Y. Soibelman – 23/1/09 – DT invts, quivers and cluster transformations

- $\mathcal{C}$ 3CY category $/k$, char $k = 0$

  $ch: K_0(\mathcal{C}) \to \Gamma$

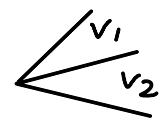
  \+ stability data + orientation data

  $\rightsquigarrow A_V^{mot} \in$ quantum torus (see Kontsevich's talks)

  $\uparrow$ basis $\hat{e}_\gamma$, $\gamma \in \Gamma$

  $$\hat{e}_{\gamma_1}\hat{e}_{\gamma_2} = q^{1/2\langle\gamma_1,\gamma_2\rangle}\hat{e}_{\gamma_1+\gamma_2}, \quad \langle\,,\rangle : \Gamma \otimes \Gamma \to \mathbb{Z}$$

  A_V^{mot} satisfies factorization property:



  $V = V_1 \cup_\partial V_2$

  $\Rightarrow A_V^{mot} = A_{V_1}^{mot} \cdot A_{V_2}^{mot}$

- $\mathrm{Ad}\, A_V^{mot}(x) := A_V^{mot}\, x \,(A_V^{mot})^{-1}$

  <u>Conj</u>: for $\sqrt{q} \to -1$, this converges to a Poisson automorphism of $T_\Gamma = \mathrm{Hom}(\Gamma, \mathbb{G}_m)$

  $\uparrow \mathbb{C}^*$

- $$A_V = \prod_{\ell \subset V}^{\rightarrow} \prod_{Z(\gamma)\in \ell} T_\gamma^{\Omega(\gamma)}, \qquad T_\gamma(e_\mu) = (1-e_\gamma)^{\langle\gamma,\mu\rangle} e_\mu$$

  $\{\Omega(\gamma)\}$ = "numerical" DT-invariants

  $\langle .,. \rangle$ on $\Gamma \rightsquigarrow T_\Gamma = \mathrm{Hom}(\Gamma, \mathbb{G}_m)$ with Poisson bracket

  $$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle}\langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$$

<u>Cluster transformations</u>:

consider $(\Gamma \oplus \Gamma^\vee, \langle .,. \rangle)$ symplectic lattice

$\mathcal{C}$ 3d CY category, $\mathcal{E} = (E_i)_{i\in I}$ spherical generators $\left(\mathrm{Ext}^\bullet(E_i, E_i) \simeq H^\bullet(S^3)\right)$

Def: The collection $\mathcal{E}$ is called <u>cluster</u> if $\mathrm{Ext}^\bullet(E_i, E_j)$ $(i \neq j)$ is concentrated in a single degree which is either 1 or 2.

Then: $a_{ij} = -\chi(\mathrm{Ext}^\bullet(E_i, E_j))$ $\begin{cases} >0 & \text{if we have } \mathrm{Ext}^1 \\ <0 & \text{—— " —— } \mathrm{Ext}^2 \\ =0 & \text{—— " —— neither} \end{cases}$

$\Rightarrow$ picture of Ext^1's forms a <u>quiver with potential (Q, W)</u>

$$Q: \quad i \xrightarrow{a_{ij} \text{ if } a_{ij} > 0} j$$

Cluster condition $\Rightarrow$ no 1-cycles or 2-cycles

$$W_E(\alpha) = \sum_{n \geq 2} \frac{1}{n+1} \langle m_n(\alpha, \dots, \alpha), \alpha \rangle$$

<u>Associated algebra</u>: $A_W = \mathbb{C}Q / \langle \partial W \rangle$

$\mathbb{C}Q$: path algebra of quiver

∂W: cyclic words in path algebra.

<u>Thm:</u>

$(\mathcal{C}, \mathcal{E})$ up to A_∞-equivalence preserving $\mathcal{E}$

3d CY cat + cluster collection

$\Updownarrow$ <u>1-1 correspondence</u>

(Q, W) up to gauge equiv^{ce}.

quiver w/ potential, minimal (ie. no quadratic terms in W)

- <u>Mutation</u> of (Q, W) (Zelevinsky, ...)

  $I = Q^0$ vertices, choose a vertex $0 \in I$

  Say $i < 0$ if $a_{i0} > 0$ ie. $i \rightarrow 0$

  $i > 0$ if $a_{i0} < 0$ ie. $0 \rightarrow i$

At category level,

Def: A <u>mutation at E_0</u> is the transformation $\mathcal{E}=(E_i)\longrightarrow \mathcal{E}'=(E_i')$ defined by
$$R_{E_0}(E_0)=E_0[-1]$$
$$R_{E_0}(E_i)=E_i \quad \text{for } i<0$$
$$R_{E_0}(E_i)=\mathrm{Cone}\left(E_0\otimes \mathrm{Ext}^1(E_0,E_i)\to E_i\right),\quad i>0$$

<u>Can show</u>: outside of a countable union of closed subvarieties in the space of $\{(Q,W)\}$, this preserves the cluster condition.

In K-theory: $\quad K_0(\mathcal{C})=\mathbb{Z}^I=\Gamma \ni [E_i]=q_i,\ [E_i']=q_i'$

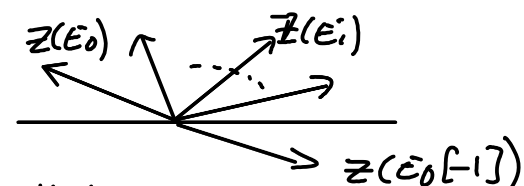
$$\begin{cases} q_i'=q_i & \text{for } i<0\\ q_0'=q_0 & \\ q_i'=q_i-\langle q_0,q_i\rangle q_0=q_i+a_{0i}q_0, & i>0\end{cases}$$

or on matrix (a_{ij}):
$$\begin{cases} a_{ij}'=a_{ij}+a_{i0}a_{0j} & \text{if } i<0<j\\ a_{i0}'=-a_{i0},\ a_{0i}'=-a_{0i} & \\ a_{ij}'=a_{ij} & \text{otherwise}\end{cases}$$

[Fomin-Zelevinsky, Cluster algebras 1-4
Fock-Goncharov, Gekhtman-Schapiro-Vainstein]

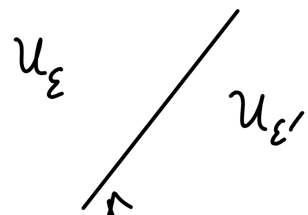
$(\mathcal{C},\mathcal{E})\qquad \mathcal{E}=(E_i)_{i\in I}\ \rightsquigarrow\ Z_i=Z(ch(E_i))\in\mathbb{H}$ upper half plane

$\mathcal{U}_{\mathcal{E}}\subset \mathrm{Stab}(\mathcal{C})=$ stab. condo. st.



$\mathcal{U}_{\mathcal{E}'}\subset \mathrm{Stab}(\mathcal{C})$ similarly for mutated collection

These chambers have no common interior points; in fact



separated by a wall of 2^{nd} type, where $Z(E_0) \in \mathbb{R}_{<0}$

(ie: E_i remain stable, but E_0 moves out of the t-structure).

$$R_{\Gamma\oplus\Gamma^\vee, q} \xrightarrow{\sim} \mathcal{D}(R_{Q,q})$$

$$e_i,\ e_i^\vee \quad \text{with} \quad \hat{e}_i \hat{e}_j = q^{a_{ji}} \hat{e}_j \hat{e}_i$$

$$\hat{e}_i^\vee \hat{e}_j = q^{-\delta_{ij}} \hat{e}_j \hat{e}_i^\vee$$

$$\rotatebox{90}{$\in$}$$

$$A_V^{mot} \rightsquigarrow E_Q$$

$V =$ "big" strict sector

($V \subsetneq \mathbb{H}$ including all $Z(E_i)$'s)

$$Ad_{E_Q} \xrightarrow[\sqrt{q} \to -1]{} \text{Poisson autom. of } T_{\Gamma\oplus\Gamma^\vee}$$

<u>Properties of E_Q:</u>

- for $|I| = 1$, $E_Q = E(ch(\mathcal{E}_{i_0}))$ where

$$E(x) = \prod_{n\geq 0} \left(1 + q^{\frac{2n+1}{2}} x\right)^{-1}$$

$$E(qx) = (1 + q^{1/2} x)\, E(x)$$

- $I = I_1 \sqcup I_2$

Q_1 vertices I_1 $\quad$ Q_2 vertices . $\quad$ 2 distinct quivers $\Rightarrow E_Q = E_{Q_1} \cdot E_{Q_2}$

For a mutation $(\mathcal{C}, \mathcal{E}) \xrightarrow[\mu_{E_0}]{} (\mathcal{C}, \mathcal{E}')$

Assumption: $\mathrm{Ad}\, A_{\mathcal{E},q}$ is a birational transformation of the skew-field associated to the quantum torus

Let $\Phi_{\mathcal{E}}(x) = (\mathrm{Ad}_{\mathcal{E},q} \circ \tau)(x)$ acting on quantum torus

$\gamma \to -\gamma$

on quiver side, Φ_Q birational transf. of the skew-field of $\mathcal{D}(R_{Q,q})$

Prop: $\mathrm{Ad}^{-1}_{\mathbb{E}(\hat{e}_{ch(E_0)})} \circ \Phi_{\mathcal{E}} \circ \mathrm{Ad}_{\mathbb{E}(ch(\hat{e}_0))} = \Phi_{\mathcal{E}'}$

Corollary: $\exists\, C_{Q,0}$ s.t. $C_{Q,0} \circ \Phi_Q = \Phi_{Q'} \circ C_{Q,0}$

★ Formulas for $C_{Q,0}$:

the $C_{Q,0}$ are given by standard cluster transformations

$$\mu_8 : \begin{cases} y_i \longmapsto \dfrac{y'_i}{(1 - 1/y'_0)^{\alpha_{i0}}} & \text{for } i<0 \\ y_0 \longmapsto 1/y'_0 & \\ y_i \longmapsto y'_i \,(1 - 1/y'_0)^{a_{0i}} & \text{for } i>0 \end{cases}$$

similar formulas for x_i.

Point: motivic DT invts are in fact invariants of the derived category, but different t-structures give different formulas for them.

Cluster collections $\rightsquigarrow$ sets of coordinates on torus
Changes of co-ordinates are given by $C_{Q,0}$, which turn out to be cluster transformations.

- Cluster transformations also appear in recent work of Gaitto-Moore-Neitzke:

study moduli space of rk 2 Higgs bundles / punctured curve C with monodromy at punctures provided by a parameter $z \in \mathbb{C}$

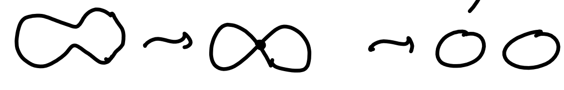
WKB theory $\rightsquigarrow$ triangulation of the spectral curve $\mathcal{S}: y^2 = P(x)$ & hence of the given curve C

Fock-Goncharov: $\rightsquigarrow$ construct cluster variety structure on $\mathcal{M}_{Higgs}$ from such data

$\Bigg($
- central charge: $\int y\, dx$ on $H_1(\mathcal{S}, \mathbb{Z})$
- analogue of Slags (BPS states): "WKB curves" on $\mathcal{S}$:= curves w/ maximal decay of WKB approx.

$\Bigg)$

Varying parameter z $\rightsquigarrow$ wall-crossing, since WKB curves change.
(acquire singularities then change homology class)



The corresponding transform$^{\underline{ns}}$ on $\mathcal{M}_{Higgs}$ are again given by wall-crossing theory $\rightsquigarrow$ cluster transformations

Voros: (long ago) general WKB theory for complex Schrödinger operator
$\rightarrow$ similar picture of WKB curves & wall-crossing.