(g^{-1}\gamma g)$ is equal to the i-th component of $\psi_k(\gamma)$. □

Lemma 5.3. $K_k \subset \operatorname{Ker}\psi_k$, *i.e.* ψ_k *factors through the stabilized group.*

Proof. Recall from Definition 2.2 that K_k is generated by commutators $[\gamma_1, \gamma_2]$ of geometric generators that are mapped to disjoint transpositions by θ_k. If γ_1 is a geometric generator, then $n-2$ of its lifts to X are contractible closed loops in $X - L_k - R_k$, while the two other lifts are not closed; and similarly for γ_2. However, if $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are disjoint, then all the lifts of $[\gamma_1, \gamma_2]$ are contractible loops in $X - L_k - R_k$; therefore $[\gamma_1, \gamma_2] \in \operatorname{Ker}\psi_k$. □

It is worth noting that, similarly, if γ_1 and γ_2 are geometric generators mapped by θ_k to adjacent (non-commuting) transpositions, then $(\gamma_1\gamma_2\gamma_1)(\gamma_2\gamma_1\gamma_2)^{-1} \in \operatorname{Ker}\psi_k$ (only one of the lifts of this loop is possibly non-trivial, but its algebraic linking numbers with L_k and R_k are both equal to zero).

Lemma 5.4. *For any $\gamma \in \operatorname{Ker}\theta_k$, the n-tuple $\psi_k(\gamma) = ((\lambda_i, \rho_i))_{1\le i\le n}$ has the property that $(\sum \lambda_i, \sum \rho_i) \equiv (0, \delta_k(\gamma)) \mod \Lambda_k$.*

Proof. $\gamma \in \pi_1(\mathbb{C}^2 - D_k)$ is homotopically trivial in $\mathbb{C}^2$, so there exists a topological disk $\Delta \subset \mathbb{C}^2$ such that $\partial\Delta = \gamma$. Now observe that $\partial(f_k^{-1}(\Delta)) = \sum \gamma_i$; therefore $(\sum \lambda_i, \sum \rho_i)$ is equal (mod Λ_k) to the algebraic intersection numbers of $f_k^{-1}(\Delta)$ with L_k and R_k. We have $f_k^{-1}(\Delta) \cdot L_k = 0$ since $f_k^{-1}(\Delta) \subset f_k^{-1}(\mathbb{C}^2) = X - L_k$, and $f_k^{-1}(\Delta) \cdot R_k = \Delta \cdot D_k = \delta_k(\gamma)$. □

Lemma 5.5. *For any geometric generator $\gamma \in \Gamma_k$, $\psi_k(\gamma^2) = ((\lambda_i, \rho_i))_{1\le i\le n}$ is given by $(\lambda_i, \rho_i) = (0, 1)$ if i is one of the two indices exchanged by the transposition $\theta_k(\gamma)$, and $(\lambda_i, \rho_i) = (0, 0)$ otherwise.*

Proof. All lifts of γ^2 are homotopically trivial, except for two of them which are freely homotopic to each other and circle once around the ramification curve R_k. □

Lemma 5.6. *There exist two geometric generators $\gamma_1, \gamma_2 \in \Gamma_k$ such that $\theta_k(\gamma_1) = \theta_k(\gamma_2)$ and $\psi_k(\gamma_1\gamma_2) = ((-1, 0), (1, 2), (0, 0), \dots, (0, 0))$.*

Proof. Consider a generic line $\ell \subset \mathbb{CP}^2$ intersecting D_k transversely in $d = \deg D_k$ points, and let $\Sigma = f_k^{-1}(L)$. The restriction $f_{k|\Sigma} : \Sigma \to \ell = \mathbb{CP}^1$ is a connected simple branched cover of degree n with d branch points, with monodromy described by the morphism $\theta_k \circ i_* : \pi_1(\ell - \{d \text{ points}\}) \to S_n$. It is a classical fact that the moduli space of all connected simple branched covers of $\mathbb{CP}^1$ with fixed degree and number of branch points is connected, i.e. up to a suitable reordering of the branch points we can assume that the monodromy of $f_{k|\Sigma}$ is described by any given standard S_n-valued morphism.

So we can find an ordered system of generators $\gamma_1, \dots, \gamma_d$ of the free group $\pi_1(\ell \cap (\mathbb{C}^2 - D_k))$ such that $\theta_k(\gamma_1) = \theta_k(\gamma_2) = (12)$ and all the other transpositions $\theta_k(\gamma_i)$ for $i \ge 3$ are elements of $S_{n-1} = \operatorname{Aut}\{2, \dots, n\}$. The loop $\gamma_1\gamma_2$ then belongs to $\operatorname{Ker}\theta_k$, and admits only two non-trivial lifts g_1 and g_2 in Σ, those which start in the first two sheets of the branched cover. The loops g_1 and g_2 bound a topological annulus A which intersects R_k in two points (projecting to the first two intersection points of ℓ with D_k). This annulus separates Σ into two components, a "large" component consisting of the sheets numbered from 2 to n, and a disk Δ corresponding to the first sheet of the cover, which does not intersect R_k but contains one of the n preimages of the intersection point of ℓ with the line at infinity in $\mathbb{CP}^2$. The lift g_1 bounds Δ with reversed orientation; since $\Delta \cdot R_k = 0$ and $\Delta \cdot L_k = 1$, the first component of $\psi_k(\gamma_1\gamma_2)$ is $(-1, 0)$. The lift g_2 bounds $\Delta \cup A$; since $A \cdot R_k = 2$ and $A \cdot L_k = 0$, the second component of $\psi_k(\gamma_1\gamma_2)$ is $(1, 2)$. □

Proof of Theorem 1.5. By Lemma 5.4, ψ_k maps the kernel of $\theta_k^+ : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ into the subgroup $\Gamma = \{(\lambda_i, \rho_i),\ \sum \lambda_i = \sum \rho_i = 0\} \simeq (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_n$ of $(\mathbb{Z}^2/\Lambda_k)^n$. By Lemma 5.3, ψ_k factors through the quotient $\operatorname{Ker}\theta_k^+/K_k = G_k^0(X, \omega)$, and gives rise to a map $\phi_k : G_k^0(X, \omega) \to \Gamma \simeq (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_n \simeq (\mathbb{Z}^2/\Lambda_k)^{n-1}$. Since Γ is abelian, $[G_k^0, G_k^0] \subset \operatorname{Ker}\phi_k$, so ϕ_k factors through the abelianization $\operatorname{Ab} G_k^0(X, \omega)$, as announced in the statement of Theorem 1.5.

We now show that ϕ_k is surjective, i.e. that ψ_k maps $\operatorname{Ker}\theta_k^+$ *onto* Γ. First, let γ and γ' be two geometric generators of $\pi_1(\mathbb{C}^2 - D_k)$ corresponding to adjacent transpositions in S_n: then $\gamma^2\gamma'^{-2} \in \operatorname{Ker}\theta_k^+$, and Lemma 5.5 implies that $\psi_k(\gamma^2\gamma'^{-2})$ has only two non-zero entries, one equal to $(0,1)$ and the other equal to $(0,-1)$. Recalling from §2 that θ_k is surjective, and using Lemma 5.2, by considering suitable conjugates of $\gamma^2\gamma'^{-2}$ we can find elements g_{ij} of $\operatorname{Ker}\theta_k^+$ such that $\psi_k(g_{ij})$ has only two non-zero entries, $(0,1)$ at position i and $(0,-1)$ at position j.

Next, consider the geometric generators γ_1, γ_2 given by Lemma 5.6: the element $\gamma_1\gamma_2^{-1}$ belongs to $\operatorname{Ker}\theta_k^+$, and $\psi_k(\gamma_1\gamma_2^{-1}) = ((-1,-1),(1,1),(0,0),\dots,(0,0))$. Therefore $\psi_k(g_{12}\gamma_1\gamma_2^{-1}) = ((-1,0),(1,0),(0,0),\dots,(0,0))$. So, using the surjectivity of θ_k and Lemma 5.2, we can find elements g'_{ij} of $\operatorname{Ker}\theta_k^+$ such that $\psi_k(g'_{ij})$ has only two non-zero entries, $(1,0)$ at position i and $(-1,0)$ at position j. We now conclude that $\psi_k(\operatorname{Ker}\theta_k^+) = \Gamma$ by observing that the $2n-2$ elements $\psi_k(g_{in})$ and $\psi_k(g'_{in})$, $1 \le i \le n-1$, generate Γ. □

We finish this section by mentioning two conjectures related to Conjecture 1.6. First of all, we mention that Conjecture 1.6 implies a result about the fundamental groups of Galois covers associated to branched covers of $\mathbb{CP}^2$. More precisely, given a complex surface X and a generic projection $X \to \mathbb{CP}^2$ of degree n with branch curve D_k, the associated Galois cover $\tilde{X}_k$ is obtained by compactification of the n-fold fibered product of X with itself above $\mathbb{CP}^2$: the complex surface $\tilde{X}_k$ is a degree $n!$ cover of $\mathbb{CP}^2$ branched along D_k. Moishezon and Teicher have constructed many interesting examples of complex surfaces by this method, and computed their fundamental groups (see e.g. [13], [16], [11]). Given an ordered system of geometric generators $\gamma_1, \dots, \gamma_d$ of $\pi_1(\mathbb{C}^2 - D_k)$, the fundamental group $\pi_1(\tilde{X}_k)$ is known to be isomorphic to the quotient of $\operatorname{Ker}(\theta : \pi_1(\mathbb{C}^2 - D_k) \to S_n)$ by the subgroup generated by $\gamma_1^2, \dots, \gamma_d^2$, and $\prod \gamma_i$ (see e.g. [16], §4).

By Lemma 5.5, the elements γ_i^2 and their conjugates map under ψ_k to elements of $(\mathbb{Z}^2/\Lambda_k)^n$ with only two non-trivial entries $(0,1)$; therefore, assuming Conjecture 1.6, quotienting by all squares of geometric generators leads to quotienting the image of ψ_k by $\{(0,\rho_i),\ \sum \rho_i \text{ is even}\} \subset (\mathbb{Z}^2/\Lambda_k)^n$. Because of Lemma 5.4, and observing that δ_k takes only even values on $\operatorname{Ker}\theta_k$, we are left with only the first factor in each summand $\mathbb{Z}^2/\Lambda_k$. Moreover, one easily checks that $\psi_k(\prod \gamma_i) = ((1,0),(1,0),\dots,(1,0)) \equiv ((1,0),\dots,(1,0),(1-n,d)) \mod \Lambda_k$; and by Lemma 5.4, the sum of the first factors is always zero, so we end up with a group isomorphic to $(\mathbb{Z}_{ks})^{n-2}$, where ks is the divisibility of L_k in $H_2(X,\mathbb{Z})$. Moreover, if we also assume that property $(*)$ holds in addition to Conjecture 1.6, it can easily be checked that the commutator subgroup $[G_k^0, G_k^0]$ is contained in the subgroup generated by the γ_i^2. Therefore, we have the following conjecture, satisfied by the examples in §4:

Conjecture 5.7. *If X is a simply connected complex surface and k is large enough, then the fundamental group of the Galois cover $\tilde{X}_k$ associated to a generic projection $f_k : X \to \mathbb{CP}^2$ defined by sections of $L^{\otimes k}$ is $\pi_1(\tilde{X}_k) = (\mathbb{Z}_{ks})^{n_k-2}$, where ks is the divisibility of L_k in $H_2(X,\mathbb{Z})$ and $n_k = \deg f_k$.*

Also, a careful observation of the examples in §4 suggests the following possible structure for the commutator subgroup $[G_k^0, G_k^0]$, which is worth mentioning in spite of the rather low amount of supporting evidence:

Conjecture 5.8. *If the symplectic manifold X is simply connected and k is large enough, then the commutator subgroup $[G_k^0, G_k^0]$ is isomorphic to $\Gamma_1 \times \Gamma_2$, where $\Gamma_1 = \mathbb{Z}_2$ if X is spin and 1 otherwise, and $\Gamma_2 = \mathbb{Z}_2$ if $L_k \equiv K_X \mod 2$ and 1 otherwise.*

6. Moishezon-Teicher techniques for ruled surfaces

6.1. **Overview of Moishezon-Teicher techniques.** Moishezon and Teicher have developed a general strategy, consisting of two main steps [8, 9, 13], in order to compute the group $\pi_1(\mathbb{C}^2 - D)$ when D is the branch curve of a generic projection to $\mathbb{CP}^2$ of a given projective surface $X \subset \mathbb{CP}^N$. First, one computes the braid factorization (see §2) associated to the curve D. This calculation involves a degeneration of the surface X to a singular configuration X_0 consisting of a union of planes intersecting along lines in $\mathbb{CP}^N$, and a careful analysis of the "regeneration" process which produces the generic branch curve D out of the singular configuration [8]. As explained in §2, the braid factorization explicitly provides, via the Zariski-Van Kampen theorem, a (rather complicated) presentation of the group $\pi_1(\mathbb{C}^2 - D)$. In a second step, one attempts to obtain a simpler description by reorganizing the relations in a more orderly fashion and by constructing morphisms between subgroups of $\pi_1(\mathbb{C}^2 - D)$ and groups related to $\tilde{B}_n$. This process is carried out in [9] for the case $X \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, and in subsequent papers for other examples.

6.1.1. *Degenerations and braid monodromy calculations.* The starting point of the calculation is a degeneration of the projective surface $X \subset \mathbb{CP}^N$ to an arrangement X_0 of planes in $\mathbb{CP}^N$ intersecting along lines. The degeneration process in the case of manifolds like $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ is described in detail in [8]. For example, in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ embedded by the linear system $O(p, q)$, one first degenerates the surface X of degree $2pq$ to a sum of q copies of $\mathbb{CP}^1 \times \mathbb{CP}^1$ embedded by $O(p, 1)$ (each of degree $2p$) inside $\mathbb{CP}^N$; then each of these surfaces is degenerated into p quadric surfaces ($\mathbb{CP}^1 \times \mathbb{CP}^1$ embedded by $O(1, 1)$); finally, each of the pq quadric surfaces is degenerated into a union of two planes intersecting along a line. The resulting arrangement can be represented by the diagram in Figure 1.

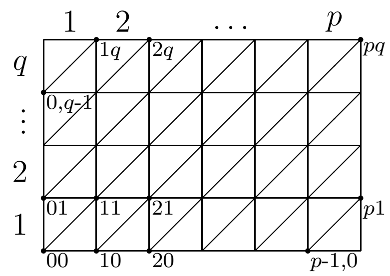


FIGURE 1

Each triangle in the diagram represents a plane. Each edge separating two triangles represents an intersection line L_i between the corresponding planes; note that the outer edges of the diagram are not part of the configuration. The branch curve for the projection $X_0 \to \mathbb{CP}^2$ is an arrangement of lines in $\mathbb{CP}^2$ (the projections of the various intersection lines L_i); however, in the regeneration process each of these lines acquires multiplicity 2, and the vertices where two or more lines intersect in X_0 turn into certain standard local configurations.

Therefore the braid factorization for D can be computed by looking at the local contributions of the various vertices in the diagram. Since the regeneration process turns a local configuration into a branch curve of degree $2m$, where m is the number of edges meeting at the given vertex, the local contribution of a vertex is naturally described by a word in the braid group B_{2m}. Moreover, because projecting X_0 to $\mathbb{CP}^2$ creates extra intersection points between the projections of the lines L_i whenever they do not intersect in X_0 (i.e. when they do not correspond to edges with a common vertex in the diagram), the branch curve D contains a number of additional nodes besides the local vertex configurations.

The major difficulty is to arrange the various local configurations and the additional nodes into a single braid factorization describing the curve D: given a linear projection $\pi : \mathbb{CP}^2 - \{\mathrm{pt}\} \to \mathbb{CP}^1$, one needs to fix a base point in $\mathbb{CP}^1$ and to choose an ordered system of loops in $\mathbb{CP}^1 - \mathrm{crit}\, \pi_{|D}$ in order to obtain a braid factorization. This choice determines in particular how the local braid monodromy (in B_{2m}) for each vertex of the grid is embedded into the braid monodromy of D (in B_d, $d = \deg D$). A careless setup leads to local embeddings $B_{2m} \hookrightarrow B_d$ that may be extremely difficult to determine.

An important observation of Moishezon is that the construction has sufficient flexibility to allow the images in $\mathbb{CP}^2$ of the various lines and intersection points to be chosen freely. This makes it possible to use the following very convenient setup [8]. First choose an ordering of the vertices in the diagram describing X_0; for example, for $\mathbb{CP}^1 \times \mathbb{CP}^1$ Moishezon chooses an ordering first by row, then by column, starting from the lower-left corner of the diagram: 00, 10, 20, ..., 01, 11, ..., pq. This determines a lexicographic ordering of the edges of the diagram: observing that each line L_i passes through two vertices v_i and v_i' ($v_i < v_i'$), the ordering is given by $L_i < L_j$ iff either $v_i' < v_j'$, or $v_i' = v_j'$ and $v_i < v_j$. It is then possible to choose a configuration where the projections of the lines L_i are given by equations with real coefficients, with slopes increasing according to the chosen lexicographic ordering, so that the intersection of the arrangement of lines in $\mathbb{CP}^2$ with a real slice $\mathbb{R}^2$ looks as in Figure 2.

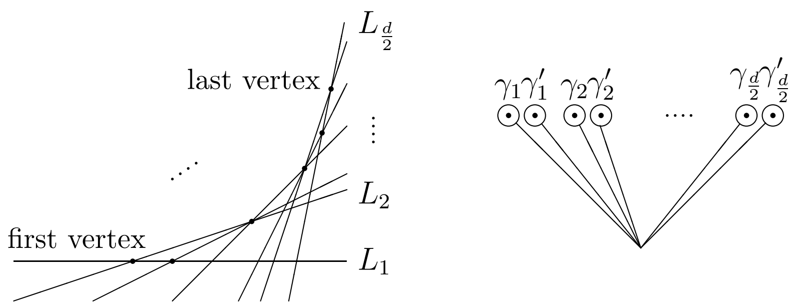


FIGURE 2

The choice of the slopes of the lines ensures that the intersection points of D with the reference fiber of π (chosen to be $\{x = A\}$ for some real number $A \gg 0$) are ordered in the natural way along the real axis, thus yielding a natural set of geometric generators $\{\gamma_i, \gamma_i'\}$ for $\pi_1(\mathbb{C}^2 - D)$, as shown on the right of Figure 2;

recall that each line L_i has multiplicity 2 and hence yields two generators, and note that the correct ordering of these generators counterclockwise around the base point is $\gamma'_{d/2}, \gamma_{d/2}, \dots, \gamma'_1, \gamma_1$. Moreover, the various vertices of the diagram describing X_0 appear, in sequence, for increasing values of x (from left to right).

Since all the contributions to the braid monodromy of D are now localized along the real x-axis, it is a fairly straightforward task to choose a set of generating loops in the base $\mathbb{CP}^1$ of the fibration π and enumerate accordingly the various contributions to the braid monodromy of D (standard configurations at the vertices of the diagram and extra nodes coming from pairs of edges without a common vertex). Going through the list of vertices in decreasing sequence ("from right to left") yields the simplest formula (Proposition 1 of [8]):

Proposition 6.1 (Moishezon)**.** *With the above setup, the braid monodromy of D is given by the factorization $\prod_{i=\nu}^{1}(C_i \cdot F_i)$, where ν is the number of vertices in the diagram, C_i is a product of contributions from nodal intersections between parts of D corresponding to non-adjacent edges, and F_i is the braid monodromy corresponding to the i-th vertex, obtained as the image of a standard local configuration under the embedding $B_{2m_i} \hookrightarrow B_d$ which maps the standard half-twists generating B_{2m_i} to half-twists along arcs that remain* below *the real axis.*

Proposition 6.1 makes it fairly simple to obtain a presentation of $\pi_1(\mathbb{C}^2 - D)$ in terms of the "global" generators $\{\gamma_i, \gamma'_i\}$: the nature of the local embeddings $B_{2m} \hookrightarrow B_d$ implies that the relations coming from each vertex are obtained from standard "local" relations (determined by the local braid monodromy) simply by renaming each of the $2m$ local geometric generators into the corresponding global generator. Additionally, the extra nodes yield various commutation relations among geometric generators.

The local configurations for the various types of vertices have been analyzed by Moishezon in [8], leading to explicit formulas for the local contributions to the braid factorization. The easiest case is that of "2-points" such as the corner points 00 and pq in the diagram for $\mathbb{CP}^1 \times \mathbb{CP}^1$. The only line that passes through the vertex locally regenerates to a conic in $\mathbb{C}^2$, presenting a single vertical tangency near the origin; hence the local braid monodromy is a single half-twist in B_2, giving rise to an equality relation between the two corresponding geometric generators of $\pi_1(\mathbb{C}^2 - D)$.

The next case is that of "3-points" such as those occurring on the boundary of the diagram for $\mathbb{CP}^1 \times \mathbb{CP}^1$. During the first step of "regeneration", which turns X_0 into a union of pq quadric surfaces, the lines corresponding to the diagonal edges are replaced by conics (the branch curve of a bidegree $(1,1)$ map from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^2$). For the vertices along the top and right sides of the diagram (labelled pj or iq), the partially regenerated configuration in $\mathbb{CP}^2$ therefore consists of a portion of conic tangent to a line, with the line having the greatest slope; after further regeneration, the line acquires multiplicity 2 and the tangent intersection is replaced by three cusps. The local contribution to braid monodromy can therefore be expressed by the product $\tilde{Z}^3_{1'2} \cdot Z^3_{1'2'} \cdot Z^3_{1'2} \cdot \hat{Z}_{11'}$, where the various factors are powers of half-twists along the paths represented in Figure 3 (cf. [8] and equation (2.4) in [9]). The first three factors correspond to cusps arising from the tangent intersection between the conic and the line, while the last factor corresponds to the vertical tangency of the conic.

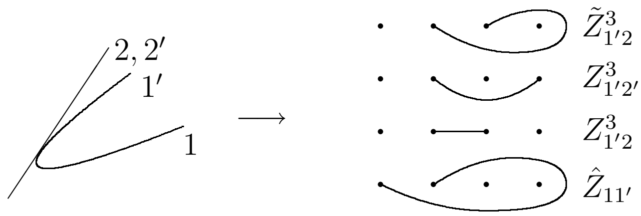


Figure 3

The 3-points on the bottom and left sides of the diagram give rise to a very similar local configuration, except for the ordering of the various components. Finally, the interior vertices of the diagram for $\mathbb{CP}^1 \times \mathbb{CP}^1$ are all of the same type ("6-points" in Moishezon's terminology); a careful analysis of their regeneration yields a certain braid factorization in B_{12}, accounting for the 6 vertical tangencies, 24 nodes and 24 cusps in the local model, as described in [8]. The local contributions to the relations defining $\pi_1(\mathbb{C}^2 - D)$ have also been calculated by Moishezon for these various standard models in §2 of [9] (see also below).

6.1.2. *Fundamental group calculations.* The setup described in §6.1.1 provides an explicit presentation of $\pi_1(\mathbb{C}^2 - D)$ in terms of geometric generators $\{\gamma_i, \gamma_i'\}$, $i = 1, \dots, \frac{d}{2}$. By Proposition 6.1, the relations consist on one hand of standard relations given by local models for the various vertices of the diagram describing the degenerated surface X_0, and on the other hand of commutation relations coming from non-adjacent edges of the diagram. The goal is then to simplify this presentation and ultimately identify $\pi_1(\mathbb{C}^2 - D)$ with a certain quotient of $\tilde{B}_n^{(2)}$ (or $\tilde{B}_n \ltimes \tilde{P}_{n,0}$). In the remainder of this section, we describe the recipes used by Moishezon for the case $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, following §3 of [9]; these methods also apply to other complex surfaces admitting similar degenerations, such as $X = \mathbb{CP}^2$ [10] or $X = \mathbb{F}_1$ (§6.2).

A first observation of Moishezon is that, after a slight change in the choice of generators, many of the local relations at the vertices can be expressed in terms of half of the generators only. More precisely, for each value of i, define a *twisting* action ρ_i on the two generators γ_i, γ_i' by the formula $\rho_i(\gamma_i) = \gamma_i'$ and $\rho_i(\gamma_i') = \gamma_i'\gamma_i\gamma_i'^{-1}$. Choose integers l_i satisfying the following compatibility conditions: if $i < j$ are the labels of the two diagonal edges meeting at a 6-point vertex of the diagram, then $l_j = l_i - 1$; if $i < j$ are the labels of the two vertical edges meeting at a 6-point, then $l_j = l_i + 1$; finally, if $i < j$ are the labels of the two horizontal edges meeting at a 6-point, then $l_j = l_i$. Now let $e_i = \rho_i^{l_i}(\gamma_i)$ and $e_i' = \rho_i^{l_i}(\gamma_i')$. Because of the *invariance properties* of the local models [8], the local relations corresponding to 2-points and 3-points have the same expressions in terms of $\{e_i, e_i'\}$ as in terms of $\{\gamma_i, \gamma_i'\}$, independently of the amount of twisting, and those for 6-points are also independent of the l_i as long as the compatibility relations hold. On the other hand, if $i_1 < \dots < i_6$ are the labels of the edges meeting at a 6-point (i_1 and i_6 are the two diagonal edges), then it is possible to eliminate either e_{i_1} or e_{i_6} from the list of generators, because the local relations imply that

$$e_{i_6} = (e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1})^{-1}e_{i_1}(e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1}). \tag{6.1}$$

The second important observation of Moishezon is that, in many cases (assuming the diagram is "large enough", i.e. in the case of a bidegree (p,q) linear system on

$\mathbb{CP}^1 \times \mathbb{CP}^1$ that $p, q \geq 2$), the relations coming from cusps and nodes of D can all be reformulated into a very nice pattern (cf. Lemma 14 of [9]). If the two edges i and j bound a common triangle in the diagram, then the local relations at their common vertex imply that

$$e_ie_je_i = e_je_ie_j,\ e_ie'_je_i = e'_je_ie'_j,\ e'_ie_je'_i = e_je'_ie_j,\ \text{and}\ e'_ie'_je'_i = e'_je'_ie'_j. \tag{6.2}$$

Otherwise, if there is no triangle having i and j as edges, or equivalently if the two transpositions $\theta(e_i) = \theta(e'_i)$ and $\theta(e_j) = \theta(e'_j) \in S_n$ are disjoint, then we have

$$[e_i, e_j] = [e_i, e'_j] = [e'_i, e_j] = [e'_i, e'_j] = 1. \tag{6.3}$$

Looking at $e_1, \dots, e_{\frac{d}{2}}$, among which there are only $n-1$ independent generators (by (6.1), many of the e_i corresponding to diagonal edges can be expressed in terms of the others), a first consequence of the relations (6.2–6.3) is the following (Proposition 8 of [9]):

Lemma 6.2 (Moishezon)**.** *In the case of the linear system $O(p,q)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ $(p, q \geq 2)$, the subgroup $\mathcal{B}$ of $\pi_1(\mathbb{C}^2 - D)$ generated by $e_1, \dots, e_{d/2}$ is isomorphic to a quotient of $\tilde{B}_n$ $(n = 2pq)$. More precisely, there exists a surjective morphism $\tilde{\alpha} : \tilde{B}_n \to \mathcal{B}$ with the property that each e_i is the image of a half-twist in $\tilde{B}_n$, and $\theta \circ \tilde{\alpha} = \sigma$ (i.e. the end points of the half-twists agree with the transpositions $\theta(e_i)$).*

We now need to add to this description the other generators e'_i, or equivalently the elements $a_i = e'_ie_i^{-1}$. In the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, we relabel these elements as d_{ij} for the diagonal edge in position ij $(1 \leq i \leq p,\ 1 \leq j \leq q$, see Figure 1), v_{ij} for the vertical edge in position ij $(1 \leq i < p,\ 1 \leq j \leq q)$, and h_{ij} for the horizontal edge in position ij $(1 \leq i \leq p,\ 1 \leq j < q)$. We are especially interested in $a_2 = v_{11}$. Moishezon's next observation is that, as a consequence of relations (6.2–6.3) and of the local relations of the lower-left-most 6-point in the diagram, the subgroup generated by v_{11} and the conjugates $g^{-1}v_{11}g$, $g \in \mathcal{B}$, is naturally isomorphic to a quotient of $\tilde{P}_{n,0}$ ([9], Definition 5 and Lemma 17). Moreover, the subgroup of $\pi_1(\mathbb{C}^2 - D)$ generated by the e_i and by v_{11} is similarly isomorphic to a quotient of the semi-direct product $\tilde{B}_n \ltimes \tilde{P}_{n,0}$, or equivalently (as seen in §4) $\tilde{B}_n^{(2)}$.

The most important relations in $\pi_1(\mathbb{C}^2 - D)$ are those coming from the vertical tangencies of D, which we now list for the various types of vertices. If the edge labelled i passes through a 2-point, then the local relation $e_i = e'_i$ can be rewritten in the form $a_i = 1$. If $i < j$ are the labels of the two edges meeting at a 3-point, then we have $e'_i = e_j^{-1}{e'_j}^{-1}e_ie'_je_j$, or equivalently $e'_j = e_i^{-1}{e'_i}^{-1}e_je'_ie_i$. Using (6.2) this relation can be rewritten as

$$a_j = e_i^{-1}e_je'_ie_j^{-1}e_ie_j^{-1} = e_i^{-2}(e_ie_j)a_i(e_j^{-1}e_i^{-1})e_je_i^2e_j^{-1}. \tag{6.4}$$

Finally, if $i_1 < \dots < i_6$ are the labels of the edges meeting at a 6-point (according to the ordering rules, i_1 and i_6 are diagonal, i_2 and i_5 are vertical, and i_3 and i_4 are horizontal), then, besides (6.1), we also have

$$\begin{cases} a_{i_6} = (e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1})^{-1}a_{i_1}(e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1}) \\ a_{i_5} = (e_{i_1}^{-1}e_{i_3}e_{i_4}^{-1}e_{i_6})^{-1}a_{i_2}(e_{i_1}^{-1}e_{i_3}e_{i_4}^{-1}e_{i_6}) \\ a_{i_4} = (e_{i_1}^{-1}e_{i_2}e_{i_5}^{-1}e_{i_6})^{-1}a_{i_3}(e_{i_1}^{-1}e_{i_2}e_{i_5}^{-1}e_{i_6}) \end{cases} \tag{6.5}$$

$$(6.6)\qquad \begin{cases} a_{i_3} = (e_{i_3}e_{i_1})^{-1}\, a_{i_2}a_{i_1}(e_{i_1}a_{i_2}^{-1}e_{i_1}^{-1})\,(e_{i_3}e_{i_1}) \\ a_{i_2} = (e_{i_2}e_{i_1})^{-1}\, a_{i_3}a_{i_1}(e_{i_1}a_{i_3}^{-1}e_{i_1}^{-1})\,(e_{i_2}e_{i_1}) \end{cases}$$

A first consequence of relations (6.4–6.6) is that, going inductively through the various vertices of the grid, all a_i can be expressed in terms of the $e_1,\dots,e_{d/2}$ and of $a_2 = v_{11}$. Therefore $\pi_1(\mathbb{C}^2 - D)$ is generated by the e_i and by v_{11}; hence it is isomorphic to a quotient of $\tilde{B}_n^{(2)}$. In other words, we have a surjective homomorphism $\alpha : \tilde{B}_n^{(2)} \to \pi_1(\mathbb{C}^2 - D)$, extending the morphism $\tilde{\alpha} : \tilde{B}_n \to \mathcal{B}$ of Lemma 6.2.

From this point on, the results in §3 make it possible to present Moishezon's argument in a simpler and more illuminating way. Observe that by Lemma 6.2 each e_i is the image by α of a half-twist in the diagonally embedded subgroup $\tilde{B}_n \subset \tilde{B}_n^{(2)}$. Moreover, it is a general fact about irreducible plane curves that all geometric generators are conjugate to each other in $\pi_1(\mathbb{C}^2 - D)$; therefore each of the geometric generators e_i, e'_i is the image of a pair of half-twists in $\tilde{B}_n^{(2)}$. Alternately this can be seen directly from the above-listed relations; these relations also imply that each a_i belongs to the normal subgroup of pure degree 0 elements $\alpha(\tilde{P}_{n,0} \times \tilde{P}_{n,0})$, and therefore that the half-twists corresponding to the geometric generators e'_i have the correct end points as prescribed by the S_n-valued monodromy representation morphism θ. Therefore $\pi_1(\mathbb{C}^2 - D)$ has the property (*) defined in §3.

In view of Lemmas 3.3 and 3.4, at this point in the argument we can discard all the relations in $\pi_1(\mathbb{C}^2 - D)$ coming from nodes and cusps of D since they automatically hold in quotients of $\tilde{B}_n^{(2)}$, and focus on the relations (6.4–6.6) instead.

By Lemma 3.2, pairs of half-twists in $\tilde{B}_n^{(2)}$ with fixed end points can be classified by two integers. More precisely, fix an ordering of the n sheets of the branched cover f, e.g. from left to right and from bottom to top in the diagram. This provides an ordering of the end points of the half-twists corresponding to e_i and e'_i; we can find an element $g \in \tilde{B}_n^{(2)}$ such that $e_i = \alpha(g^{-1}(x_1, x_1)g)$, with ordering of the end points preserved. Then by Lemma 3.2 there exist integers k and l such that $e'_i = \alpha(g^{-1}(x_1u_1^{-k}\eta^{-k(-k-1)/2}, x_1u_1^{-l}\eta^{-l(-l-1)/2})g)$, i.e. $a_i = \alpha(g^{-1}(u_1^k\eta^{k(k-1)/2}, u_1^l\eta^{l(l-1)/2})g)$. One easily checks by Lemma 3.1 that reversing the ordering of the end points changes k into $-k$ and l into $-l$.

Since α is a priori not injective, the integers k and l are not necessarily unique, and there may exist another pair of integers $(k', l') = (k+\kappa, l+\lambda)$ with the same property, i.e. such that $\mu = (u_1^\kappa\eta^{k'(k'-1)/2-k(k-1)/2}, u_1^\lambda\eta^{l'(l'-1)/2-l(l-1)/2}) \in \operatorname{Ker}\alpha$. If κ is odd, then the normal subgroup generated by μ contains the commutator of μ with $(u_2, 1)$, which is equal to $(\eta, 1)$; so $(\eta, 1) \in \operatorname{Ker}\alpha$. If κ is even, then $\eta^{k'(k'-1)/2-k(k-1)/2} = \eta^{\kappa/2} = \eta^{\kappa(\kappa-1)/2}$ (recall that $\eta^2 = 1$). Similarly, if λ is odd then $(1, \eta) \in \operatorname{Ker}\alpha$, otherwise $\eta^{l'(l'-1)/2-l(l-1)/2} = \eta^{\lambda(\lambda-1)/2}$. In both cases we arrive to the conclusion that $\tilde{\mu} = (u_1^\kappa\eta^{\kappa(\kappa-1)/2}, u_1^\lambda\eta^{\lambda(\lambda-1)/2}) \in \operatorname{Ker}\alpha$. In fact, μ and $\tilde{\mu}$ generate the same normal subgroups, so we also have the converse implication.

Therefore the set of all possible values for (κ, λ) forms a subgroup $\Lambda \subset \mathbb{Z}^2$; in fact $\Lambda = \{(\kappa, \lambda),\ (u_1^\kappa\eta^{\kappa(\kappa-1)/2}, u_1^\lambda\eta^{\lambda(\lambda-1)/2}) \in \operatorname{Ker}\alpha\}$, and the pair of integers (k, l) is only defined mod Λ. So, to e_i and e'_i we can associate an element $\bar{a}_i = (k, l) \in \mathbb{Z}^2/\Lambda$. This element $\bar{a}_i$ contains all the relevant information about e_i and e'_i apart from the end points. Indeed, because of Lemma 3.5, up to composition of α with an automorphism of $\tilde{B}_n^{(2)}$ we can assume e_i to be the image by α of any given pair of half-twists with the correct end points. And, by Lemma 3.2, if two half-twists $x, y \in \tilde{B}_n$ have the same end points, then $x^2y^{-2} \in \{1, \eta\}$, so up to a factor of η the product $e'_i e_i = a_i e_i^2$ is determined by $\bar{a}_i$; that ambiguity can in fact be lifted by arguing that e_i and e'_i are images of half-twists.

The subgroup Λ can be determined by looking at the relations in $\pi_1(\mathbb{C}^2 - D)$ coming from vertical tangencies of D, which determine the kernel of α. We now reformulate these relations in terms of the $\bar{a}_i$. First, at a 2-point, the relation $a_i = 1$ becomes $\bar{a}_i = (0, 0)$. What happens at a 3-point depends on the ordering of the sheets of f (i.e., of the triangles of the diagram): the relation (6.4) becomes

$$\pm\bar{a}_i + \pm\bar{a}_j = (1, 1), \tag{6.7}$$

where the first sign is $+$ if the triangle T which has both i and j among its edges comes *after* the other triangle bounded by the edge i and $-$ otherwise, and the second sign is $+$ if T comes *after* the other triangle bounded by the edge j and $-$ otherwise. In the case of a 6-point with the standard ordering used by Moishezon, (6.5) and (6.6) become

$$\bar{a}_{i_6} = \bar{a}_{i_1}, \qquad \bar{a}_{i_5} = \bar{a}_{i_2}, \qquad \bar{a}_{i_4} = \bar{a}_{i_3}, \qquad \bar{a}_{i_1} - \bar{a}_{i_2} + \bar{a}_{i_3} = 0. \tag{6.8}$$

In the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, denoting by $\bar{d}_{ij}$, $\bar{v}_{ij}$ and $\bar{h}_{ij}$ the elements of $\mathbb{Z}^2/\Lambda$ corresponding to d_{ij}, v_{ij} and h_{ij}, the relations become (listing the vertices from left to right and bottom to top): $\bar{d}_{1,1} = (0, 0)$, $\bar{v}_{i,1} - \bar{d}_{i+1,1} = (1, 1)$, $\bar{h}_{1,j} + \bar{d}_{1,j+1} = (1, 1)$; $\bar{d}_{i+1,j+1} = \bar{d}_{i,j}$, $\bar{v}_{i,j+1} = \bar{v}_{i,j}$, $\bar{h}_{i+1,j} = \bar{h}_{i,j}$, $\bar{d}_{i,j} - \bar{v}_{i,j} + \bar{h}_{i,j} = 0$; $-\bar{d}_{p,j} - \bar{h}_{p,j} = (1, 1)$, $\bar{d}_{i,q} - \bar{v}_{i,q} = (1, 1)$, $\bar{d}_{p,q} = (0, 0)$. Moreover, by construction $\bar{v}_{11} = (0, 1)$ (because v_{11} was identified to a generator of $\tilde{P}_{n,0}$).

Working inductively from the lower-left corner of the diagram, these equations yield the formulas

$$\bar{d}_{i,j} = (j - i, 0), \qquad \bar{v}_{i,j} = (1 - i, 1), \qquad \bar{h}_{i,j} = (1 - j, 1) \tag{6.9}$$

(compare with Proposition 10 of [9], recalling that the identification between $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ and $\tilde{B}_n^{(2)}$ is given by $(x, u) \mapsto (x, xu)$). Moreover, we are left with the relations $(p - 1, -1) = (1, 1)$ and $(q - 1, -1) = (1, 1)$. In other words, Λ is the subgroup of $\mathbb{Z}^2$ generated by $(2 - p, 2)$ and $(2 - q, 2)$.

Because all relations in $\pi_1(\mathbb{C}^2 - D)$ coming from vertical tangencies correspond to equality relations between pairs of half-twists in $\tilde{B}_n^{(2)}$, by the above remarks $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by a certain number of elements of the form $(u_1^\kappa \eta^{\kappa(\kappa-1)/2}, u_1^\lambda \eta^{\lambda(\lambda-1)/2})$, and therefore it is completely determined by the subgroup $\Lambda \subset \mathbb{Z}^2$. In our case, $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by $(u_1^{2-p}\eta^{(2-p)(1-p)/2}, u_1^2\eta)$ and $(u_1^{2-q}\eta^{(2-q)(1-q)/2}, u_1^2\eta)$. We can now finish the proof of Theorem 4.1, observing that $H^0_{p,q} = (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\operatorname{Ker}\alpha$. Recalling from Lemma 3.1 that $\tilde{P}_{n,0}$ has commutator subgroup $\{1, \eta\} \simeq \mathbb{Z}_2$ and that $\operatorname{Ab}\tilde{P}_{n,0} \simeq \mathbb{Z}^{n-1}$, we have two cases to consider. First, if e.g. p is odd, then by considering the commutator of $(u_1^{2-p}\eta^{(2-p)(1-p)/2}, u_1^2\eta)$ with $(u_2, 1)$ we obtain that $(\eta, 1) \in \operatorname{Ker}\alpha$ (and similarly if q is odd); but one easily checks that $(1, \eta) \notin \operatorname{Ker}\alpha$. On the other hand, if p and q are both even, then no non-trivial element of $C = \{1, \eta\} \times \{1, \eta\}$ belongs to $\operatorname{Ker}\alpha$. Therefore, $[H^0_{p,q}, H^0_{p,q}] \simeq C/(C \cap \operatorname{Ker}\alpha)$ is isomorphic to $\mathbb{Z}_2$ if p or q is odd, and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if p and q are even. Moreover, we have $\operatorname{Ab} H^0_{p,q} \simeq (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\langle C, \operatorname{Ker}\alpha\rangle \simeq$

$(\mathbb{Z}^2/\Lambda)^{n-1}$, which one easily shows to be isomorphic to $(\mathbb{Z}_2 \oplus \mathbb{Z}_{p-q})^{n-1}$ or $(\mathbb{Z}_{2(p-q)})^{n-1}$ depending on the parity of p and q. This completes the proof of Theorem 4.1. The computations for $\mathbb{CP}^2$ (Theorem 4.2) and other algebraic surfaces admitting similar degenerations can be carried out by the same method; for example, the case of the Hirzebruch surface $\mathbb{F}_1$ is treated in §6.2 below.

6.2. **The Hirzebruch surface $\mathbb{F}_1$.** In this section, we prove Theorem 4.5 using the method outlined in the preceding section. Consider the projective embedding of $\mathbb{F}_1$ defined by sections of the linear system $O(pF+qE)$, $p > q \geq 2$ (recall F is the fiber and E is the exceptional section). This projective surface can be degenerated in the same manner as the Veronese surface of which it is a blow-up (the projective embedding of $\mathbb{CP}^2$ defined by sections of $O(p)$), following the procedure described in §3 of [8]. This surface of degree $n = (2p-q)q$ can be first degenerated into a sum of q Hirzebruch surfaces, of degrees respectively $2p-1$, $2p-3$, ... , $2(p-q)+1$. Each of these Hirzebruch surfaces can then be degenerated into the union of a plane and a certain number of quadric surfaces, which in turn can each be degenerated to two planes. The resulting diagram is pictured in the right half of Figure 4.

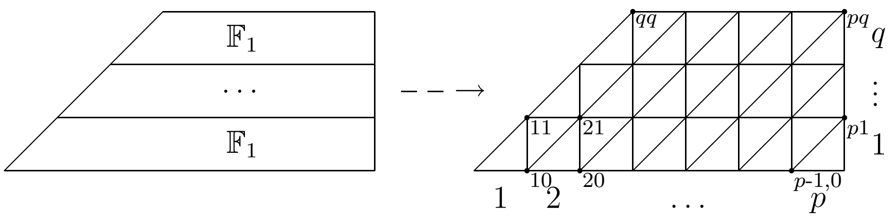


FIGURE 4

One uses the same setup as in §6.1.1, ordering the vertices from left to right and bottom to top, and the edges accordingly. The braid monodromy is given by Proposition 6.1. It follows from Moishezon's work that all vertices correspond to well-known configurations: the two vertices qq and pq are 2-points, while the other boundary vertices are 3-points and the interior vertices are 6-points.

As in §6.1.2, one replaces the natural set of geometric generators $\{\gamma_i, \gamma_i'\}$ by twisted generators $e_i = \rho_i^{l_i}(\gamma_i)$ and $e_i' = \rho_i^{l_i}(\gamma_i')$, where the integers l_i satisfy the required compatibility conditions, in order to have (6.1) at all 6-points. Moreover, relations (6.2) and (6.3) hold for all pairs of edges ((6.2) if the edges bound a common triangle, (6.3) otherwise), by the same argument as for $\mathbb{CP}^2$: the proof of Lemma 1 of [10] (see also Lemma 14 of [9]) applies almost without modification.

Eliminating redundant diagonal edges as allowed by (6.1), we are left with exactly $n-1$ independent generators among the e_i. As in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, relations (6.2) and (6.3) imply that the subgroup $\mathcal{B}$ generated by the e_i is isomorphic to a quotient of $\tilde{B}_n$, and Lemma 6.2 extends to the case of the Hirzebruch surface $\mathbb{F}_1$.

As previously, we let $a_i = e_i' e_i^{-1}$, and we relabel these elements as d_{ij}, v_{ij} and h_{ij}. We are now interested in $a_1 = v_{11}$: one can again show that the subgroup generated by v_{11} and the conjugates $g^{-1} v_{11} g$, $g \in \mathcal{B}$ is isomorphic to a quotient of $\tilde{P}_{n,0}$, by Lemma 5 of [10] (the argument is the same for $\mathbb{F}_1$ as for $\mathbb{CP}^2$); the subgroup of $\pi_1(\mathbb{C}^2 - D)$ generated by the e_i and by a_1 is again isomorphic to a quotient of $\tilde{B}_n \ltimes \tilde{P}_{n,0} \simeq \tilde{B}_n^{(2)}$.

Relations (6.4–6.6) imply that, going through the various 3-points and 6-points of the diagram, all the a_i can be expressed in terms of $e_1, \dots, e_{d/2}$ and $a_1 = v_{11}$; therefore $\pi_1(\mathbb{C}^2 - D)$ is generated by $e_1, \dots, e_{d/2}$ and a_1, so that we again obtain a surjective morphism $\alpha : \tilde{B}_n^{(2)} \to \pi_1(\mathbb{C}^2 - D)$. As in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, the various geometric generators are images by α of pairs of half-twists with correct end points, so that property (*) holds once more. Using the classification of half-twists in $\tilde{B}_n$ (Lemma 3.2), we can consider pairs of integers $\bar{a}_i$ instead of the elements a_i; once again, the $\bar{a}_i$ are only defined modulo a certain subgroup $\Lambda \subset \mathbb{Z}^2$.

The various relations between the $\bar{a}_i$ are now the following: $\bar{v}_{i,1} - \bar{d}_{i+1,1} = (1,1)$, $\bar{v}_{i,i} - \bar{h}_{i+1,i} = (1,1)$; $\bar{d}_{i+1,j+1} = \bar{d}_{i,j}$, $\bar{v}_{i,j+1} = \bar{v}_{i,j}$, $\bar{h}_{i+1,j} = \bar{h}_{i,j}$, $\bar{d}_{i,j} - \bar{v}_{i,j} + \bar{h}_{i,j} = 0$; $-\bar{d}_{p,j} - \bar{h}_{p,j} = (1,1)$, $\bar{v}_{q,q} = (0,0)$, $\bar{d}_{i,q} - \bar{v}_{i,q} = (1,1)$, $\bar{d}_{p,q} = (0,0)$. Moreover, $\bar{v}_{1,1} = (0,1)$. Therefore, $\bar{d}_{i,j} = (2j-2i+1, j-i+1)$, $\bar{v}_{i,j} = (2-2i, 2-i)$ and $\bar{h}_{i,j} = (1-2j, 1-j)$ (compare with Proposition 4 of [10]), and we are left with two additional relations: $(2p-2, p-2) = (1,1)$ and $(2-2q, 2-q) = (0,0)$. Therefore, Λ is the subgroup of $\mathbb{Z}^2$ generated by $(2p-3, p-3)$ and $(2q-2, q-2)$, and $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by $(u_1^{2p-3}\eta^{(2p-3)(2p-4)/2}, u_1^{p-3}\eta^{(p-3)(p-4)/2})$ and $(u_1^{2q-2}\eta^{(2q-2)(2q-3)/2}, u_1^{q-2}\eta^{(q-1)(q-2)/2})$.

Considering the commutator of the first generator with $(u_2, 1)$, we obtain that $(\eta, 1) \in \operatorname{Ker}\alpha$. Moreover, if either p is even or q is odd, then considering the commutator of one of the generators with $(1, u_2)$, we obtain that $(1, \eta) \in \operatorname{Ker}\alpha$. On the contrary, if p is odd and q is even then $(1,\eta) \notin \operatorname{Ker}\alpha$. We conclude that $[H^0_{p,q}, H^0_{p,q}] \simeq C/(C \cap \operatorname{Ker}\alpha)$ is trivial or isomorphic to $\mathbb{Z}_2$ depending on the parity of p and q, and that $\operatorname{Ab} H^0_{p,q} \simeq (\mathbb{Z}^2/\Lambda)^{n-1} \simeq (\mathbb{Z}^2/\langle (p,3), (q,2)\rangle)^{n-1} \simeq (\mathbb{Z}_{3q-2p})^{n-1}$.

7. Double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$

In this section, we sketch the proof of Theorem 4.6, which combines the methods described in §6 with ideas similar to those in [3].

7.1. **Generic perturbations of iterated branched covers.** Let C be a smooth algebraic curve of degree $(2a, 2b)$ in $Y = \mathbb{CP}^1 \times \mathbb{CP}^1$, and let $X_{a,b}$ be the double cover of Y branched along C. Then one can construct a map $f^0 : X_{a,b} \to \mathbb{CP}^2$ simply by composing the double cover $\pi : X_{a,b} \to Y$ with a generic projective map $g : Y \to \mathbb{CP}^2$ determined by sections of $O(p,q)$. The map f^0 is not generic : its ramification curve is the union of the ramification curve of π and the preimage by π of the ramification curve of g, and so the branch curve D^0 of f^0 is the union of $g(C)$ (with multiplicity 1) and the branch curve D_g of g (with multiplicity 2).

This situation is extremely similar to that considered in [3] for the composition of a generic map from a symplectic 4-manifold to $\mathbb{CP}^2$ with a quadratic map from $\mathbb{CP}^2$ to itself. The local behavior of the map f^0 is generic everywhere except at the intersection points of C with the ramification curve of g ; assuming that C and g are chosen generically, a local model for f^0 near these points is $(x,y) \mapsto (-x^2+y, -y^2)$, for which a generic local perturbation is given e.g. by $(x,y) \mapsto (-x^2+y, -y^2+\epsilon x)$ where ϵ is a small non-zero constant (cf. also [3]). There are several ways in which the map f^0 can be perturbed and made generic. If the linear system $\pi^*O(p,q)$ is sufficiently ample, then f^0 can be deformed within the holomorphic category into a generic projective map which no longer factors through the double cover π. Another possibility, if p and q are sufficiently large, is to use approximately holomorphic methods (Theorem 1.1) to deform f^0 into a map with generic local models (cf. [3]).

In both cases, the effect of the perturbation on the topology of the branch curve of f^0 is pretty much the same. First, the local model near an intersection point of C with the ramification curve of g is perturbed as described above (up to isotopy), which transforms a tangent intersection of $g(C)$ with the branch curve of g in $\mathbb{CP}^2$ into a standard configuration with three cusps [3]. Secondly, the two copies of the branch curve of g, which make up the multiplicity two component of D^0, are separated and made transverse to each other; this deformation of D_g is performed either within the holomorphic category or resorting to approximately holomorphic perturbations. In the second case, the perturbation process can be performed in a very flexible manner, which in some cases may create negative intersections ; restricting oneself to algebraic perturbations is a convenient way to avoid this phenomenon, but makes the global perturbation harder to describe explicitly. In any case, up to isotopy and creation or cancellation of pairs of intersections between the two deformed copies of the branch curve of g, the topology of the resulting generic branch curve D is uniquely determined and can be computed easily from that of D^0. In fact, the approximately holomorphic perturbation process can always be carried out, even for small values of p and q for which neither the holomorphic construction nor Theorem 1.1 are able to yield generic projective maps ; in this situation, we can still study the topology of the curve D, but Theorem 4.6 only describes a "virtual" generic projective map.

As in §6, the study of the curve D relies on a degeneration process: one first degenerates the curve C in $Y = \mathbb{CP}^1 \times \mathbb{CP}^1$ into a union of two sets of parallel lines, $2a$ along one factor and $2b$ along the other factor. Parallel lines are then merged, so that the resulting configuration $C_0 \subset Y$ consists of only two components, a $(1,0)$-line of multiplicity $2a$ and a $(0,1)$-line of multiplicity $2b$. Finally, one degenerates the projective embedding of Y given by the linear system $O(p,q)$ into an arrangement Y_0 of planes intersecting along lines, as in §6.1. The fully degenerated branch curve is a union of lines, some of which correspond to the intersections between the planes in Y_0 (each contributing with multiplicity 4, since the branch curve of g is counted with multiplicity 2), while the others are the images of the $p+q$ components into which C_0 degenerates (some of these components contribute with multiplicity $2a$, others with multiplicity $2b$).

The curve D can be recovered from this arrangement of lines by the converse "regeneration" process, which first yields the union $D_g \cup g(C_0)$ (by deforming Y_0 into the smooth surface Y), then $D_g \cup g(C) = D^0$ (by separating the multiple components of C_0 and smoothing the resulting curve), and finally D (by performing the prescribed local perturbation at the intersection points of the two ramification curves and by perturbing the two copies of D_g in a generic way).

7.2. **Braid monodromy calculations.** The braid monodromy for the curve $D_g \cup g(C_0)$ (and for the subsequent regenerations D^0 and D) can be computed using the same methods as in §6.1.1. The diagram describing the degenerated configuration is as represented on Figure 5, which differs from Figure 1 only by the addition of edges corresponding to C_0 along the top and right boundaries of the diagram.

Thanks to Proposition 6.1, we only need to understand the local behavior of the curves $D_g \cup g(C_0)$, D^0 and D near the various vertices of the diagram. At all vertices

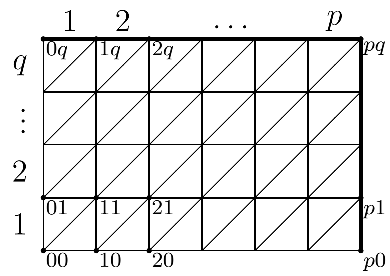


FIGURE 5

except those through which C_0 passes (top and right sides of the diagram), the local description of $D_g \cup g(C_0)$ and D^0 is exactly the same as that of D_g, which has already been discussed in §6.1 : the various vertices are standard 2-points, 3-points and 6-points as in Moishezon's work [9]. Moreover, the local configuration for D at such a vertex simply consists of two copies of the local configuration for D_g, shifted apart from each other by a generic translation. The two components, which correspond to the two preimages of the ramification curve of g under the branched cover π, may intersect at nodal points of either orientation ; we won't be overly concerned by the details of these intersections, since the various possible configurations only differ by isotopies and creations or cancellations of pairs of nodes, which do not affect the stabilized fundamental group in any way.

We now consider a vertex along the top boundary of the diagram, at position iq with $1 \leq i \leq p-1$. The local configuration for $D_g \cup g(C_0)$ at such a point is as shown on Figure 6. The parts labelled $1, 1', 2, 2'$ correspond to D_g, and form a standard 3-point (cf. §6.1.1 and Figure 3), presenting three cusp singularities near the point A. The parts labelled 3 and 4 correspond to $g(C_0)$, obtained by "regeneration" of the two lines associated to the horizontal edges of the diagram passing through the vertex. The curve $g(C_0)$ presents tangent intersections with the two lines 2 and $2'$ near the point B, and with the conic $1, 1'$ at the point C. The two intersections of the line labelled 4 with the conic $1, 1'$ in $\mathbb{CP}^2$ remain as nodes since the corresponding curves fail to intersect in Y.

The local description of the curve $D^0 = D_g \cup g(C)$ is obtained from that of $D_g \cup g(C_0)$ by separating C_0 into $2b$ parallel components ; this yields $2b$ copies of the lines labelled 3 and 4 in Figure 6, and the local configuration near the points B and C becomes as shown in the right half of Figure 6 (the pictures correspond to the case $b = 2$). Finally, in order to obtain D we must perturb D^0 in the manner explained in §7.1: the multiplicity two component $D_g \subset D^0$ (corresponding

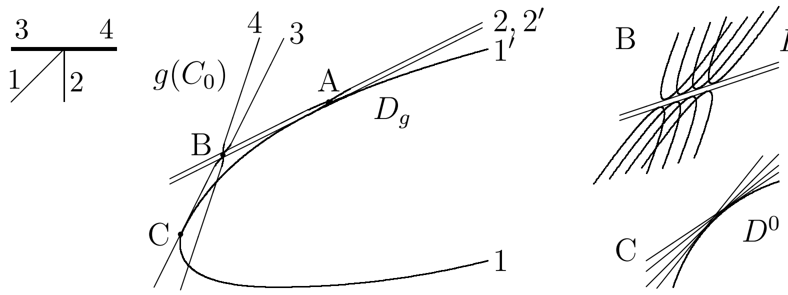


FIGURE 6

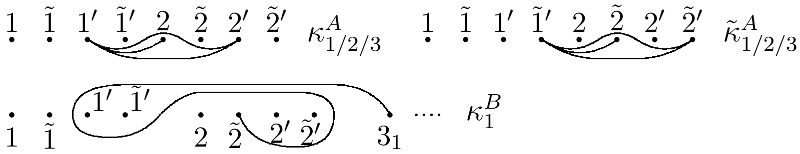


FIGURE 7

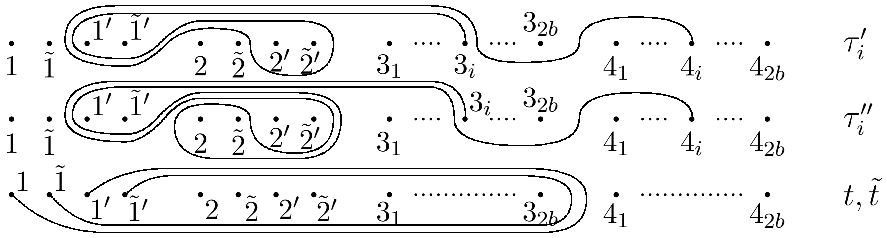


FIGURE 8

to the parts labelled $1, 1', 2, 2'$ in Figure 6) is separated into two distinct copies (in particular the point A is duplicated), while each tangent intersection of $g(C)$ with D_g (such as those near points B and C) gives rise to three cusps. It is then possible to write explicitly the local braid monodromy for D, with values in B_{4b+8} by enumerating carefully the $4b + 2$ vertical tangencies, $18b + 6$ cusps, and nodes of the local model (the exact number of nodes depends on the choice of boundary values for the local perturbation of D^0).

In fact, since we only aim to compute *stabilized* fundamental groups of branch curve complements, we shall not concern ourselves with the nodes of D, since these only yield commutation relations which by definition always hold in the stabilized group. Moreover, for reasons that will be apparent later in the argument, the cusp points are also of limited relevance for our purposes; those which will play a role in the argument, namely the six cusps near point A and one of the $12b$ cusps near point B of Figure 6, give rise to braid monodromies equal to the cubes of the half-twists represented in Figure 7. Actually, the truly important information is contained in the vertical tangencies, which correspond to the half-twists $\tau'_1, \dots, \tau'_{2b}, \tau''_1, \dots, \tau''_{2b}, t, \tilde t \in B_{4b+8}$ represented in Figure 8. As in §6.1, the reference fiber of π is $\{x = A\}$ for A a large positive real constant, and the chosen generating paths in the base (x-plane) remain under the real axis except near their end points; the labels $1, 1', 2, 2', \tilde 1, \tilde 1', \tilde 2, \tilde 2'$ and $3_1, \dots, 3_{2b}, 4_1, \dots, 4_{2b}$ correspond respectively to the two copies of D_g and to $g(C)$.

We now turn to vertices along the right boundary of the diagram, at positions pj with $1 \le j \le q - 1$. The local geometric configuration is very similar to that for the vertices along the top boundary, except for the local description of the curve $g(C)$ which now involves $2a$ parallel copies of $g(C_0)$ instead of $2b$. Another difference is that, due to the ordering of the vertices and edges of the diagram, the slope of some of the line components to which $g(C)$ degenerates becomes smaller than that of some of the components to which D_g degenerates, so that the braid monodromy has to be calculated again, with results very similar to those above. In fact, it can easily be checked that, up to a Hurwitz equivalence, the only effect of the change of ordering on the local braid monodromy is the simultaneous conjugation of all

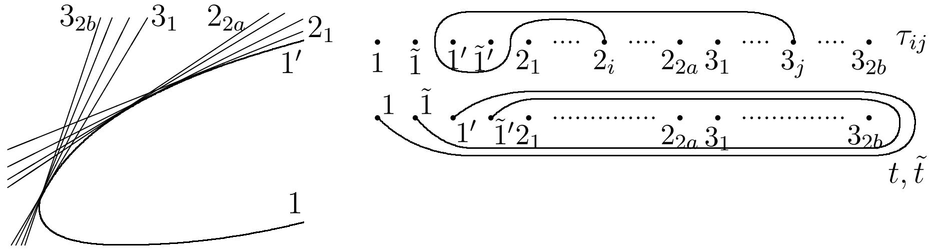


FIGURE 9

contributions by a braid that exchanges the groups of points labelled $2, \tilde{2}, 2', \tilde{2}'$ and $3_1, \dots, 3_{2a}$ by moving them around each other counterclockwise.

The last vertex that remains to be investigated is the corner vertex at position pq. The local configuration for $D^0 = D_g \cup g(C)$ is obtained from that represented in Figure 9 (left) by smoothing the $4ab$ mutual intersections between the lines labelled $2_1, \dots, 2_{2a}$ and $3_1, \dots, 3_{2b}$. Indeed, the local configuration for D_g is simply a conic (labelled $1, 1'$ in Figure 9), while $g(C_0)$ consists of two lines tangent to that conic, and $g(C)$ is obtained by "thickening" these two lines into respectively $2a$ and $2b$ components ($2_1, \dots, 2_{2a}$ corresponding to the vertical edge of the diagram, and $3_1, \dots, 3_{2b}$ corresponding to the horizontal edge of the diagram) and smoothing their mutual intersections. The curve D is then obtained from D^0 by separating the multiplicity 2 component D_g into two distinct copies, while each tangent intersection of D_g with $g(C)$ gives rise to three cusps.

The braid monodromy for the corner vertex can be deduced explicitly from this description. We are particularly interested in the $8ab + 2$ vertical tangencies of the local model, for which the corresponding half-twists τ_{ij} ($1 \le i \le 2a$, $1 \le j \le 2b$, each appearing twice), t and $\tilde{t}$ in $B_{2a+2b+4}$ are represented in Figure 9 (right).

7.3. **Fundamental group calculations.** As in §6, the Zariski-Van Kampen theorem provides an explicit presentation of $\pi_1(\mathbb{C}^2 - D)$ in terms of the braid monodromy. The main difference is that there are now four generators for each interior edge of the diagram (Figure 5), because the regeneration process involves two copies of the branch curve of g; we denote by γ_i, γ'_i and $\tilde{\gamma}_i, \tilde{\gamma}'_i$ the four generators corresponding to the i-th interior edge. Moreover, each edge along the top boundary of the diagram contributes $2b$ generators (denoted by $z_{i,1}, \dots, z_{i,2b}$ for the horizontal edge in position iq, where $1 \le i \le p$), and similarly each edge along the right boundary contributes $2a$ generators ($y_{j,1}, \dots, y_{j,2a}$ for the vertical edge in position pj, where $1 \le j \le q$).

We are in fact interested in the stabilized quotient G of $\pi_1(\mathbb{C}^2 - D)$ (see Definition 2.2), which can be expressed in terms of the same generators by adding suitable commutation relations. Let Γ be the subgroup of G generated by the γ_i, γ'_i, and let $\tilde{\Gamma}$ be the subgroup generated by the $\tilde{\gamma}_i, \tilde{\gamma}'_i$. By definition, the elements of Γ always commute with those of $\tilde{\Gamma}$, because the images by the geometric monodromy representation θ of the geometric generators γ_i, γ'_i and $\tilde{\gamma}_i, \tilde{\gamma}'_i$ act on two disjoint sets of $n/2 = 2pq$ sheets of the branched cover f.

As in §6, we introduce twisted generators e_i, e'_i and $\tilde{e}_i, \tilde{e}'_i$ for Γ and $\tilde{\Gamma}$, by choosing integers l_i satisfying the same compatibility conditions at the inner vertices as in §6, and setting as previously $e_i = \rho_i^{l_i}(\gamma_i)$, $e'_i = \rho_i^{l_i}(\gamma'_i)$, $\tilde{e}_i = \tilde{\rho}_i^{l_i}(\tilde{\gamma}_i)$ and $\tilde{e}'_i = \tilde{\rho}_i^{l_i}(\tilde{\gamma}'_i)$, with the obvious definition for ρ_i and $\tilde{\rho}_i$. Even though this could be avoided by proving a suitable invariance property, we will assume that $l_i = 1$ for every diagonal edge in the top-most row or in the right-most column of the diagram (so $e_i = \gamma'_i$, $\tilde{e}_i = \tilde{\gamma}'_i$), and $l_j = 0$ for every vertical edge in the top-most row and every horizontal edge in the right-most column (so $e_j = \gamma_j$, $\tilde{e}_j = \tilde{\gamma}_j$). Finally, as in §6.1 we let $a_i = e'_i e_i^{-1}$ and $\tilde{a}_i = \tilde{e}'_i \tilde{e}_i^{-1}$, and we relabel these elements as d_{ij}, v_{ij}, h_{ij} (resp. $\tilde{d}_{ij}, \tilde{v}_{ij}, \tilde{h}_{ij}$) according to their position in the diagram.

Lemma 7.1. *The subgroup $\mathcal{B}_\Gamma \subset \Gamma$ generated by the e_i and the subgroup $\mathcal{B}_{\tilde{\Gamma}} \subset \tilde{\Gamma}$ generated by the $\tilde{e}_i$ are naturally isomorphic to quotients of $\tilde{B}_{n/2}$. Moreover, the subgroups Γ and $\tilde{\Gamma}$ of G are naturally isomorphic to quotients of $\tilde{B}^{(2)}_{n/2}$, with geometric generators corresponding to pairs of half-twists. Furthermore, Γ is generated by the elements of $\mathcal{B}_\Gamma$ and v_{11}, and $\tilde{\Gamma}$ is generated by the elements of $\mathcal{B}_{\tilde{\Gamma}}$ and $\tilde{v}_{11}$.*

Proof. We first look at relations corresponding to the interior vertices of the diagram (Figure 5) and to the vertices along the bottom and left boundaries. Since the local description of D at these vertices simply consists of two superimposed copies of D_g, and since the generators of Γ commute with those of $\tilde{\Gamma}$, one easily checks that the local configurations yield relations among the e_i, e'_i that are exactly identical to those discussed in §6 in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$; additionally, an identical set of relations also holds among the $\tilde{e}_i, \tilde{e}'_i$.

Next we consider the local configuration at a vertex along the top boundary of the diagram, and more precisely the cusp singularities present near the point labelled A on Figure 6, as pictured on Figure 7. Denoting by i and j respectively the labels of the diagonal and vertical edges meeting at the given vertex, the relations corresponding to these six cusps are

$$(7.1)\qquad \begin{aligned} &\gamma'_i\gamma_j\gamma'_i = \gamma_j\gamma'_i\gamma_j,\ \gamma'_i\gamma'_j\gamma'_i = \gamma'_j\gamma'_i\gamma'_j,\ \gamma'_i(\gamma_j^{-1}\gamma'_j\gamma_j)\gamma'_i = (\gamma_j^{-1}\gamma'_j\gamma_j)\gamma'_i(\gamma_j^{-1}\gamma'_j\gamma_j),\\ &\tilde{\gamma}'_i\tilde{\gamma}_j\tilde{\gamma}'_i = \tilde{\gamma}_j\tilde{\gamma}'_i\tilde{\gamma}_j,\ \tilde{\gamma}'_i\tilde{\gamma}'_j\tilde{\gamma}'_i = \tilde{\gamma}'_j\tilde{\gamma}'_i\tilde{\gamma}'_j,\ \tilde{\gamma}'_i(\tilde{\gamma}_j^{-1}\tilde{\gamma}'_j\tilde{\gamma}_j)\tilde{\gamma}'_i = (\tilde{\gamma}_j^{-1}\tilde{\gamma}'_j\tilde{\gamma}_j)\tilde{\gamma}'_i(\tilde{\gamma}_j^{-1}\tilde{\gamma}'_j\tilde{\gamma}_j). \end{aligned}$$

It can easily be checked that these relations satisfy a property of invariance under twisting similar to that of 3-points. In fact, replacing the various generators by their images under arbitrary powers of the twisting actions $\rho_i, \tilde{\rho}_i, \rho_j, \tilde{\rho}_j$ amounts to a conjugation of the relations (7.1) by braids belonging to the local monodromy (either the entire local monodromy, or two of the six cusps near A, or combinations thereof), and thus always yields valid relations.

Therefore, the twisted generators e_i, e'_i, e_j, e'_j of Γ satisfy the relations (6.2), and similarly for $\tilde{e}_i, \tilde{e}'_i, \tilde{e}_j, \tilde{e}'_j$ in $\tilde{\Gamma}$. One easily checks that a similar conclusion holds for pairs of inner edges meeting at a vertex along the right boundary of the diagram (recall that the local braid monodromy only differs by a simple conjugation). Finally, because we are looking at the stabilized fundamental group, the commutation relations discussed in §6 automatically hold in Γ and $\tilde{\Gamma}$.

So, except for the equality relations arising from vertical tangencies at the vertices along the top and right boundaries of the diagram, all the relations described in §6.1 for the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ simultaneously hold in Γ and in $\tilde{\Gamma}$. Therefore, the structure of Γ and $\tilde{\Gamma}$ can be studied by the same argument as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ ([9], see also §6), which yields the desired result. ∎

Lemma 7.2. *The equality $z_{r,i} = z_{r,1}$ holds for every $1 \le r \le p$, $1 \le i \le 2b$; similarly, $y_{r,i} = y_{r,1}$ for every $1 \le r \le q$, $1 \le i \le 2a$. Moreover, the $y_{r,i}$ and the $z_{r,i}$ are all conjugates of $y_{q,1}$ under the action of elements of $\mathcal{B}_\Gamma$ and $\mathcal{B}_{\tilde\Gamma}$.*

Proof. First consider the corner vertex at position pq, and more precisely the half-twists τ_{ij} arising from the vertical tangencies of the local model near this vertex (Figure 9). Denoting by μ the label of the diagonal edge in position pq, the half-twist τ_{1i} yields the relation $(y_{q,1}^{-1} \dots y_{q,2a}^{-1} z_{p,1}^{-1} \dots z_{p,i-1}^{-1}) z_{p,i} (z_{p,i-1} \dots z_{p,1} y_{q,2a} \dots y_{q,1}) = \tilde\gamma'_\mu \gamma'_\mu y_{q,1} {\gamma'_\mu}^{-1} {\tilde\gamma'_\mu}^{-1}$. It follows that the quantity $(z_{p,1}^{-1} \dots z_{p,i-1}^{-1}) z_{p,i} (z_{p,i-1} \dots z_{p,1})$ is independent of i, which by an easy induction on i implies that $z_{p,i} = z_{p,1}$ for all i. Observing that $y_{q,1}, \dots, y_{q,2a}$ and $z_{p,1}, \dots, z_{p,2b}$ are mapped by θ to disjoint transpositions and hence commute in G, we in fact have $z_{p,i} = \tilde\gamma'_\mu \gamma'_\mu y_{q,1} {\gamma'_\mu}^{-1} {\tilde\gamma'_\mu}^{-1}$ for all i. Since by assumption the twisting parameter l_μ is equal to 1, the generators $\gamma'_\mu = e_\mu$ and $\tilde\gamma'_\mu = \tilde e_\mu$ belong to $\mathcal{B}_\Gamma$ and $\mathcal{B}_{\tilde\Gamma}$ respectively. This proves the claims made about the $z_{p,i}$.

Similarly comparing the relations corresponding to the half-twists τ_{i1}, it can be seen immediately that the quantity $(y_{q,1}^{-1} \dots y_{q,i-1}^{-1}) y_{q,i} (y_{q,i-1} \dots y_{q,1})$ is independent of i, which implies that $y_{q,i} = y_{q,1}$ for all i.

We now proceed by induction : assume that $z_{r+1,i} = z_{r+1,1}$ for all i, and that $z_{r+1,1}$ is a conjugate of $y_{q,1}$ under the action of $\mathcal{B}_\Gamma$ and $\mathcal{B}_{\tilde\Gamma}$. Let μ and ν be the labels of the diagonal and vertical edges meeting at the vertex in position rq, and let $\psi_r = \tilde\gamma'_\nu \gamma'_\nu \tilde\gamma_\nu \gamma_\nu \tilde\gamma'_\mu \gamma'_\mu \gamma_\nu^{-1} \tilde\gamma_\nu^{-1} {\gamma'_\nu}^{-1} {\tilde\gamma'_\nu}^{-1}$. Define $\zeta_r = \psi_r \gamma_\nu \psi_r^{-1}$, $\zeta'_r = \psi_r \gamma'_\nu \psi_r^{-1}$, $\tilde\zeta_r = \psi_r \tilde\gamma_\nu \psi_r^{-1}$, and $\tilde\zeta'_r = \psi_r \tilde\gamma'_\nu \psi_r^{-1}$. Recalling that the elements of Γ commute with those of $\tilde\Gamma$, the relations (7.1) imply that $\zeta'_r = \gamma'_\nu \gamma_\nu \gamma'_\mu (\gamma_\nu^{-1} \gamma'_\nu \gamma_\nu) {\gamma'_\mu}^{-1} \gamma_\nu^{-1} {\gamma'_\nu}^{-1} = \gamma'_\nu \gamma_\nu (\gamma_\nu^{-1} \gamma'_\nu \gamma_\nu)^{-1} \gamma'_\mu (\gamma_\nu^{-1} \gamma'_\nu \gamma_\nu) \gamma_\nu^{-1} {\gamma'_\nu}^{-1} = \gamma_\nu \gamma'_\mu \gamma_\nu^{-1} = {\gamma'_\mu}^{-1} \gamma_\nu \gamma'_\mu$. Similar calculations for the other elements yield that

(7.2)
$$\zeta_r = {\gamma'_\mu}^{-1} (\gamma_\nu^{-1} \gamma'_\nu \gamma_\nu) \gamma'_\mu,\ \tilde\zeta_r = {\tilde\gamma'_\mu}^{-1} (\tilde\gamma_\nu^{-1} \tilde\gamma'_\nu \tilde\gamma_\nu) \tilde\gamma'_\mu,\ \zeta'_r = {\gamma'_\mu}^{-1} \gamma_\nu \gamma'_\mu,\ \tilde\zeta'_r = {\tilde\gamma'_\mu}^{-1} \tilde\gamma_\nu \tilde\gamma'_\mu.$$

Due to the choice of twisting parameters $l_\mu = 1$ and $l_\nu = 0$, $\zeta'_r \in \mathcal{B}_\Gamma$ and $\tilde\zeta'_r \in \mathcal{B}_{\tilde\Gamma}$.

Since the $z_{r,i}$ commute with the $z_{r+1,i}$ in G (they are mapped to disjoint transpositions by θ), and since by assumption $z_{r+1,i} = z_{r+1,1}$ for all i, we have

$$(z_{r,1}^{-1} \dots z_{r,i}^{-1} z_{r+1,1}^{-1} \dots z_{r+1,i-1}^{-1}) z_{r+1,i} (z_{r+1,i-1} \dots z_{r+1,1} z_{r,i} \dots z_{r,1}) = z_{r+1,1}$$

for all i. Therefore, the relation arising from the vertical tangency τ'_i (Figure 8) at the vertex rq can be written in the form

$$z_{r+1,1} = \tilde\zeta'_r \zeta'_r (z_{r,1}^{-1} \dots z_{r,i-1}^{-1}) z_{r,i} (z_{r,i-1} \dots z_{r,1}) {\zeta'_r}^{-1} {\tilde\zeta'_r}^{-1}.$$

In particular, the value of $(z_{r,1}^{-1} \dots z_{r,i-1}^{-1}) z_{r,i} (z_{r,i-1} \dots z_{r,1})$ does not depend on i, which implies that $z_{r,i} = z_{r,1}$ for all i. Moreover, we have $z_{r,i} = {\zeta'_r}^{-1} {\tilde\zeta'_r}^{-1} z_{r+1,1} \tilde\zeta'_r \zeta'_r$. So, by induction on decreasing values of r, we obtain the desired results about $z_{r,i}$. The case of $y_{r,i}$ is handled using exactly the same argument, going inductively through the vertices along the right boundary of the diagram. Indeed, observe that the local braid monodromy at one of these vertices simply differs from that at a vertex along the top boundary by a conjugation which exchanges the positions of two groups of geometric generators ; however, because the corresponding transpositions in S_n are disjoint, these generators commute with each other in G, so that the relations induced by the local braid monodromy can be expressed in exactly the same form. □

Lemma 7.3. *The element $\tilde{v}_{11}$ belongs to the subgroup of G generated by Γ, $\mathcal{B}_{\tilde{\Gamma}}$, and $y_{q,1}$.*

Proof. Consider the local relations for the vertex at position $1q$, and more precisely the equality relation corresponding to the half-twist labelled τ_1'' in Figure 8 : with the same notations as in the proof of Lemma 7.2, we have $z_{2,1} = \zeta_1^{-1}\tilde{\zeta}_1^{-1}z_{1,1}\tilde{\zeta}_1\zeta_1$. Moreover, the cusp point with monodromy κ_1^B pictured on Figure 7 yields the relation $\tilde{\zeta}_1 z_{1,1}\tilde{\zeta}_1 = z_{1,1}\tilde{\zeta}_1 z_{1,1}$. It follows that $z_{2,1} = \zeta_1^{-1}z_{1,1}\tilde{\zeta}_1 z_{1,1}^{-1}\zeta_1$. Therefore, using formula (7.2) for $\tilde{\zeta}_1$, we obtain $\tilde{\gamma}'_\nu = \tilde{\gamma}_\nu\tilde{\gamma}'_\mu z_{1,1}^{-1}\zeta_1 z_{2,1}\zeta_1^{-1}z_{1,1}\tilde{\gamma}'_\mu{}^{-1}\tilde{\gamma}_\nu^{-1}$, where μ and ν are the labels of the two interior edges meeting at the considered vertex.

Observe that, since $l_\nu = 0$ and $l_\mu = 1$, the generators $\tilde{\gamma}_\nu = \tilde{e}_\nu$ and $\tilde{\gamma}'_\mu = \tilde{e}_\mu$ belong to $\mathcal{B}_{\tilde{\Gamma}}$. Moreover, it is obvious from (7.2) that $\zeta_1 \in \Gamma$. Using the result of Lemma 7.2 to express $z_{1,1}$ and $z_{2,1}$ in terms of $y_{q,1}$, it follows that $\tilde{\gamma}'_\nu = \tilde{e}'_\nu$ belongs to the subgroup of G generated by Γ, $\mathcal{B}_{\tilde{\Gamma}}$, and $y_{q,1}$. Therefore, $\tilde{v}_{1,q} = \tilde{e}'_\nu\tilde{e}_\nu^{-1}$ also belongs to this subgroup. Finally, the local relations analogous to (6.5) for the $\tilde{e}_i$ and $\tilde{a}_i$ at the vertex in position $1r$ imply that $\tilde{v}_{1,r}$ and $\tilde{v}_{1,r+1}$ are conjugates of each other under the action of elements of $\mathcal{B}_{\tilde{\Gamma}}$. Therefore, by induction $\tilde{v}_{1,1}$ can be expressed in terms of $\tilde{v}_{1,q}$ and elements of $\mathcal{B}_{\tilde{\Gamma}}$, which completes the proof. □

Lemma 7.4. *The subgroup $\mathcal{B}$ of G generated by $\mathcal{B}_\Gamma$, $\mathcal{B}_{\tilde{\Gamma}}$ and $y_{q,1}$ is naturally a quotient of $\tilde{B}_n$, with geometric generators corresponding to half-twists.*

Proof. We construct a surjective map $\alpha : \tilde{B}_n \to \mathcal{B}$ as follows (recall that $n = 4pq$). First observe that the subgroup of $\tilde{B}_n$ generated by the half-twists $x_1, \dots, x_{2pq-1}$ is naturally isomorphic to $\tilde{B}_{n/2}$, which by Lemma 7.1 admits a surjective homomorphism to $\mathcal{B}_\Gamma$ mapping half-twists to geometric generators. We use this homomorphism to define $\alpha(x_i)$ for $1 \le i \le 2pq-1$. Any two half-twists in $\tilde{B}_{n/2}$ are conjugate to each other; therefore, after a suitable conjugation we can assume that $\alpha(x_{2pq-1}) = e_\mu$, where μ is the label of the diagonal edge at position pq in the diagram, and that the other $\alpha(x_i)$ ($i \le 2pq-2$) are geometric generators mapped by θ to transpositions disjoint from $\theta(y_{q,1})$. Because of the stabilization process, this last requirement implies that $\alpha(x_i)$ commutes with $y_{q,1}$ for $i \le 2pq-2$.

Similarly, the subgroup of $\tilde{B}_n$ generated by $x_{2pq+1}, \dots, x_{n-1}$ is naturally isomorphic to $\tilde{B}_{n/2}$ and admits a surjective homomorphism to $\mathcal{B}_{\tilde{\Gamma}}$, which we use to define $\alpha(x_i)$ for $2pq+1 \le i \le n-1$. Once again, without loss of generality we can assume that $\alpha(x_{2pq+1}) = \tilde{e}_\mu$ and that the other $\alpha(x_i)$ commute with $y_{q,1}$. Finally, we define $\alpha(x_{2pq}) = y_{q,1}$.

All that remains to be checked is that α can be made into a group homomorphism (obviously surjective by construction), i.e. that the relations defining $\tilde{B}_n$ are also satisfied by the chosen images $\alpha(x_i)$ in $\mathcal{B}$. Since α is built out of two group homomorphisms and since the elements of $\mathcal{B}_\Gamma$ commute with those of $\mathcal{B}_{\tilde{\Gamma}}$, the only relations to be checked are those involving x_{2pq}.

Consider the corner vertex at position pq in the diagram: the cusp singularities arising from the regeneration of the rightmost tangent intersection of D_g with $g(C)$ in Figure 9 imply the relations $\gamma'_\mu y_{q,1}\gamma'_\mu = y_{q,1}\gamma'_\mu y_{q,1}$ and $\tilde{\gamma}'_\mu y_{q,1}\tilde{\gamma}'_\mu =$

$y_{q,1}\tilde{\gamma}'_\mu y_{q,1}$. Since $l_\mu = 1$, we have $\gamma'_\mu = e_\mu$ and $\tilde{\gamma}'_\mu = \tilde{e}_\mu$, so that these relations can be rewritten as $\alpha(x_{2pq-1})\alpha(x_{2pq})\alpha(x_{2pq-1}) = \alpha(x_{2pq})\alpha(x_{2pq-1})\alpha(x_{2pq})$ and $\alpha(x_{2pq+1})\alpha(x_{2pq})\alpha(x_{2pq+1}) = \alpha(x_{2pq})\alpha(x_{2pq+1})\alpha(x_{2pq})$. Finally, for all i such that $|i-2pq| \geq 2$, the relation $[\alpha(x_{2pq}), \alpha(x_i)] = 1$ holds by construction. Therefore, α defines a surjective group homomorphism from $\tilde{B}_n$ to $\mathcal{B}$, mapping half-twists to geometric generators. □

Proposition 7.5. *The morphism α extends to a surjective group homomorphism from $\tilde{B}_n^{(2)} \simeq \tilde{B}_n \ltimes \tilde{P}_{n,0}$ to G mapping pairs of half-twists to geometric generators. In particular, the group G has property $(*)$.*

Proof. Lemma 7.2 implies that G is generated by Γ, $\tilde{\Gamma}$, and $y_{q,1}$. Therefore, by Lemma 7.1, G is generated by $\mathcal{B}$, v_{11} and $\tilde{v}_{11}$, while Lemma 7.3 implies that $\tilde{v}_{11}$ can be eliminated from the list of generators. Since Lemma 7.4 identifies $\mathcal{B}$ with a quotient of $\tilde{B}_n$, the main remaining task is to check that the subgroup $\mathcal{P}$ generated by the $g^{-1}v_{11}g$, $g \in \mathcal{B}$, is naturally isomorphic to a quotient of $\tilde{P}_{n,0}$. This can be done by proving that $\mathcal{P}$ is a *primitive $\tilde{B}_n$-group* (Definition 5 of [9]), as it follows from the discussion in §1 of [9] that every such group is a quotient of $\tilde{P}_{n,0}$ (compare Propositions 1, 2, 3 of [9] with the presentation of $\tilde{P}_{n,0}$ given in Lemma 3.1).

As stated in Lemma 7.1, the arguments of [9] show that the subgroup generated by the $g^{-1}v_{11}g$, $g \in \mathcal{B}_\Gamma$, is a primitive $\tilde{B}_{n/2}$-group (and hence a quotient of $\tilde{P}_{n/2,0}$). The desired result about $\mathcal{P}$ then follows simply by observing that v_{11} commutes with $y_{q,1}$ and with the generators of $\mathcal{B}_{\tilde{\Gamma}}$ and using a criterion due to Moishezon (Proposition 6 of [9]); indeed, an obvious corollary of this criterion is that, upon enlarging the conjugation action from $\tilde{B}_{n/2}$ to $\tilde{B}_n$, it is sufficient to check that the additional half-twist generators act trivially on the given prime element (v_{11}).

Since G is obviously generated by its subgroups $\mathcal{B}$ and $\mathcal{P}$, and since $\mathcal{P}$ is normal, it is naturally a quotient of $\tilde{B}_n \ltimes \tilde{P}_{n,0} \simeq \tilde{B}_n^{(2)}$. Moreover, the geometric generators of G are all mutually conjugate (because the curve D is irreducible), and by construction the e_i (and $\tilde{e}_i$) correspond to pairs of half-twists in $\tilde{B}_n^{(2)}$, so the same is true of all geometric generators. Finally, by going carefully over the construction, it is not hard to check that the end points of the half-twists (x, y) corresponding to a given geometric generator γ are always the natural ones, in the sense that $\sigma(x) = \sigma(y) = \theta(\gamma)$. Therefore, G has property $(*)$. □

At this point, the only remaining task in the proof of Theorem 4.6 is to characterize the kernel of the surjective morphism $\alpha : \tilde{B}_n^{(2)} \to G$ given by Proposition 7.5. As a consequence of Lemmas 3.3 and 3.4, the commutation relations induced either by nodes in the branch curve D or by the stabilization process, as well as the relations induced by the cusp points of D, automatically hold, so that $\operatorname{Ker}\alpha$ is generated by equality relations between pairs of half-twists induced by the vertical tangencies of D. Moreover, as in §6.1.2 the classification of half-twists in $\tilde{B}_n$ (Lemma 3.2) allows us to associate to every a_i (resp. $\tilde{a}_i$) a pair of integers $\bar{a}_i$ (resp. $\bar{\tilde{a}}_i$), well-defined modulo the subgroup $\Lambda = \{(\kappa, \lambda),\ (u_1^\kappa \eta^{\kappa(\kappa-1)/2}, u_1^\lambda \eta^{\lambda(\lambda-1)/2}) \in \operatorname{Ker}\alpha\} \subset \mathbb{Z}^2$. Recall however from §6.1.2 that this construction requires us to choose an ordering of the $n = 4pq$ sheets of the branched cover; in our case, these split into two sets of $2pq$ sheets, the first one on which the $\theta(e_i), \theta(e'_i)$ act by permutations, and the second one on which the $\theta(\tilde{e}_i), \theta(\tilde{e}'_i)$ act by permutations. The ordering we will consider is obtained by enumerating first the first set of $2pq$ sheets, and then the second one. In each set, the sheets are naturally in correspondence with the $2pq$ triangles of the diagram in Figure 5: the ordering we choose for each of the two sets of $2pq$ sheets is obtained as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ [9] by enumerating the $2pq$ triangles of the diagram from left to right and from bottom to top.

We have seen above that the relations coming from the vertical tangencies at the inner vertices of the diagram and at those along the lower and left boundaries are exactly the same as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, except they simultaneously apply to the generators of Γ and to those of $\tilde{\Gamma}$. Therefore, as in §6.1.2, these relations do not contribute to $\operatorname{Ker}\alpha$ by themselves, but they translate into equalities between the $\bar{a}_i$ (and similarly between the $\bar{\tilde{a}}_i$), which yield the following formulas (with the obvious notations) : $\bar{d}_{i,j} = \bar{\tilde{d}}_{i,j} = (j-i,0)$, $\bar{v}_{i,j} = \bar{\tilde{v}}_{i,j} = (1-i,1)$, $\bar{h}_{i,j} = \bar{\tilde{h}}_{i,j} = (1-j,1)$ (compare with (6.9)).

Next, we consider the corner vertex at position pq, for which the braid monodromy contribution of the vertical tangencies is represented in Figure 9. Recall that some of the half-twists τ_{ij} were used in the proof of Lemma 7.2 to eliminate $y_{q,2},\dots,y_{q,2a}$ and $z_{p,1},\dots,z_{p,2b}$ from the list of generators by expressing them in terms of $y_{q,1}$; however, since these relations imply that $y_{q,i} = y_{q,1}$ and $z_{p,i} = z_{p,1}$ (cf. Lemma 7.2), all the other relations coming from the τ_{ij} become redundant. Therefore these equality relations do not make any contributions to the kernel of α. We are left with the two half-twists $t, \tilde{t}$ of Figure 9. Denote by μ the label of the diagonal edge passing through the corner vertex. Because G has property $(*)$, and using the results of §3, we can find an element $g \in \tilde{B}_n^{(2)}$ such that $z_{p,1} = \alpha(g^{-1}(x_1,x_1)g)$, $e_\mu = \gamma'_\mu = \alpha(g^{-1}(x_2,x_2)g)$, and $y_{q,1} = \alpha(g^{-1}(x_3,x_3)g)$. Recalling that $\bar{d}_{p,q} = (q-p,0)$ and observing that the conjugation by g preserves the ordering of the end points for e_μ, by definition of $\bar{d}_{p,q}$ we have $e'_\mu = \alpha(g^{-1}(x_2u_2^{p-q}\eta^{(p-q)(p-q-1)/2},x_2)g)$, and therefore $\gamma_\mu = e_\mu^{-1}e'_\mu e_\mu = \alpha(g^{-1}(x_2u_2^{q-p}\eta^{(q-p)(q-p-1)/2},x_2)g)$. The half-twist t yields the relation $\gamma_\mu = z_{p,1}^{2b}y_{q,1}^{2a}\gamma'_\mu y_{q,1}^{-2a}z_{p,1}^{-2b}$; an easy computation shows that the right-hand side of this relation is equal to $\alpha(g^{-1}(x_2u_2^{a-b}\eta^{(a-b)(a-b-1)/2},x_2u_2^{a-b}\eta^{(a-b)(a-b-1)/2})g)$. Comparing the two formulas for γ_μ, we conclude that the relation introduced by the half-twist t is equivalent to the property that $(a-b+p-q,a-b) \in \Lambda$. A similar calculation shows that the relation introduced by $\tilde{t}$ can also be rewritten in the form $(a-b+p-q,a-b) \in \Lambda$.

We now consider the vertex at position rq ($1 \le r \le p-1$), and investigate in the same manner the equality relations coming from the vertical tangencies $\tau'_i, \tau''_i, t, \tilde{t}$ represented in Figure 8. Recall that the relations induced by τ'_i were used in the proof of Lemma 7.2 to show that $z_{r,i} = \zeta_r'^{-1}\tilde{\zeta}_r'^{-1}z_{r+1,1}\tilde{\zeta}'_r\zeta'_r$ and consequently eliminate the $z_{r,i}$ from the list of generators; these relations are therefore already accounted for. Next, we turn to the relation induced by τ''_i, which taking into account that $z_{r,i} = z_{r,1}$ and $z_{r+1,i} = z_{r+1,1}$ can be written in the form $z_{r+1,1} = \zeta_r^{-1}\tilde{\zeta}_r^{-1}z_{r,1}\tilde{\zeta}_r\zeta_r$. Using the expression of $z_{r,1}$ in terms of $z_{r+1,1}$, this identity can also be expressed by the commutation relation $[z_{r+1,1}, \tilde{\zeta}'_r\zeta'_r\tilde{\zeta}_r\zeta_r] = 1$. By (7.2), we have $\tilde{\zeta}'_r\zeta'_r\tilde{\zeta}_r\zeta_r = e_\mu^{-1}\tilde{e}_\mu^{-1}\tilde{e}'_\nu\tilde{e}_\nu e'_\nu e_\nu\tilde{e}_\mu e_\mu$, where μ and ν are the labels of the two interior edges meeting at position rq. Since $z_{r+1,1}$ commutes with e_μ and $\tilde{e}_\mu$, the relation can then be rewritten as $[z_{r+1,1}, \tilde{e}'_\nu\tilde{e}_\nu e'_\nu e_\nu] = 1$. Taking into account the ordering of the sheets of the branched cover, an easy calculation in $\tilde{B}_n^{(2)}$ shows that this relation automatically holds as a consequence of the equality $\bar{v}_{r,q} = \tilde{\bar{v}}_{r,q}$.

The relation induced by the half-twist t (Figure 8) can be expressed as $\gamma_\mu = z_{r,1}^{2b}\gamma'_\nu\gamma_\nu\gamma'_\mu\gamma_\nu^{-1}{\gamma'_\nu}^{-1}z_{r,1}^{-2b}$. Using property $(*)$ and recalling that $\bar{d}_{r,q} = (q-r,0)$ and $\bar{v}_{r,q} = (1-r,1)$, we can find $g \in \tilde{B}_n^{(2)}$, preserving the ordering of the end points for e_μ and e_ν, such that $z_{r,1} = \alpha(g^{-1}(x_1,x_1)g)$, $\gamma'_\mu = e_\mu = \alpha(g^{-1}(x_2,x_2)g)$, $\gamma_\mu = e_\mu^{-1}e'_\mu e_\mu = \alpha(g^{-1}(x_2u_2^{q-r}\eta^{(q-r)(q-r-1)/2}, x_2)g)$, $\gamma_\nu = e_\nu = \alpha(g^{-1}(x_3,x_3)g)$, and $\gamma'_\nu = e'_\nu = \alpha(g^{-1}(x_3u_3^{r-1}\eta^{(r-1)(r-2)/2}, x_3u_3^{-1}\eta)g)$. So $z_{r,1}^{2b}\gamma'_\nu\gamma_\nu\gamma'_\mu\gamma_\nu^{-1}{\gamma'_\nu}^{-1}z_{r,1}^{-2b}$ is equal to $\alpha(g^{-1}(x_2u_2^{2-r-b}\eta^{(2-r-b)(1-r-b)/2}, x_2u_2^{2-b}\eta^{(2-b)(1-b)/2})g)$. Comparing this with the expression for γ_μ, it becomes apparent that the relation induced by t is in fact equivalent to the condition $(q+b-2,b-2) \in \Lambda$. A similar calculation for the half-twist $\tilde{t}$ shows that the relation it induces can also be expressed in the form $(q+b-2,b-2) \in \Lambda$.

Finally, the case of the vertices along the right boundary of the diagram can be studied by exactly the same argument; the relations corresponding to the vertical tangencies of the local model can be expressed by the single requirement that $(p+a-2,a-2) \in \Lambda$.

Therefore, $\Lambda \subset \mathbb{Z}^2$ is the subgroup generated by $(p+a-2,a-2)$ and $(q+b-2,b-2)$, and $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by the two elements $g_1 = (u_1^{p+a-2}\eta^{\lambda(p+a-2)}, u_1^{a-2}\eta^{\lambda(a-2)})$ and $g_2 = (u_1^{q+b-2}\eta^{\lambda(q+b-2)}, u_1^{b-2}\eta^{\lambda(b-2)})$, where $\lambda(i) = i(i-1)/2$. Observe that $G^0_{p,q} = (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\operatorname{Ker}\alpha$, and recall from Lemma 3.1 that $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] = \{1,\eta\} \simeq \mathbb{Z}_2$ and $\operatorname{Ab}\tilde{P}_{n,0} \simeq \mathbb{Z}^{n-1}$.

We first consider the commutator subgroup $[G^0_{p,q}, G^0_{p,q}] \simeq C/(C \cap \operatorname{Ker}\alpha)$, where $C = \{1,\eta\} \times \{1,\eta\}$. First of all, if $a+p$ is odd, then considering the commutator of g_1 with $(u_2,1)$ we obtain that $(\eta,1) \in \operatorname{Ker}\alpha$, and similarly if $b+q$ is odd; otherwise, one easily checks that $(\eta,1) \notin \operatorname{Ker}\alpha$. Moreover, if a is odd, then considering the commutator of g_1 with $(1,u_2)$ we obtain that $(1,\eta) \in \operatorname{Ker}\alpha$, and similarly if b is odd; when a and b are both even, $(1,\eta) \notin \operatorname{Ker}\alpha$. Also, it is easy to check that $\operatorname{Ker}\alpha$ only contains (η,η) if it also contains $(\eta,1)$ and $(1,\eta)$. The claim made in the statement of Theorem 4.6 about the structure of $[G^0_{p,q}, G^0_{p,q}]$ follows.

Finally, we have $\operatorname{Ab}G^0_{p,q} \simeq (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\langle C, \operatorname{Ker}\alpha\rangle \simeq (\mathbb{Z}^2/\Lambda)^{n-1}$. Observing that $\mathbb{Z}^2/\Lambda = \mathbb{Z}^2/\langle (p+a-2,a-2),(q+b-2,b-2)\rangle \simeq \mathbb{Z}^2/\langle (p,a-2),(q,b-2)\rangle$, this completes the proof of Theorem 4.6.

Acknowledgements. The authors wish to thank M. Gromov for prompting their interest in this problem and for useful advice and suggestions. The first and third authors wish to thank respectively Ecole Polytechnique and Imperial College for their hospitality.

References

[1] D. Auroux, *Symplectic maps to projective spaces and symplectic invariants*, Proc. 7th Gökova Geometry-Topology Conference (2000), International Press, 2001, pp. 1–42.

[2] D. Auroux, L. Katzarkov, *Branched coverings of* $\mathbb{CP}^2$ *and invariants of symplectic* 4*-manifolds*, Inventiones Math. **142** (2000) 631–673.

[3] D. Auroux, L. Katzarkov, *The degree doubling formula for braid monodromies and Lefschetz pencils*, preprint.

[4] G. Dethloff, S. Orevkov, M. Zaidenberg, *Plane curves with a big fundamental group of the complement*, Voronezh Winter Math. Schools, Amer. Math. Soc. Transl. Ser. 2, **184**, Amer. Math. Soc., 1998, pp. 63–84.

[5] S.K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Diff. Geom. **44** (1996) 666–705.

[6] L. Katzarkov, T. Pantev, *Fundamental groups of complements of curves in $\mathbb{CP}^2$ with wild topology*, preprint.

[7] B. Moishezon, *Stable branch curves and braid monodromies*, Algebraic Geometry (Chicago, 1980), Lecture Notes in Math. **862**, Springer, 1981, pp. 107–192.

[8] B. Moishezon, *Algebraic surfaces and the arithmetic of braids II*, Combinatorial methods in topology and algebraic geometry (Rochester, 1982), Contemp. Math. **44**, Amer. Math. Soc., 1985, pp. 311–344.

[9] B. Moishezon, *On cuspidal branch curves*, J. Algebraic Geom. **2** (1993) 309–384.

[10] B. Moishezon, *Topology of generic polynomial maps in complex dimension two*, preprint.

[11] B. Moishezon, A. Robb, M. Teicher, *On Galois covers of Hirzebruch surfaces*, Math. Annalen **305** (1996) 493–539.

[12] A. Robb, *On branch curves of algebraic surfaces*, Singularities and Complex Geometry (Beijing, 1994), AMS/IP Stud. Adv. Math. **5**, Amer. Math. Soc., 1997, pp. 193–221.

[13] M. Teicher, *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math. **62** (part 1), Amer. Math. Soc., 1997, pp. 127–150.

[14] M. Teicher, *New invariants for surfaces*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math. **231**, Amer. Math. Soc., 1999, pp. 271–281.

[15] M. Teicher, *The fundamental group of a $\mathbb{CP}^2$ complement of a branch curve as an extension of a solvable group by a symmetric group*, Math. Ann. **314** (1999) 19–38.

[16] M. Teicher, *Hirzebruch surfaces: degenerations, related braid monodromy, Galois covers*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math. **241**, Amer. Math. Soc., 1999, pp. 305–325.

[17] O. Zariski, *On the Poincaré group of rational plane curves*, Amer. J. Math. **58** (1936) 607–619.

Department of Mathematics, M.I.T., Cambridge MA 02139, USA
E-mail address: auroux@math.mit.edu

Department of Mathematics, Imperial College, London SW7 2BZ, U.K.
E-mail address: s.donaldson@ic.ac.uk

Department of Mathematics, University of California, Irvine, CA 92697, USA
E-mail address: lkatzark@math.uci.edu

Department of Mathematics, University of California, Irvine, CA 92697, USA
E-mail address: myotov@math.uci.edu

LUTTINGER SURGERY ALONG LAGRANGIAN TORI AND NON-ISOTOPY FOR SINGULAR SYMPLECTIC PLANE CURVES

D. AUROUX, S. K. DONALDSON, AND L. KATZARKOV

Abstract. We discuss the properties of a certain type of Dehn surgery along a Lagrangian torus in a symplectic 4-manifold, known as Luttinger's surgery, and use this construction to provide a purely topological interpretation of a non-isotopy result for symplectic plane curves with cusp and node singularities due to Moishezon [9].

1. Introduction

It is an important open question in symplectic topology to determine whether, in a given symplectic manifold, all (connected) symplectic submanifolds realizing a given homology class are mutually isotopic. In the case where the ambient manifold is Kähler or complex projective, one may in particular ask whether symplectic submanifolds are always isotopic to complex submanifolds.

The isotopy results known so far rely heavily on the theory of pseudo-holomorphic curves and on the Gromov compactness theorem [7]. The best currently known result for smooth curves is due to Siebert and Tian [11], who have proved that smooth connected symplectic curves of degree at most 17 in $\mathbb{CP}^2$, or realizing homology classes with intersection pairing at most 7 with the fiber class in a S^2-bundle over S^2, are always symplectically isotopic to complex curves; this strongly suggests that symplectic isotopy holds for smooth curves in $\mathbb{CP}^2$ and S^2-bundles over S^2. Isotopy results have also been obtained for certain singular configurations; e.g., Barraud has obtained a result for certain arrangements of pseudo-holomorphic lines in $\mathbb{CP}^2$ [4].

On the other hand, Fintushel and Stern [6] and Smith [12] have constructed infinite families of pairwise non-isotopic smooth connected symplectic curves representing the same homology classes in certain symplectic 4-manifolds. In both cases, the construction starts from parallel copies of a given suitable embedded curve of square zero and modifies them by a *braiding* construction in order to yield connected symplectic curves; the constructed submanifolds are distinguished by the diffeomorphism types of the corresponding double branched covers, either using Seiberg-Witten theory in the argument of Fintushel and Stern, or by more topological methods in Smith's argument. It is also worth mentioning that other examples have recently been obtained by Vidussi using link surgery [15].

These constructions are predated by a result of Moishezon concerning singular curves with nodes and cusps in $\mathbb{CP}^2$ [9]. More precisely, the construction yields infinite families of inequivalent cuspidal braid monodromies, but as observed by Moishezon the result can be reformulated in terms of singular plane curves, which one can in fact assume to be symplectic (cf. e.g. Theorem 3 of [2]). The statement can be expressed as follows:

Theorem 1.1 (Moishezon [9]). *There exists an infinite set $\mathcal{N}$ of positive integers such that, for each $m \in \mathcal{N}$, there exist integers ρ_m, d_m and an infinite family of symplectic curves $S_{m,k} \subset \mathbb{CP}^2$ ($k \geq 0$) of degree m with ρ_m cusps and d_m nodes, such that whenever $k_1 \neq k_2$ the curves S_{m,k_1} and S_{m,k_2} are not smoothly isotopic.*

In particular, because a finiteness result holds for complex curves, infinitely many of the symplectic curves $S_{m,k}$ are not isotopic to any complex curve.

Moishezon's argument relies on the observation that the fundamental groups $\pi_1(\mathbb{CP}^2 - S_{m,k})$ are mutually non-isomorphic. However this requires the heavy machinery of braid monodromy techniques, and in particular the calculation of the fundamental group of the complement of the branch curve of a generic polynomial map from $\mathbb{CP}^2$ to itself, carried out in [10] (see also [14]) and preceding papers. The curious reader is referred to §6 of [3] (see also [13]) for an overview of Moishezon-Teicher braid monodromy techniques.

The aim of this paper is to provide a topological interpretation of Moishezon's construction, along with an elementary proof of Theorem 1.1; this reformulation shows that Moishezon's result is very similar to those of Fintushel-Stern and Smith, in the sense that it also reduces to a braiding process where the various constructed curves are distinguished by the topology of associated branched covers. We also show that these constructions can be thought of in terms of Luttinger surgery [8] along Lagrangian tori in a symplectic 4-manifold.

We start by introducing the surgery construction and describing its elementary properties in §2; its interpretation in terms of braiding constructions for branched covers is discussed in §3, while Moishezon's examples are presented in §4.

2. Luttinger surgery along Lagrangian tori

2.1. **The surgery construction.** Let T be an embedded Lagrangian torus in a symplectic 4-manifold (X, ω), and let γ be a simple closed co-oriented loop in T.

It is well-known that a neighborhood of T in X can be identified symplectically with a neighborhood of the zero section in the cotangent bundle $T^*T \simeq T \times \mathbb{R}^2$ with its standard symplectic structure. Moreover, T itself can be identified with $\mathbb{R}^2/\mathbb{Z}^2$ in such a way that γ is identified with the first coordinate axis and its co-orientation coincides with the standard orientation of the second coordinate axis. Denoting by (x_1, x_2) the corresponding coordinates on T and by (y_1, y_2) the dual coordinates in the cotangent fibers, the symplectic form is given by $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Let $r > 0$ be such that the set $U_r = (\mathbb{R}^2/\mathbb{Z}^2) \times [-r, r] \times [-r, r] \subset (\mathbb{R}^2/\mathbb{Z}^2) \times \mathbb{R}^2$ is contained in the neighborhood of T over which the identification holds. Choose a smooth step function $\chi : [-r, r] \to [0, 1]$ such that $\chi(t) = 0$ for $t \leq -\frac{r}{3}$, $\chi(t) = 1$ for $t \geq \frac{r}{3}$, and $\int_{-r}^{r} t\, \chi'(t)\, dt = 0$ (this last condition expresses the fact that χ is *centered* around $t = 0$). Given an integer $k \in \mathbb{Z}$, define $\phi_k : U_r - U_{r/2} \to U_r - U_{r/2}$ by the formulas $\phi_k(x_1, x_2, y_1, y_2) = (x_1 + k\chi(y_1), x_2, y_1, y_2)$ if $y_2 \geq \frac{r}{2}$ and $\phi_k = \mathrm{Id}$ otherwise.

Because the support of $d\chi$ is contained in $[-\frac{r}{3}, \frac{r}{3}]$, the map ϕ_k is actually a diffeomorphism of $U_r - U_{r/2}$; moreover, ϕ_k obviously preserves the symplectic form. Therefore, we can tentatively make the following definition:

Definition 2.1. *$X(T,\gamma,k)$ is the manifold obtained from X by removing a small neighborhood of T and gluing back the standard piece U_r, using the symplectomorphism ϕ_k to identify the two sides near their boundaries. In other terms, $X(T,\gamma,k) = (X - U_{r/2}) \cup_{\phi_k} U_r$.*

It can be easily checked that this surgery operation is equivalent to that introduced by Luttinger in [8] to study Lagrangian tori in $\mathbb{R}^4$ (see also [5]).

Forgetting about the symplectic structure, the topological description of the construction is that of a parametrized $1/k$ Dehn surgery (with Lagrangian framing): a neighborhood $T \times D^2$ of T is cut out, and glued back in place by identifying the two boundaries via a diffeomorphism of $T \times S^1$ that acts trivially on $H_1(T)$ but maps the homology class of the meridian $\mu = \{pt\} \times S^1$ to $[\tilde{\mu}] = [\mu] + k[\gamma]$.

Observe that the normal bundle to T along γ comes equipped with a natural framing, so that the loop γ can be pushed away from T in a canonical way (up to homotopy), which allows us to define the homotopy class of γ in $\pi_1(X - T)$; comparing the fundamental groups of X and $X(T,\gamma,k)$ with $\pi_1(X - T)$, we see that the surgery operation preserves the fundamental group (resp. first homology group) whenever γ^k is homotopically (resp. homologically) trivial in $X - T$.

The fact that the construction is well-defined symplectically is a consequence of Moser's stability theorem. More precisely:

Proposition 2.2. *$X(T,\gamma,k)$ carries a natural symplectic form $\tilde{\omega}$, well-defined up to isotopy independently of the choices made in the construction. Moreover, deforming T among Lagrangian tori and $\gamma \subset T$ by smooth isotopies induces a deformation (pseudo-isotopy) of the symplectic structure $\tilde{\omega}$, and if the symplectic area swept by γ is equal to zero then this deformation preserves the cohomology class $[\tilde{\omega}]$ and is therefore an isotopy.*

Proof. Fixing an orientation of T (and therefore of γ), and observing that the identification of a neighborhood of T with a neighborhood of the zero section in T^*T is canonical up to isotopy, the possible choices for coordinate systems over a neighborhood of T differ by isotopies and transformations of the form $(x_1', x_2', y_1', y_2') = (x_1 + nx_2, x_2, y_1, y_2 - ny_1)$ for some integer n.

To handle the case of isotopies, thanks to Moser's stability theorem we only need to worry about the cohomology class of the symplectic form $\tilde{\omega}$ on $\tilde{X} = X(T,\gamma,k)$. Because the surgery affects only a neighborhood of T, once a loop $\delta \subset X - U_r$ homotopic to the meridian of T in $\tilde{X}$ has been fixed, the cohomology class $[\tilde{\omega}]$ is completely determined by the quantity $\int_D \tilde{\omega}$, where $D \subset \tilde{X}$ is a disk such that $\partial D = \delta$ and realizing a fixed homotopy class.

Equivalently, if one considers a family depending continuously on a parameter $t \in [0,1]$, the dependence on t of the cohomology class $[\tilde{\omega}_t]$ is exactly given by the symplectic area swept in X by the meridian loop $\tilde{\mu}_t = \phi_{k,t}(\partial\Delta)$, where $\Delta = \{(0,0,y_1,y_2),\ y_1,y_2 \in [-r,r]\} \subset U_r$ (this is because $\tilde{\mu}_t$ bounds a Lagrangian disk $\tilde{\Delta}_t$ in $\tilde{X}$). Viewing the family of loops $\tilde{\mu}_t$ as a map $\tilde{\mu} : S^1 \times [0,1] \to X$, we have

$$[\tilde{\omega}_t] = [\tilde{\omega}_0] + \Big(\int_{S^1 \times [0,t]} \tilde{\mu}^* \omega \Big)\, PD([T])$$

(the sign in this formula depends on the choice of an orientation of $S^1 \times [0,1]$). However, observing that the loop $\mu_t = \partial\Delta$, which coincides with $\tilde{\mu}_t$ on three of its four sides, bounds a Lagrangian disk in X and therefore sweeps no area, the symplectic area swept by $\tilde{\mu}_t$ is the difference between the area swept by the arc $\{(0,0,y_1,r),\ y_1 \in [-r,r]\}$ and that swept by the arc $\{(k\chi(y_1),0,y_1,r),\ y_1 \in [-r,r]\}$. Using the local expression for the symplectic form and the fact that the function χ is centered ($\int_{-r}^{r} t\,\chi'(t)\,dt = 0$), one sees that this is equal to k times the symplectic area swept by the loop $\{(x_1,0,0,0), x_1 \in [0,1]\}$, i.e. by γ. Therefore, as long as the loop γ is fixed, or that it is moved in such a way that no symplectic area is swept, we do not need to worry about continuous deformations of the construction parameters.

Observe that the coordinate change $(x_1', x_2', y_1', y_2') = (x_1 + nx_2, x_2, y_1, y_2 - ny_1)$ simply amounts to a modification of the shape of the cut-off region inside each fiber of the cotangent bundle (from a square to a parallelogram), which clearly has no effect on $\tilde{\omega}$ (e.g., by the above isotopy argument). Therefore, to complete the proof we only need to consider the effect of a simultaneous change of orientation on T and γ (recall that a co-orientation of γ inside T is fixed); this simply amounts to changing x_1 and y_1 into $-x_1$ and $-y_1$, which clearly does not affect the construction. □

Additionally, it is straightforward to check that, if γ^* is the loop γ with the opposite co-orientation, then $X(T,\gamma^*,k)$ is symplectomorphic to $X(T,\gamma,-k)$.

Example. Let $\phi : \Sigma \to \Sigma$ be a symplectomorphism of a Riemann surface (Σ, ω_Σ), and consider the mapping torus $Y(\phi) = [0,1] \times \Sigma/(1,x) \sim (0,\phi(x))$. The manifold $X = S^1 \times Y(\phi)$ fibers over $S^1 \times S^1$, with monodromy Id along the first factor and ϕ along the second factor, and carries a natural symplectic structure $\omega = d\theta \wedge dt + \omega_\Sigma$. Let γ be any simple closed loop in Σ. By picking a point $(\theta_0, t_0) \in S^1 \times S^1$, we can embed γ as a closed loop $\bar{\gamma} = \{(\theta_0,t_0)\} \times \gamma$ inside a fiber of X. Observe that $T = S^1 \times \{t_0\} \times \gamma$ is an embedded Lagrangian torus in (X,ω), containing $\bar{\gamma}$. It is easy to check that the manifold $X(T,\bar{\gamma},k)$ is exactly $S^1 \times Y(\tau^k \circ \phi)$, where τ is a Dehn twist about the loop γ (positive or negative depending on the co-orientation).

2.2. **Effect on the canonical class.** We now study the effect of the surgery procedure on the canonical class $c_1(\tilde{K})$ of $\tilde{X} = X(T,\gamma,k)$. Although there is in general no natural identification between $H^2(X,\mathbb{Z})$ and $H^2(\tilde{X},\mathbb{Z})$ (these two spaces may even have different ranks), we can compare the two canonical classes $c_1(K)$ and $c_1(\tilde{K})$ by means of the relative cohomology groups. Indeed, $H^2(X,T)$ can be identified with $H^2(\tilde{X},T)$ using excision, and we have long exact sequences

$$\cdots \longrightarrow H^1(T) \xrightarrow{\delta} H^2(X,T) \xrightarrow{\iota} H^2(X) \longrightarrow H^2(T) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^1(T) \xrightarrow{\tilde{\delta}} H^2(\tilde{X},T) \xrightarrow{\tilde{\iota}} H^2(\tilde{X}) \longrightarrow H^2(T) \longrightarrow \cdots$$

The choice of a trivialization τ of K over T determines a lift $\hat{c}_1(K,\tau)$ of $c_1(K)$ in the relative group $H^2(X,T)$: the relative Chern class with respect to the chosen trivialization.

Observe that the choice of a section σ of the Lagrangian Grassmannian $\Lambda(TX)$ over a certain subset of X determines the homotopy class of a trivialization τ_σ of K over the same subset: indeed, considering a 2-form θ_σ such that $\operatorname{Ker}\theta_\sigma = \sigma$ at every point, and given any ω-compatible almost-complex structure J, the $(2,0)$-component of θ provides a nowhere vanishing section of K. With this understood, we can fix homotopy classes of trivializations τ_T and $\tilde{\tau}_T$ of K and $\tilde{K}$ over a neighborhood of T by considering the family of Lagrangian tori $(\mathbb{R}^2/\mathbb{Z}^2) \times \{(y_1,y_2)\}$ parallel to T in either X or $\tilde{X}$ (i.e., the trivialization of the canonical bundle is given by the $(2,0)$-component of $dy_1 \wedge dy_2$).

Because the trivializations τ_T and $\tilde{\tau}_T$ of K and $\tilde{K}$ coincide over the punctured neighborhood $U_r - U_{r/2}$, the relative Chern classes $\hat{c}_1(K, \tau_T)$ and $\hat{c}_1(\tilde{K}, \tilde{\tau}_T)$ are equal to each other; therefore, we have $c_1(\tilde{K}) = \tilde{\imath}(\hat{c}_1(K, \tau_T))$. However, if we consider another trivialization τ of $K_{|T}$, differing from τ_T by an element $\nu \in H^1(T)$, then we obtain a different lift $\hat{c}_1(K, \tau) = \hat{c}_1(K, \tau_T) + \delta(\nu)$ of $c_1(K)$ in $H^2(X, T)$. It is important to observe that, even though $\delta(\nu) \in H^2(X, T)$ maps to zero in $H^2(X)$, it does not necessarily lie in the kernel of $\tilde{\imath} : H^2(\tilde{X}, T) \to H^2(\tilde{X})$. In fact, $\tilde{\imath}(\delta(\nu))$ precisely measures the obstruction for the trivialization of $\tilde{K}$ determined by τ over the subset $U_r - U_{r/2}$ (i.e., differing from $\tilde{\tau}_T$ by the element $\delta(\nu)$ in the relative cohomology) to extend over a neighborhood of T in $\tilde{X}$.

An easy computation yields that $\tilde{\imath}(\delta(\nu)) = -\langle \nu, k[\gamma] \rangle \, PD([T])$. Indeed, given any 2-cycle $\tilde{C}$ in $\tilde{X}$, we can find a relative 2-cycle C in (X, T) representing the same relative homology class; recalling that the meridians μ and $\tilde{\mu}$ differ by $k[\gamma]$ and being careful about the orientation of the boundary, one easily checks that $[\partial C] = -([\tilde{C}] \cdot [T]) \, k[\gamma]$. It follows that $\langle \tilde{\imath}(\delta(\nu)), [\tilde{C}] \rangle = \langle \nu, [\partial C] \rangle = -\langle \nu, k[\gamma] \rangle \, ([\tilde{C}] \cdot [T])$, which yields the desired result. Therefore, we conclude that $c_1(\tilde{K}) = \tilde{\imath}(\hat{c}_1(K, \tau_T)) = \tilde{\imath}(\hat{c}_1(K, \tau) - \delta(\nu)) = \tilde{\imath}(\hat{c}_1(K, \tau)) + k\langle \nu, [\gamma] \rangle \, PD([T])$.

In the special case where there is a proportionality relation of the form $c_1(K) = \lambda[\omega]$ in $H^2(X, \mathbb{R})$, it is of particular interest to study simultaneously the effect of the surgery construction on the canonical and symplectic classes, by directly considering $c_1(\tilde{K}) - \lambda[\tilde{\omega}] \in H^2(\tilde{X}, \mathbb{R})$. The assumption on $c_1(K)$ allows us choose a (Hermitian) connection ∇ on the canonical bundle K with curvature 2-form $F = -2\pi i \lambda \omega$. Since the surgery only affects a neighborhood of T, we can endow $\tilde{K}$ with a (Hermitian) connection $\tilde{\nabla}$ with curvature $\tilde{F}$ that coincides with ∇ outside of $U_{r/2}$. As before, denote by $\tilde{\mu} = \phi_k(\partial\Delta)$ the meridian of T in $\tilde{X}$, which bounds a Lagrangian disk $\tilde{\Delta}$ in $\tilde{X}$. Then we have $c_1(\tilde{K}) - \lambda[\tilde{\omega}] = \alpha \, PD([T])$, where $\alpha = \frac{i}{2\pi} \int_{\tilde{\Delta}} \tilde{F}$.

We use the above-defined trivializations τ_T and $\tilde{\tau}_T$ of K and $\tilde{K}$ over neighborhoods of T in X and $\tilde{X}$. This allows us to write locally the connection on K in the form $\nabla = d + 2\pi i \lambda (y_1 \, dx_1 + y_2 \, dx_2) + i\beta$, where β is a closed 1-form and can therefore be expressed as $\beta = a_1 \, dx_1 + a_2 \, dx_2 + dh$. The integral of $\tilde{F}$ over $\tilde{\Delta}$ is given by the integral along its boundary $\tilde{\mu}$ of the 1-form representing the connection $\tilde{\nabla}$ in the chosen trivialization of $\tilde{K}$, i.e. the holonomy of $\tilde{\nabla}$ along $\tilde{\mu}$ in the chosen trivialization (note that the choice of a homotopy class of trivialization allows us to view the holonomy as $i\mathbb{R}$-valued rather than S^1-valued). Since the chosen connections and trivializations of K and $\tilde{K}$ coincide along $\tilde{\mu}$, this is equal to the holonomy of ∇ along $\tilde{\mu}$, which is given by the formula

$$i \int_{\tilde{\mu}} (2\pi\lambda y_1 + a_1) \, dx_1 + (2\pi\lambda y_2 + a_2) \, dx_2 + dh = i \int_{-r}^{r} (2\pi\lambda y_1 + a_1) \, k\chi'(y_1) \, dy_1 = ika_1.$$

Observing that ia_1 is the holonomy of the flat connection $\nabla_{|T}$ along the loop γ in the given trivialization, we obtain the following:

Definition 2.3. *Given a loop $\delta \subset X$ and a homotopy class of trivializations τ of the canonical bundle K along δ, we define $H(\delta, \tau)$ to be the real number such that the holonomy of ∇ along δ is equal to $-2\pi i H(\delta, \tau)$ in the trivialization τ.*

Proposition 2.4. $c_1(\tilde{K}) - \lambda[\tilde{\omega}] = k\,H(\gamma, \tau_T)\,PD([T])$.

Remark 2.5. A change of homotopy class of the trivialization τ affects the quantity $H(\delta, \tau)$ by an integer, while a continuous deformation of the loop δ affects $H(\delta, \tau)$ by λ times the symplectic area swept.

3. Luttinger surgery for branched covers

In this section, we consider the case where X is a branched cover of another symplectic 4-manifold (Y, ω_Y), and show that the braiding constructions used by Fintushel-Stern [6] and Smith [12] are a special case of the surgery described in §2.

Let $f : X \to Y$ be a covering map with smooth ramification curve $R \subset X$ and simple branching at the generic points of R, such that the branch curve $\Sigma = f(R)$ is a symplectic submanifold of Y, immersed except possibly at complex cusp points. Recall that X carries a natural symplectic structure (up to isotopy), obtained from the degenerate 2-form $f^*\omega_Y$ by adding a small exact perturbation along the ramification curve R. More precisely, the local models for f near the points of R allow us to construct an exact 2-form α such that, at any point of R, the restriction of α to the kernel of df is a positive volume form. The form $\omega = f^*\omega_Y + \epsilon\alpha$ is then symplectic for $\epsilon > 0$ sufficiently small, and its isotopy class does not depend on the choice of α or ϵ (see e.g. Proposition 10 of [1] and Theorem 3 of [2]).

Consider a loop a_0 contained in the smooth part of Σ, and an annulus $V_0 \subset \Sigma$ forming a neighborhood of a_0 in Σ. Locally the manifold Y is a fibration over a neighborhood U of the origin in $\mathbb{R}^2$, with fibers $V_z \subset Y$ that are smooth symplectic annuli for all $z \in U$. This fibration carries a natural symplectic connection, given by the symplectic orthogonal to V_z at each point. Given a path $t \mapsto z(t)$ in U starting at the origin, for small enough values of t we can consider the loops $a_t \subset V_{z(t)}$ obtained by parallel transport of a_0, and by construction $\bigcup_{t\geq 0} a_t$ is a smooth Lagrangian surface in Y.

Assume that, for some value $z_0 \in U - \{0\}$, the symplectic annulus V_{z_0} is contained in the branch curve Σ. Assume moreover that, by parallel transport along a certain path $t \mapsto z(t)$ joining the origin to z_0 in U, we can construct an embedded Lagrangian annulus $A = \bigcup_{t\in[0,t_0]} a_t$ in Y, such that $A \cap \Sigma = \partial A = a_0 \cup a_{t_0} \subset V_0 \cup V_{z_0}$ (see Figure 1). Assume finally that, among the lifts of A, exactly two have boundary contained in the ramification curve R of the map f in X; these two lifts together form an embedded torus $T \subset X$, and a suitable choice of the perturbation α of the pull-back form $f^*\omega_Y$ ensures that T is Lagrangian. Because there is freedom in choosing the local fibration and the path $z(t)$, the above assumptions can be

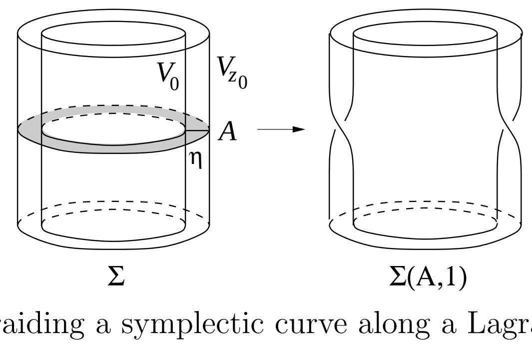


Figure 1. Braiding a symplectic curve along a Lagrangian annulus

made to hold in a rather wide range of situations, including those considered by Fintushel-Stern and Smith, but also the examples studied by Moishezon [9].

Choose a smooth arc $t \mapsto \eta(t) \in a_t$ joining the two boundary components of A, and let γ be the loop in T formed by the two lifts of η with end points lying in R. Observe that the homotopy class of the loop γ in T does not depend on the choice of the arc η. Moreover, an orientation of a_0 determines a co-orientation of γ in T.

We can perform a *braiding* construction on the two parallel annuli V_0 and V_{z_0} contained in Σ, twisting them k times around each other along the annulus A for any given integer $k \in \mathbb{Z}$. The process is described by the following local model: a neighborhood of a_0 in Y is diffeomorphic to $D^2 \times [-r, r] \times S^1$, where the factor D^2 corresponds to U and the factor $[-r, r] \times S^1$ corresponds to the annuli V_z. The branch curve Σ can be locally identified with the subset $\{\pm u\} \times [-r, r] \times S^1 \subset D^2 \times [-r, r] \times S^1$, for some $u \in D^2 - \{0\}$, while the annulus A corresponds to $[-u, u] \times \{0\} \times S^1$. Considering a step function $\chi : [-r, r] \to [0, 1]$ as in §2, the twisted curve $\Sigma(A, k)$ is obtained from Σ by replacing $\{\pm u\} \times [-r, r] \times S^1$ with $\{(\pm u \exp(i\pi k\chi(t)), t),\ t \in [-r, r]\} \times S^1$. Observe that the construction depends on the choice of an orientation of the factor $[-r, r]$, or equivalently (because Σ is symplectic) of an orientation of the loop a_0. Moreover, it is easy to check that the construction can be performed in a way that preserves the symplecticity of the twisted curve.

Recall that the double cover of D^2 branched at the two points $\pm u$ is an annulus $S^1 \times [-r, r]$; therefore a local model for X is $S^1 \times [-r, r] \times [-r, r] \times S^1$, with the torus T corresponding to $S^1 \times \{0\} \times \{0\} \times S^1$ and the loop γ corresponding to $S^1 \times \{0\} \times \{0\} \times \{pt\}$. Also recall that a half-twist exchanging the two points $\pm u$ in D^2 lifts to a Dehn twist of the annulus $S^1 \times [-r, r]$, i.e. a transformation of the form $(x_1, y_1) \mapsto (x_1 + \chi(y_1), y_1)$. Therefore, if one tries to understand the effect of the twisting construction on the branched cover X in terms of cutting out the piece $S^1 \times [-r, r] \times [-r, r] \times S^1$ and gluing it back in place via a nontrivial diffeomorphism, the difference between the gluing maps on the two sides $S^1 \times [-r, r] \times \{\pm r\} \times S^1$ must be the k-th power of a Dehn twist along the first S^1 factor; so if e.g. we take the gluing map to be the identity near $S^1 \times [-r, r] \times \{-r\} \times S^1$, then on the other side it must map (x_1, y_1, r, x_2) to $(x_1 + k\chi(y_1), y_1, r, x_2)$.

The topological modification undergone by X is therefore exactly the construction described in §2. Moreover, the symplectic structure on $X(T, \gamma, k)$ introduced in §2 and that obtained from its branched covering structure above (Y, ω_Y) coincide up to isotopy. Indeed, in both cases a neighborhood of T containing the region modified by the surgery can naturally be described as a symplectic fibration where the fibers and the base are symplectic annuli (in the model of §2 the base is the (x_2, y_2)-plane; in the branched covering model the base is the annulus V_0). After checking that the symplectic structures coincide near the boundaries and that the two symplectic forms lie in the same cohomology class, we can conclude by means of Moser's argument. We obtain the following result:

Proposition 3.1. *The branched cover of Y obtained from X by replacing the branch curve Σ with the twisted curve $\Sigma(A, k)$ is naturally symplectomorphic to $X(T, \gamma, k)$.*

Note that the construction only depends on the isotopy classes of the loop a_0 and of the arc η, even symplectically; this follows from Proposition 2.2 by observing that, when the arc η is deformed in Y, the symplectic area swept by γ is always zero (the areas swept by the two lifts of η exactly compensate each other).

Remark 3.2. When the branch curve Σ contains $n > 2$ parallel annuli V_{z_i}, one can similarly construct modified symplectic surfaces associated to arbitrary elements b of the braid group B_n. However, decomposing b into a product of the standard generators of B_n and their inverses (or any other half-twists) and starting from a suitable collection of disjoint Lagrangian annuli in Y with boundary in Σ, the general braiding construction easily reduces to an iteration of the elementary process described above. Therefore, assuming that the braid b is *liftable*, i.e. compatible with the branching data of the map f, and that it can be decomposed into liftable half-twists (note that in the case of a double cover all braids are liftable), we can describe the effect of the general braiding construction on the symplectic manifold X as a sequence of Luttinger surgeries along disjoint Lagrangian tori.

The examples studied by Fintushel and Stern [6] show that, in some cases, the non-triviality of Luttinger surgeries along Lagrangian tori can be proved using Seiberg-Witten invariants; however in many cases it is possible to conclude by much more elementary arguments, as shown in §4 for Moishezon's examples.

We finish this section by observing that, in the context of branched covers, it is possible to provide a more topological interpretation of the quantity $H(\gamma, \tau_T)$ introduced in §2.2 to describe the effect of the surgery on the canonical and symplectic classes of X in the case where they are proportional to each other.

More precisely, assume that $c_1(K) = \lambda[\omega]$ in $H^2(X, \mathbb{R})$, where ω is the symplectic form induced by a branched covering map $f : X \to Y$ with smooth ramification curve R. Assume moreover that $[\gamma] \in H_1(X, \mathbb{Z})$ is a torsion element, i.e. $m[\gamma] = 0$ for some integer $m \neq 0$, and let N be a surface with boundary such that $\partial N = \gamma_1 \cup \cdots \cup \gamma_m$, where the γ_i are parallel copies of γ all obtained as double lifts of arcs in Y. Then f_*N is a 2-cycle in Y, and we have the following:

Proposition 3.3. *$m\, H(\gamma, \tau_T) = (\lambda[\omega_Y] - c_1(K_Y)) \cdot [f_*N] - I(N, R)$, where $I(N, R)$ is the algebraic intersection number between R and N, counting the $2m$ intersection points that lie on the boundary of N with multiplicity $1/2$.*

Proof. By definition, $m\, H(\gamma, \tau_T) = \sum H(\gamma_i, \tau_T)$ can be expressed as the difference of two terms, one measuring the integral over N of the curvature of the connection ∇ on the canonical bundle K_X, and the other measuring the obstruction to extending the trivialization of K_X given over the boundary of N to a trivialization over all of N (i.e., the *relative degree* of K_X over N with respect to the given boundary trivialization). By assumption, the first term is proportional to the symplectic area of N; observing that the exact perturbation added to $f^*\omega_Y$ does not contribute to this area (in fact we could work directly with the degenerate form $f^*\omega_Y$), we obtain that it is equal to $\lambda[\omega_Y] \cdot [f_*N]$.

In order to compute the relative degree $\deg(K_X, N)$ of K_X over N (the boundary trivialization is implicit in the notation), we first deform the boundary loops γ_i inside X in order to obtain loops γ_i' bounding a surface N' in X, disjoint from R, and C^1-close to γ_i; we can assume that the immersed loops $f(\gamma_i') \subset Y$ are in fact embedded. The trivialization τ_T naturally induces a trivialization of K_X over each loop γ_i', and the relative degree is unaffected by the operation.

Recall that the trivialization τ_T of the canonical bundle over γ is defined by the Lagrangian plane field given by the tangent spaces to T along γ. Deforming to γ_i', this corresponds to a Lagrangian plane field generated by two vector fields, one tangent to γ_i' and the other almost parallel to the normal direction to γ in T. This trivialization of K_X is naturally the lift of a trivialization of K_Y along $f(\gamma_i')$, determined by two vector fields, the first one v_1 tangent to $f(\gamma_i')$ and the other v_2 pointing in a direction transverse to the arc η inside the Lagrangian annulus A.

Outside of the ramification curve, K_X is isomorphic to f^*K_Y, and a trivialization of K_Y lifts to a trivialization of K_X. However, we have in fact $c_1(K_X) = f^*c_1(K_Y) + PD([R])$, so the relative degree of K_X over N' differs from that of K_Y over f_*N' by a correction term equal to the algebraic intersection number of R with N'. On the other hand, the relative degree of K_Y over f_*N' can be evaluated by observing that each loop $f(\gamma_i')$ bounds a small disk D_i in Y (because $f(\gamma_i')$ is contained in a neighborhood of the arc η). Moreover, $f_*N' - \sum D_i$ is a 2-cycle in Y, homologous to f_*N. Therefore, $\deg(K_Y, f_*N') = c_1(K_Y) \cdot [f_*N] + \sum \deg(K_Y, D_i)$.

It can be checked explicitly that $\deg(K_Y, D_i)$, which measures the obstruction to extending our trivialization of K_Y over D_i, is equal to -1, 0 or $+1$ depending on the chosen perturbation of γ_i. More precisely, recall from above that Y admits a local fibration π over an open subset $U \subset \mathbb{R}^2$ (with fibers the symplectic annuli V_z). Considering the two vector fields v_1 and v_2 defining the trivialization of K_Y along $f(\gamma_i')$, and observing that v_2 remains almost parallel to the fibers of π along $f(\gamma_i')$ and extends to D_i, the relative degree $\deg(K_Y, D_i)$ is equal to the rotation number of the vector field $\pi_*(v_1)$ tangent to the loop $\pi(f(\gamma_i'))$ in U (we can safely assume that this loop is immersed). Recall that γ_i is the double lift of an arc joining two points of the branch curve Σ in Y, and therefore passes through exactly two points of the ramification curve R. For each of these two points, we have two inequivalent possibilities for the perturbation of γ_i into γ_i', towards one side or the other with respect to R; this yields a contribution of ± 1 to the linking number of $f(\gamma_i')$ with Σ, and $\pm\frac{1}{2}$ to the rotation number of $\pi(f(\gamma_i'))$ in U. It follows that the possible values of the rotation number are -1, 0, and $+1$, corresponding to linking numbers of $f(\gamma_i')$ with Σ equal to -2, 0, and $+2$ respectively. Therefore, $\deg(K_Y, D_i)$ is in all cases equal to half of the intersection number of D_i with the branch curve Σ, which coincides with the local difference between the intersection numbers $I(N, R)$ and $I(N', R)$ (once again, counting boundary intersections with a coefficient 1/2). As a consequence, $\sum \deg(K_Y, D_i) = I(N, R) - I(N', R)$, and so $\deg(K_X, N) = \deg(K_Y, f_*N') + I(N', R) = c_1(K_Y) \cdot [f_*N] + I(N, R)$, which completes the proof. □

4. Non-isotopic singular symplectic plane curves

4.1. **The manifolds $X_{p,0}$.** Given two symplectic manifolds Y and Z, both obtained as branched covers of the same manifold M, and assuming that the branch curves D_g and D_h of $g : Y \to M$ and $h : Z \to M$ intersect transversely and positively in M, we can construct a new symplectic manifold $X = Y \times_M Z = \{(y, z) \in Y \times Z,\ g(y) = h(z)\}$. The manifold X is naturally equipped with two branched covering structures given by the two projections; considering e.g. the projection onto the first factor, $f : X \to Y$, we obtain a branched covering map which is simply the pull-back of h via the map g. In particular, the fiber of f above a point $y \in Y$ is naturally identified to the fiber of h above the point $g(y) \in M$, the degree of f is equal to that of h, and its branch curve is $D = g^{-1}(D_h)$.

We consider the case where $M = Y = Z = \mathbb{CP}^2$, and $g : Y \to M$ is a generic map defined by three polynomials of degree 3, while $h : Z \to M$ is a generic map defined by three polynomials of degree $p \geq 2$. We define $X_{p,0} = Y \times_M Z$, and consider the projection to the first factor, $f : X_{p,0} \to Y = \mathbb{CP}^2$, which is a branched covering of degree p^2. It is worth noting that $X_{p,0}$ is in fact a complex surface. Via a suitable transformation in $PGL(3, \mathbb{C})$, we can assume that, outside of a given fixed small ball B, the branch curve D_h of the map h lies arbitrarily close to a union of $d = 3p(p-1)$ lines passing through a single point in $\mathbb{CP}^2$ (observe that $\deg D_h = d$). The cubic map g can be chosen in such a way that D_g does not intersect the ball B. The branch curve $D = g^{-1}(D_h)$ of f can then be obtained topologically from the union of d smooth cubics $C_1, \dots, C_d \subset \mathbb{CP}^2$ lying in a generic pencil, by removing a small neighborhood of each of the 9 base points where the C_i intersect and replacing it with a configuration similar to the branch curve of h (or rather to $D_h \cap B$).

The manifold $X_{p,0}$ can also be described as follows. Blow up $Y = \mathbb{CP}^2$ at the 9 intersection points of the cubics C_i, in order to obtain the rational elliptic surface $E(1)$ with twelve singular fibers and nine exceptional sections of square -1. Let W be the p^2-fold cover of $E(1)$ branched along d smooth fibers $F_1, \dots, F_d$ (the proper transforms of the cubics $C_1, \dots, C_d$). The branching pattern of the projection $q : W \to E(1)$ is prescribed by that of the map h, in the sense that W is the pullback of the elliptic fibration $E(1)$ over $\mathbb{CP}^1$ under the base change map $h_{|S} : S \to \ell = \mathbb{CP}^1$, where S is the smooth plane curve of degree p obtained as the preimage by h of a generic line $\ell \subset \mathbb{CP}^2$. In particular, W is the total space of an elliptic fibration π over the curve S of genus $(p-1)(p-2)/2$, with $12p^2$ singular fibers and nine exceptional sections $E_1, \dots, E_9$ of square $-p^2$. We can glue a copy of $\mathbb{CP}^2$ to W along each of the exceptional sections E_i, replacing a neighborhood of E_i with the complement of a smooth degree p curve in $\mathbb{CP}^2$. It is easy to check that the resulting manifold $W \#_{\cup E_i} 9\mathbb{CP}^2$ is diffeomorphic to $X_{p,0}$.

Moreover, even though the symplectic structure on W depends on an area parameter for each exceptional section E_i and hence is naturally defined only up to deformation equivalence (see below), the identification between $W \#_{\cup E_i} 9\mathbb{CP}^2$ and $X_{p,0}$ can be made to hold at the symplectic level. Indeed, if we perform the symplectic sums along E_i without any loss of symplectic volume, then up to isotopy (and scaling by a constant factor) the symplectic structure on $W \#_{\cup E_i} 9\mathbb{CP}^2$ no longer depends on the area parameters (observe that the area of E_i determines the volume of each $\mathbb{CP}^2$ summand); it is not hard to check that this symplectic structure is the same as that arising from the description of $X_{p,0}$ as a complex projective surface.

Normalize the Fubini-Study Kähler form on $Y = \mathbb{CP}^2$ so that its cohomology class is Poincaré dual to the homology class $[L]$ of a line; the natural symplectic structure ω induced on $X_{p,0}$ by the covering map f is then Poincaré dual to the homology class $[H] = [f^{-1}(L)]$.

Lemma 4.1. *The symplectic and canonical classes of $X_{p,0}$ are related by the identity $c_1(K) = \lambda_p[\omega]$ in $H^2(X_{p,0}, \mathbb{R})$, where $\lambda_p = (6p-9)/p$.*

Proof. The ramification curve R of the branched covering $f : X_{p,0} \to Y = \mathbb{CP}^2$ is the preimage under the projection to the second factor $e : X_{p,0} = Y \times_M Z \to Z$ of the ramification curve R_h of the degree p polynomial map $h : Z \to M$. The curve R_h is a smooth curve of degree $3p-3$ in $Z=\mathbb{CP}^2$; in particular, denoting by $[\ell]$ the homology class of a line in $M=\mathbb{CP}^2$, we have $p[R_h]=(3p-3)[h^{-1}(\ell)]$ in $H_2(Z,\mathbb{Z})$. Pulling back by e, we obtain the equality $p[R]=(3p-3)[(h\circ e)^{-1}(\ell)]$ in $H_2(X_{p,0},\mathbb{Z})$. Since $h\circ e=g\circ f$, and since $[g^{-1}(\ell)]=3[L]$ in $H_2(Y,\mathbb{Z})$, we conclude that $p[R]=(9p-9)[f^{-1}(L)]=(9p-9)[H]$. Since it is a general fact about branched covers of $\mathbb{CP}^2$ that $[R]=PD(c_1(K))+3[H]$, we conclude that $p(c_1(K)+3[\omega])=(9p-9)[\omega]$, or equivalently $c_1(K)=\lambda_p[\omega]$. □

It is worth noting that, because the symplectic structure on $E(1)$ depends on the choice of the volumes of the blow-up operations, the symplectic structure on W depends on the choice of the symplectic areas of the exceptional sections E_i, and is determined only up to deformation (pseudo-isotopy). The situation that we naturally want to consider is the limit as the area of E_i and consequently the symplectic volume of the copy of $\mathbb{CP}^2$ glued to W along E_i become very small; on the level of the branch curve $D\subset Y$, this means that the balls around the intersection points of the cubics $C_1,\dots,C_d$ that we delete and replace with copies of $D_h\cap B$ are very small.

4.2. **The manifolds $X_{p,k}$.** The branch curve of $q:W\to E(1)$ consists of d parallel elliptic curves $F_1,\dots,F_d$ (fibers of $E(1)$), and similarly the branch curve of $f:X_{p,0}\to\mathbb{CP}^2$ is obtained from d cubics $C_1,\dots,C_d$ in a pencil by a modification near the base points. Therefore, as discussed in §3 we can construct a Lagrangian annulus A in $E(1)$ (or $\mathbb{CP}^2$) that lifts to a Lagrangian torus T in W or $X_{p,0}$, and Luttinger surgery along T amounts to braiding the branch curve along the annulus A.

We start with the observation that the d branch points $z_1,\dots,z_d$ of the simple branched cover $h_{|S}:S\to\mathbb{CP}^1$ can be grouped into pairs of points with matching branching data; this can be done in many ways, and in fact amounts to the choice of a degeneration of S to a nodal curve with p^2 rational components intersecting in a total of $d/2$ points. In particular, we can assume that one of the components of the degenerated curve intersects only once with the others; or equivalently, we can find two branch points of $h_{|S}$, e.g. z_1 and z_2, and an arc η_0 joining them in $\mathbb{CP}^1$, such that the union of two lifts of η_0 forms a closed curve $\gamma_0\subset S$ that separates S into two components, one of genus 0 consisting of only one sheet of $h_{|S}$, and the other of genus $(p-1)(p-2)/2$ consisting of the remaining p^2-1 sheets. Equivalently, observing that W can be constructed from p^2 copies of the elliptic fibration $E(1)$ by repeatedly performing fiber sums (some of which are self-sums, to increase the genus of the base), γ_0 can also be thought of as a loop in the base S that separates one of the copies of $E(1)$ from the others.

Let a_0 be an arbitrary simple closed loop in the fiber F_1 above z_1, representing a non-zero homology class in F_1 and avoiding the 9 points where F_1 intersects the exceptional sections of $E(1)$. In fact, the choice made by Moishezon in [9] amounts to choosing a degeneration of the pencil of cubics containing $C_1,\dots,C_d$ so that each C_i becomes close to a union of three lines in $\mathbb{CP}^2$, and taking a_0 to be one of the vanishing cycles for the corresponding degeneration of the fiber F_1; but other choices for a_0 are equally suitable. As in §3, use parallel transport above the arc η_0 to construct a Lagrangian annulus $A\subset E(1)$ joining a_0 to a similar loop in the fiber F_2 above z_2. Note that we equip $E(1)$ with a symplectic form which coincides with that of Y outside of a small neighborhood of the exceptional sections; moreover, we can assume that z_1 and z_2 are arbitrarily close to each other, so that the construction is well-defined and the annulus A remains away from the exceptional sections. In fact, we could also construct A directly as an embedded Lagrangian annulus joining the cubics C_1 and C_2 in $Y = \mathbb{CP}^2$.

By construction, the annulus A lifts to an embedded Lagrangian torus T in $W - \bigcup E_i \subset X_{p,0}$, and an embedded arc $\eta \subset A$ which projects to the arc $\eta_0 \subset \mathbb{CP}^1$ lifts to an embedded loop $\gamma \subset T$ such that $\pi(\gamma) = \gamma_0 \subset S$. Let $D_{p,k} = D(A, k)$ be the singular plane curve obtained from the branch curve D of f by twisting k times along the annulus A, and let $X_{p,k} = X_{p,0}(T, \gamma, k)$ be the symplectic manifold obtained from $X_{p,0}$ by twisting k times along the loop γ in the Lagrangian torus T. By Proposition 3.1, $X_{p,k}$ is naturally a symplectic branched cover of $Y = \mathbb{CP}^2$, with branch curve $D_{p,k}$.

Although the description that we give here is very different from that given by Moishezon in [9], it is an interesting exercise left to the reader to check that the two constructions are actually identical. In fact, because the two loops γ_0 and a_0 can be viewed as vanishing cycles for degenerations of the base and fiber of π, the operation of partial conjugation of the braid monodromy described by Moishezon exactly amounts to the braiding construction described in §3.

Lemma 4.2. *The homology class $[T] \in H_2(X_{p,k}, \mathbb{Z})$ is not a torsion class. Moreover, if $p \not\equiv 0 \mod 3$ or $k \equiv 0 \mod 3$ then $[T]$ is primitive.*

Proof. By Poincaré duality, $[T]$ is a non-torsion class if and only if we can find a 2-cycle that has non-trivial intersection pairing with T; this is possible if and only if the meridian of T represents a torsion class in $H_1(X_{p,k} - T, \mathbb{Z})$.

Recall from §2 that the class $[\tilde{\mu}]$ of a meridian of T in $X_{p,k}$ can be expressed as $[\mu] + k[\gamma]$, where $[\mu]$ is the class of a meridian of T in $X_{p,0}$; since the complements of T in $X_{p,0}$ and $X_{p,k}$ are diffeomorphic, it is therefore sufficient to prove that both $[\mu]$ and $[\gamma]$ are torsion classes in $H_1(X_{p,0} - T, \mathbb{Z})$.

We first show that $[\mu]$ is trivial in $H_1(X_{p,0} - T, \mathbb{Z})$. Consider an arc ξ_0 in $\mathbb{CP}^1$ which joins the image z_0 of a singular fiber of $E(1)$ to the branch point z_1 of $h_{|S}$ and does not intersect η_0 in any other point. Starting from the singular point in the fiber above z_0 and using parallel transport along ξ_0, we can construct a (Lagrangian) disk $D \subset E(1)$, lying above ξ_0 and with boundary δ contained in the smooth fiber F_1 above z_1 (δ is the vanishing cycle associated to the chosen singular fiber and the arc ξ_0). If the point z_0 and the arc ξ_0 are chosen in a suitable way, we can assume that the intersection number of δ with a_0 in F_1 is equal to 1 (recall that by assumption a_0 represents a primitive class in $H_1(F_1, \mathbb{Z})$), and that the disk D does not intersect the exceptional sections of $E(1)$. The two lifts via $h_{|S}$ of ξ_0 which pass through the ramification point above z_1 form a single arc ξ in S that joins two of the critical values of the elliptic fibration $\pi : W \to S$ and intersects the loop γ_0 transversely in a single point. Similarly, the disk D lifts to a sphere (of self-intersection -2) in $W - \bigcup E_i$; by construction, the intersection number of this sphere with the torus T is equal to 1. Removing a complement of the intersection with T from the sphere, we have realized the meridian μ as a boundary in $X_{p,0} - T$, and therefore $[\mu] = 0$.

We next consider the loop γ, which we push away from T by moving it slightly along the fibers of the elliptic fibration $\pi : W \to S$. In fact, we can keep moving γ along the fibers until it lies in a neighborhood of one of the exceptional sections E_i. Recall that $\pi(\gamma) = \gamma_0$ bounds a disk Δ in S, corresponding to one of the sheets of $h_{|S}$; however, the monodromy of the fibration π along γ_0 is not quite trivial, but differs from the identity by a Dehn twist around each of the nine points where the fiber intersects the exceptional sections E_i. In other terms, the normal bundle to E_i, with its natural trivialization over the boundary γ_0, has degree -1 over the disk Δ. Therefore, there is an obstruction to collapsing γ inside $W - \bigcup E_i$, but γ is homologous to $-\nu$, where ν is a small meridian loop around E_i in W.

In $X_{p,0}$, a neighborhood of E_i is replaced by the complement $\mathbb{CP}^2 - S$ of a smooth plane curve of degree p. Considering a generic line in $\mathbb{CP}^2$ and removing neighborhoods of its p intersection points with S, we conclude that $-p[\nu]$ is homologically trivial in $X_{p,0} - T$, and therefore that $[\gamma]$ is a torsion element in $H_1(X_{p,0} - T, \mathbb{Z})$, which completes the proof that $[T]$ is not torsion in $H_2(X_{p,k}, \mathbb{Z})$.

Another way to look at the loop γ is to view W as a fiber sum of p^2 copies of $E(1)$, with the loop γ_0 in the base S separating one of the $E(1)$'s from the others. Therefore, $W - \bigcup E_i$ contains a subset U diffeomorphic to the complement of a fiber and of the 9 exceptional sections in the rational elliptic surface $E(1)$; the loop γ then corresponds to the meridian of the removed fiber in U (with reversed orientation). However, U can be identified with the complement of a smooth cubic in $\mathbb{CP}^2$, so $\pi_1(U) = \mathbb{Z}/3$, and therefore $3[\gamma] = 0$ in $H_1(X_{p,0} - T, \mathbb{Z})$.

If $p \not\equiv 0 \mod 3$, then we conclude that $[\gamma] = 0$, and so $[\tilde{\mu}] = [\mu] + k[\gamma] = 0$, i.e. the meridian $\tilde{\mu}$ is a boundary in $X_{p,k} - T$. Therefore we can find a 2-cycle in $X_{p,k}$ which intersects T once, i.e. $[T]$ is primitive. When p is a multiple of 3, the same argument holds provided that k is also a multiple of 3. □

4.3. **Proof of Theorem 1.1.** Our strategy to prove Theorem 1.1 is to show that the manifolds $X_{p,k}$ are not symplectomorphic to each other by using Proposition 2.4. We start with a computation of the quantity $H(\gamma, \tau_T)$ introduced in §2.2 in the case of the Lagrangian torus T and the loop γ constructed in §4.2:

Lemma 4.3. *In $X_{p,0}$, we have $H(\gamma, \tau_T) = (2p-3)/p$.*

Proof. We use Proposition 3.3. The ramification curve R of $f : X_{p,0} \to Y = \mathbb{CP}^2$ is obtained by gluing together the ramification curve of $q : W \to E(1)$, which consists of $d = 3p(p-1)$ fibers of π, with the ramification curve of a polynomial map of degree p, i.e. a smooth curve of degree $3p-3$, inside each of the nine copies of $\mathbb{CP}^2$ glued to W along the exceptional sections E_i.

Recall from above that the loop γ is homotopic inside $W - \bigcup E_i$ to a small loop $\bar{\nu}$ that is the reversed meridian to one of the exceptional sections; this deformation can be performed without crossing R (except along γ itself). Inside $X_{p,0}$, the loop $\bar{\nu}$ can also be viewed as the meridian of the smooth degree d curve S removed from $\mathbb{CP}^2$ prior to gluing with W; therefore, taking p copies of $\bar{\nu}$ we obtain the (reversed) boundary of a punctured line in $\mathbb{CP}^2 - S$, which intersects the ramification curve in $3p-3$ points. Therefore, p copies of γ bound a surface N in $X_{p,0}$ such that $I(N, R) = p - (3p-3)$ (recall that the $2p$ boundary intersections only count with coefficient $1/2$). Moreover, because the image by f of $\mathbb{CP}^2 - S$ is contained in a small ball around one of the base points of the pencil of cubics on Y, one easily checks that the homology class $[f_* N]$ is trivial. Therefore we have $p\, H(\gamma, \tau_T) = -I(N, R) = 2p-3$, which gives the result.

Alternately, remember that W is the fiber sum of p^2 copies of $E(1)$, and so $W - \bigcup E_i$ contains a subset U diffeomorphic to the complement of a fiber and of the 9 exceptional sections in $E(1)$, corresponding to one sheet of the branched cover $q : W \to E(1)$. Also, γ corresponds to the meridian of the removed fiber in U. Therefore, three copies of γ bound a punctured line $\mathcal{N}$ in U, which does not intersect the ramification curve anywhere except on the boundary, so $I(\mathcal{N}, R) = 3$; moreover, one easily checks that $f_*\mathcal{N}$ has degree 1 in $\mathbb{CP}^2$. Recalling that since $Y = \mathbb{CP}^2$ we have $c_1(K_Y) = -3[\omega_Y]$, we obtain $3\,H(\gamma, \tau_T) = (\lambda_p + 3) - 3 = \lambda_p = (6p-9)/p$, which again gives the result. □

Proposition 4.4. *For a fixed value of $p \not\equiv 0 \mod 3$, the manifolds $X_{p,k}$ $(k \geq 0)$ are pairwise non-symplectomorphic. The same result remains true for $p \equiv 0 \mod 3$ if we restrict ourselves to values of k that are multiples of 3.*

Proof. The manifolds $X_{p,k}$ are distinguished by the periods of the cohomology class $\alpha_{p,k} = c_1(K_{X_{p,k}}) - \lambda_p[\omega_{X_{p,k}}]$ evaluated on elements of $H_2(X_{p,k}, \mathbb{Z})$. Indeed, by Proposition 2.4 and Lemma 4.3 we have $\alpha_{p,k} = k(2p-3)/p\ PD([T])$, and by Lemma 4.2 the homology class $[T] \in H_2(X_{p,k}, \mathbb{Z})$ is primitive, so the evaluation of $\alpha_{p,k}$ on integer homology classes yields all integral multiples of $k(2p-3)/p$. □

In fact, the difference between the branched covering maps $f_{p,k} : X_{p,k} \to Y = \mathbb{CP}^2$ can be seen on a purely topological level, without considering symplectic structures. Indeed, defining $[L]$ to be the homology class of a line in Y, the cohomology class of the symplectic form on $X_{p,k}$ is the Poincaré dual of $[f_{p,k}^{-1}(L)]$; and the canonical class of $X_{p,k}$ is related to the homology class of the ramification curve R of $f_{p,k}$ by the formula $[R_{p,k}] = c_1(K_{X_{p,k}}) + 3[f_{p,k}^{-1}(L)]$. Therefore, the cohomology class $\alpha_{p,k}$ is in fact a smooth invariant of the branched covering structure, and the maps $f_{p,k} : X_{p,k} \to \mathbb{CP}^2$ are not even smoothly isotopic as branched covers.

The branch curves $D_{p,k}$ are symplectic curves of degree $m = 3d = 9p(p-1)$ in $\mathbb{CP}^2$, and by construction they all have the same numbers of nodes and cusps (in fact there are $27(p-1)(4p-5)$ cusps and $27(p-1)(p-2)(3p^2+3p-8)/2$ nodes, as can be checked e.g. using the Plücker formulas, cf. [9]).

In order to conclude that the curves $D_{p,k}$ are not smoothly isotopic, we need to study the possible p^2-fold covers of $\mathbb{CP}^2$ branched along $D_{p,k}$. These are given by homomorphisms from the fundamental group $\pi_1(\mathbb{CP}^2 - D_{p,k})$ to the symmetric group S_{p^2}, satisfying certain compatibility relations. Because $\pi_1(\mathbb{CP}^2 - D_{p,k})$ is finitely generated and S_{p^2} is a finite group, there are only finitely many such morphisms, i.e. $\mathbb{CP}^2$ admits only finitely many p^2-fold covers branched over $D_{p,k}$. Because we have infinitely many inequivalent branched covers $X_{p,k}$, we conclude that infinitely many of the curves $D_{p,k}$ are not smoothly isotopic. This completes the proof of Theorem 1.1.

Remark 4.5. The number of p^2-fold covers of $\mathbb{CP}^2$ branched above $D_{p,k}$ can be bounded explicitly by observing that $\pi_1(\mathbb{CP}^2 - D_{p,k})$ is generated by $m = \deg D_{p,k}$ small meridian loops, all of which must be mapped to transpositions in S_{p^2}. However, the structure of $\pi_1(\mathbb{CP}^2 - D_{p,k})$, as described by Moishezon in [9] using braid monodromy techniques, implies that there is in fact only one possible branched covering structure for each of the curves $D_{p,k}$ as soon as $p \geq 3$. It then follows immediately from the non-isotopy of the branched covers $f_{p,k} : X_{p,k} \to \mathbb{CP}^2$ that the curves $D_{p,k}$ are all different.

Remark 4.6. The fact that the homology class $[T]$ fails to be primitive when $p \equiv 0 \mod 3$ and $k \not\equiv 0 \mod 3$ is directly related to the first homology groups of the manifolds $X_{p,k}$. Indeed, whereas it can be easily checked that $H_1(X_{p,k},\mathbb{Z})$ is trivial whenever p is not a multiple of 3, it appears that $H_1(X_{p,0},\mathbb{Z})=\mathbb{Z}/3$ (generated e.g. by $[\gamma]$ or by $[\nu]$) when $p\equiv 0 \mod 3$; as a consequence, when p is a multiple of 3 the group $H_1(X_{p,k},\mathbb{Z})$ is isomorphic to $\mathbb{Z}/3$ for $k\equiv 0 \mod 3$ and trivial otherwise.

Remark 4.7. The construction presented here can be modified in various manners, e.g. by starting with other pairs of branched covers $g:Y\to M$ and $h:Z\to M$, or by twisting the branch curves in different ways. This potentially leads to many more examples of non-isotopic singular symplectic curves in symplectic 4-manifolds. However, it remains unknown whether it is possible to construct examples of non-isotopic smooth connected symplectic curves representing a homology class of positive square inside a given compact symplectic 4-manifold.

Remark 4.8. The relation between our strategy to prove Theorem 1.1 (by comparing the canonical and symplectic classes of the branched covers $X_{p,k}$) and the strategy used by Moishezon in [9] (by comparing the fundamental groups $\pi_1(\mathbb{CP}^2-D_{p,k})$) becomes more apparent if one considers the observations and conjectures made in [3] about the structure of fundamental groups of branch curve complements. Indeed, Moishezon's argument relies on a computation showing that, while the fundamental group $\pi_1(\mathbb{CP}^2-D_{p,0})$ is always infinite, the groups $\pi_1(\mathbb{CP}^2-D_{p,k})$ are finite as soon as $p\geq 3$ and $k\neq 0$, and have different ranks for different values of k. On the other hand, Conjecture 1.6 in [3] states that, at least for "sufficiently ample" simply connected branched covers of $\mathbb{CP}^2$, the fundamental group of the complement of the branch curve is directly related to the numerical properties of the symplectic and canonical classes. In particular, it follows from Theorem 1.5 in [3] that, if the canonical and symplectic classes are proportional to each other, then the fundamental group of the branch curve complement must be infinite; the converse implication is conjectured to hold as well (assuming again that the branched cover is simply connected and "sufficiently ample"). The fact that Theorem 1.1 can be proved either by considering fundamental groups of complements or numerical relations in the homology of the branched covers can be considered as additional evidence for these conjectures.

Acknowledgements. The authors wish to thank Ron Stern, Ivan Smith, Tom Mrowka, Fedor Bogomolov and Miroslav Yotov for helpful comments. The first and third authors wish to thank respectively Ecole Polytechnique and Imperial College for their hospitality.

References

[1] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$, Inventiones Math. **139** (2000) 551–602.

[2] D. Auroux, L. Katzarkov, *Branched coverings of* $\mathbb{CP}^2$ *and invariants of symplectic* 4*-manifolds*, Inventiones Math. **142** (2000) 631–673.

[3] D. Auroux, S. K. Donaldson, L. Katzarkov, M. Yotov, *Fundamental groups of complements of plane curves and symplectic invariants*, preprint, math.GT/0203183.

[4] J. F. Barraud, *Courbes pseudo-holomorphes équisingulières en dimension 4*, Bull. Soc. Math. Fr. **128** (2000) 179–206.

[5] Y. Eliashberg, L. Polterovich, *New applications of Luttinger's surgery*, Comment. Math. Helv. **69** (1994) 512–522.

[6] R. Fintushel, R. Stern, *Symplectic surfaces in a fixed homology class*, J. Diff. Geom. **52** (1999) 203–222.

[7] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Inventiones Math. **82** (1985) 307–347.
[8] K.M. Luttinger, *Lagrangian tori in* $\mathbb{R}^4$, J. Diff. Geom. **42** (1995) 220–228.
[9] B. Moishezon, *The arithmetic of braids and a statement of Chisini*, Geometric Topology (Haifa, 1992), Contemp. Math. **164**, Amer. Math. Soc., Providence, 1994, 151–175.
[10] B. Moishezon, *Topology of generic polynomial maps in complex dimension two*, preprint.
[11] B. Siebert, G. Tian, *On the holomorphicity of genus two Lefschetz fibrations*, preprint.
[12] I. Smith, *Symplectic submanifolds from surface fibrations*, Pacific J. Math. **198** (2001) 197–205.
[13] M. Teicher, *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math. **62** (part 1), Amer. Math. Soc., 1997, pp. 127–150.
[14] M. Teicher, *The fundamental group of a* $\mathbb{CP}^2$ *complement of a branch curve as an extension of a solvable group by a symmetric group*, Math. Ann. **314** (1999) 19–38.
[15] S. Vidussi, *Nonisotopic symplectic tori in the fiber class of elliptic surfaces*, preprint.

Department of Mathematics, M.I.T., Cambridge MA 02139, USA
E-mail address: auroux@math.mit.edu

Department of Mathematics, Imperial College, London SW7 2BZ, U.K.
E-mail address: s.donaldson@ic.ac.uk

Department of Mathematics, University of California, Irvine, CA 92697, USA
E-mail address: lkatzark@math.uci.edu