Yongbin Ruan – 20/1/09 – Singularity theory and integrable hierarchies

GW theory $\xleftrightarrow[\text{mirror}]{}$ integrable hierarchies

(first: Kontsevich 1991, point $\leftrightarrow$ KdV)

1) $\Lambda^\infty V$ and KP (Kyoto school)

$W \subset \mathbb{C}^n$ k-dim^{l} subspace, $v_1 \dots v_k$ basis $\rightsquigarrow$ $v_1 \wedge \dots \wedge v_k \in \Lambda^k \mathbb{C}^n$

$\Rightarrow Gr(k,n) \xhookrightarrow[i]{} \mathbb{P}(\Lambda^k \mathbb{C}^n)$
Plücker embedding

(1) $GL(n, \mathbb{C})$ acts on $G(k,n)$

In fact it acts on $\mathbb{P}(\Lambda^k \mathbb{C}^n)$ and $i(G(k,n))$ is a single orbit)

(2) $i(G(k,n))$ is def^{d} by quadratic relations

• Now, ∞-dim^{l} case: $V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} z^j$

semi-∞ wedge: $\Lambda^\infty V$: basis $e_{i_0} \wedge e_{i_1} \wedge \dots$ / $i_n = -n$ for $n \gg 0$

vacuum $|0\rangle = e_0 \wedge e_{-1} \wedge \dots$

Facts: (1) $\Lambda^\infty V$ is a highest weight rep^n of central ext of gl_∞

(2) GL_∞ acts on V & hence on $\Lambda^\infty V$

(3) $\Omega = GL_\infty \cdot |0\rangle$ single orbit is grassmannian defined by "infinite Plücker relation"

Boson-Fermion correspondence: $\Lambda^\infty V \simeq \mathbb{C}[x_1, x_2, \dots]$

$\tau \in \Omega \leftrightarrow$ power series in $x_1, x_2, \dots$

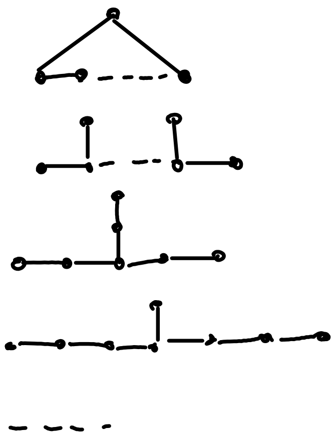
"infinite Plucker eqn" $\leftrightarrow$ Hirota eq^n of KP-hierarchy

2) Drinfeld-Sokolov / Kac-Wakimoto hierarchies:

Input data: $\left.\begin{array}{l} \mathfrak{g} \quad \infty\text{-dim}^{l} \text{ Lie alg} \\ V \text{ integrable highest weight rep.} \\ R \text{ vertex op. construction of } V \end{array}\right\} \Rightarrow$ integrable hierarchy

Affine Kac-Moody algebras

$A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, ------



$\Rightarrow$ Drinfeld-Sokolov & Kac-Wakimoto hierarchies

Hierarchies in geometry (Witten)

- **$A_1^{(1)}$ or KdV-hierarchy:**

$\overline{M}_{g,n} = \{$ isom-class of nodal stable curves $\}$

$L_i \to \overline{M}_{g,n}$ with fiber $(L_i)_{(C,x_i)} = T_{x_i}^* C$; $\psi_i = c_1(L_i)$

$$\rightsquigarrow \langle \tau_{l_1} \dots \tau_{l_n} \rangle_g := \int_{\overline{M}_{g,n}} \psi_1^{l_1} \wedge \dots \wedge \psi_n^{l_n}$$

<u>generating function</u>:

$$\mathcal{F}_g(t_0, t_1, t_2, \dots) = \sum_{\substack{n \geq 0 \\ l_1 \dots l_n}} \frac{t_{l_1} \dots t_{l_n}}{n!} \langle \tau_{l_1} \dots \tau_{l_n} \rangle_g$$

$$\mathcal{D} = \exp\Big(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_g \Big)$$

<u>Witten conj (Kontsevich)</u>: $\mathcal{D}$ is a tau-function for KdV-hierarchy

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<u>generalize this?</u> Kontsevich-Manin: <u>cohomological field theory</u> :=

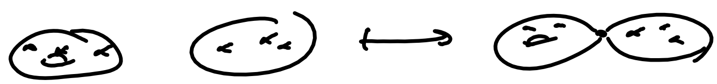
- state space $\mathcal{H}$ with unit $\mathbb{1}$ & pairing
- a class $\Lambda_{g,k} : \mathcal{H}^{\otimes k} \to H^*(\overline{M}_{g,k}; \mathbb{Q})$

- intersection number

$$\langle \tau_{\ell_1}(\alpha_1) \cdots \tau_{\ell_k}(\alpha_k) \rangle_{\Lambda_{g,k}} = \int_{\overline{M}_{g,k}} \psi_1^{\ell_1} \cdots \psi_k^{\ell_k} \, \Lambda_{g,k}(\alpha_1 \ldots \alpha_k)$$

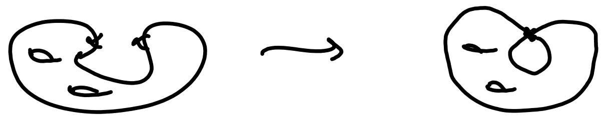
$$\mathcal{F}_{\Lambda_{g,k}}, \ \mathcal{D}_{\Lambda_{g,k}} \quad \text{generating series}$$

- <u>axiom:</u> $\Lambda_{g,k}$ behaves naturally for pullback via



$$\overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1} \rightarrow \overline{M}_{g_1+g_2,k_1+k_2}$$

and



$$\overline{M}_{g,k+2} \longrightarrow \overline{M}_{g+1,k}$$

<u>Witten's generalization:</u>

- introduce a new PDE assoc. to a quasihomogeneous singularity

$$W : \mathbb{C}^N \to \mathbb{C}, \quad W(\lambda^{q_1} x_1, \ldots, \lambda^{q_N} x_N) = \lambda W(x_1 \ldots x_N) \quad \forall \lambda \in \mathbb{C}^*$$

for some $q_1 \ldots q_N \in \mathbb{Q}$

$$G \subset G^{max} := \{ (u_1 \ldots u_N) \in (\mathbb{C}^*)^N / \ W(u_1 x_1, \ldots, u_N x_N) = W(x_1, \ldots x_N) \}$$

<u>Witten eqn:</u> $\boxed{\bar{\partial} s + \overline{\partial_i W(s_1 \ldots s_N)} = 0}$

eqn for $s_i \in \Gamma(L_i)$, L_i orbifold line bundles$/C$

C moduli Riem. surface

- $\overline{M}_{W,G}$ moduli space of solutions to Witten's eqn $\Rightarrow$

$\Rightarrow$ generating series $\mathcal{F}_g^{W,G}$, $\mathcal{D}^{W,G}$

<u>Conj:</u> If W is ADE-singularity then $\mathcal{D}^{W,\langle J \rangle}$ is a tau-function of Drinfeld-Sokolov/Kac-Wakimoto ADE hierarchy.

here $\langle J \rangle = \langle \mathrm{diag}(\exp(2\pi i q_1) \dots \exp(2\pi i q_N)) \rangle$

- A solution of Witten's conj. (joint w/ Huijun Fan & Tyler Jarvis)

Hard part: construct $\overline{M}_{W,G}$ & virt. fund cycle

(involves: compactif^n, Fredholm theory, asymptotic behavior, constr^n of virtual cycle, dependence on parameter)

Then: Thm:

- Witten conj. holds for A_n sing. (Faber-Shadrin-Zvonkine 2007)
- —— " —— for $E_{6,7,8}$ (F-J-R.)
- Witten conj doesn't hold for D_n !!! (F-J-R.).

Thm (FJR)

- For D_n, $\mathcal{D}^{W,\langle J \rangle}$ satisfies A_{2n-3} hierarchy
- For $D_N^T = x^{n-1}y + y^2$, $\mathcal{D}^{W,\langle J \rangle}$ satisfies D_n-hierarchy

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Mirror symmetry for integrable hierarchies: $\Rightarrow$ LG-mirror symmetry

$A_n = x^{n+1}$, $D_n = x^{n-1} + xy^2$, $E_6 = x^3 + y^4$, $E_7 = x^3 + xy^3$, $E_8 = x^3 + y^5$

Invertible singularity :=

- # variables = # monomials
- then $\exists$ matrix $A_W = (a_{ij})$ s.t. $W = \sum_i \left(\prod_j x_j^{a_{ij}} \right)$
  (entries = exponents!)
  (Berglund-Hübsch)
- say sing. is invertible if A_W is invertible.

$A_{A_n} = [n+1]$, $A_{D_n} = \begin{bmatrix} n-1 & 1 \\ 0 & 2 \end{bmatrix}$, $A_{E_{6/8}} = \begin{bmatrix} 3 & 0 \\ 0 & 4/5 \end{bmatrix}$, $A_{E_7} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

Then MS exchanges $W \longleftrightarrow W^T$

corresponding to $A_W \longleftrightarrow A_W^t$ transpose

• ADE-sings. are self-transpose, except $D_n \leftrightarrow D_n^T \cong A_{2n-3}$.

Also, group:

$$\begin{array}{ccc} G_{max} & \longleftrightarrow & \{1\} \\ \cup & & \cap \\ \langle J \rangle & \longleftrightarrow & SL = \{(u_1, \dots u_N) / \prod u_i = 1\} \end{array}$$

$J = \mathrm{diag}(e^{2\pi i q_j})$

Dual group (Krawitz): $\gamma \in G \longleftrightarrow \alpha \in G^T$ satisfies relation $\gamma A^{-1} \alpha \in \mathbb{Z}$

Thm: The Quantum ring of (W, G) is isomorphic to the Milnor ring of (W^T, G^T)

As a particular case,

Witten eqn for $(W, G^{max}) \leftrightarrow (W^T, \{1\})$ Milnor ring

$g=0$ theory $\Leftrightarrow$ miniversal deform^{n} of sing.

can express in the language of semisimple Frobenius mflds (Dubrovin-Zhang-Givental-Teleman)

Conjecture — beyond ADE:

For any invertible singularity W, the A-model theory (W. eqn) of (W, G_{max}) satisfies W^T-hierarchy (assoc^d to Dynkin diagram of W^T).

NB: Witten's eqn behaves similarly to σ-model with target of $c_1 = 0$, $\dim = \hat{c}_W = \sum_{i=1}^{N} (1 - 2q_i)$

In fact, $\{W = 0\} \subset \mathbb{P}(c_1, \dots, c_N)$ defines a CY orbifold $\Leftrightarrow \sum q_i = 1$.

proportional to $q_1 \dots q_N$

Other important application: LG/CY correspondence

Conj: A-model singularity theory of $(W, \langle J \rangle)$ $\iff$ GW-theory of $X_W = \{W=0\}$

Witten eqn $\iff$ holom. curve eqn.

$\bar{\partial} s_i + \overline{\partial_i W(s_1 .. s_N)} = 0$ $\qquad$ $\bar{\partial} f = 0$

more generally, $(W, G) \iff X_W / (G/\langle J \rangle)$ orbifold.