S. Cautis – 20/1/09 – Equivalences and sl_2 actions

<u>Motivation:</u> <u>Conj:</u> any birational map $Y \dashrightarrow Y'$ between hyperkähler varieties should factor as a composition of stratified Mukai flops

Study: <u>Q^n:</u> construct an equivalence $D(Y) \xrightarrow{\sim} D(Y')$ if Y, Y' differ by a stratified Mukai flop

<u>Recall:</u> Mukai flop:

$$\begin{tikzcd} & W \arrow[dl, "\text{blowup zero section}"'] & \simeq & W' \arrow[dr, "\text{blowup zero section}"] & \\ T^*\mathbb{P}V & & & & T^*\mathbb{P}V' \end{tikzcd}$$

or more generally, <u>stratified</u> Mukai flop:

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ T^*Gr(k,N) & & T^*Gr(N-k,N) \end{tikzcd}$$

<u>Namely:</u> $T^*Gr(k,N) = \left\{ (x,V) / 0 \subset V^k \subset \mathbb{C}^N,\ x \in End(\mathbb{C}^N),\ x(\mathbb{C}^N) \subset V,\ x(V) = 0 \right\}$

(ie. V + endom. which is strictly upper triangular for flag $0 \subset V \subset \mathbb{C}^N$)

Now: $W = \left\{ (x, V, V') / 0 \subset V^k \subset V'^{N-k} \subset \mathbb{C}^N \right\}$ [assume $k \leq \frac{N}{2}$]

$$\begin{tikzcd} W \arrow[d, "\pi_1"', "\text{forget } V'"] \arrow[dr, "\pi_2"', "\text{forget } V"] & \\ Y = T^*Gr(k,N) = \left\{ (x,V) / 0 \subset V^k \subset \mathbb{C}^N \right\} \arrow[dr, "p_1"'] & Y' = T^*Gr(N-k,N) = \left\{ (x,V') / 0 \subset V' \subset \mathbb{C}^N \right\} \arrow[d, "p_2"] \\ & B = \left\{ x / x^2 = 0,\ \dim \ker x \geq N-k \right\} \end{tikzcd}$$

These maps are all <u>birational</u>

<u>e.g.</u> for π_1: generically $Im(x) = V$, so $\dim \ker(x) = N-k$; in that case (x,V) determine $V' = \ker(x)$

similarly for others: generically x determines $V = im(x)$, $V' = \ker(x)$.

Guess #1: $\pi_{2*}\pi_1^*$ is an equivalence?

This is not true (Namikawa, Kawamata)

Guess #2: use, instead of W,

$$\begin{tikzcd} & Y \times_B Y' \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\ Y & & Y' \end{tikzcd}$$

: $\pi'_{2*} \circ \pi_1'^{*}$ equivalence?

Namikawa-Kawamata: for $k=1$ this is an equivalence

but... Namikawa: this fails for $T^*Gr(2,4)$

$\rightarrow$ dealt with by Kawamata.

Recall: rep of $SL(2)$ break up into weight spaces

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{tikzcd} V_{-N} \arrow[r, bend left, "e"] & V_{-N+1} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & \cdots \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V_N \arrow[l, bend left, "f"] \end{tikzcd}$$

& $\exists$ natural isom. $V_{-\lambda} \xrightarrow[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}]{} V_\lambda$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{e^\lambda}{\lambda!} - \frac{e^{\lambda+1} f}{(\lambda+1)!\,1!} + \frac{e^{\lambda+2} f^2}{(\lambda+2)!\,2!} + \cdots$$

Categorify this:

Let $D_\lambda := D(T^*Gr(l,N))$ where N fixed, $N - 2l = \lambda$

Then want functors

$$\begin{tikzcd} D(T^*Gr(l+1,N)) \arrow[r, bend left, "E"] & D(T^*Gr(l,N)) \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"] \end{tikzcd}$$

- E is given by

$$W = \{(x, V, V') : 0 \subset V^l \subset V'^{\,l+1} \subset \mathbb{C}^N\}$$

$$\begin{tikzcd} & W \arrow[dl, "\text{forget } V"'] \arrow[dr, "\text{forget } V'"] & \\ T^*Gr(l+1,N) & & T^*Gr(l,N) \end{tikzcd}$$

Similarly for F.

Relation $ef - fe = h$ categorifies to: $E \circ F = F \circ E \oplus \mathrm{Id}^{\oplus \lambda}$
(for $\lambda > 0$)

More generally, $E^{(m)} : D(T^* Gr(\ell+m, N)) \to D(T^* Gr(\ell, N))$

via $W = \{(x, V, V') \mid 0 \subset V^{\ell} \subset V'^{\,\ell+m} \subset \mathbb{C}^N\}$

categorifies $e^{(m)} = \frac{e^m}{m!}$

We have relation $E^{(m)} \circ E = E^{(m+1)} \otimes_{\mathbb{C}} H^*(\mathbb{P}^m)$

categorifies $e^{(m)} \cdot e = \frac{e^{m+1}}{m!} = (m+1)\, e^{(m+1)}$

Now we can build the terms $E^{(\lambda)}$, $E^{(\lambda+1)} \circ F^{(1)}$, $E^{(\lambda+2)} \circ F^{(2)}$, ...
needed to categorify $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e^{(\lambda)} + e^{(\lambda+1)} f + e^{(\lambda+2)} f^{(2)} + \cdots$

Construct a <u>complex</u> $(*)$ $E^{(\lambda)} \to E^{(\lambda+1)} \circ F \to E^{(\lambda+2)} \circ F^{(2)} \to \cdots$

$\to$ each of these functors has a kernel which is a <u>sheaf</u> on the appropriate W, so we construct the kernel by constructing natural maps between the relevant kernels, which come from the fact that E, F are mutually <u>adjoints</u>.

Now, consider $\mathrm{Cone}(E^{(\lambda)} \to E^{(\lambda+1)} \circ F \to E^{(\lambda+2)} \circ F^{(2)} \to \cdots) =: T(\lambda)$.

This $T(\lambda)$ gives the desired equivalence for the stratified flop.

<u>Thm</u> (Cautis–Kamnitzer–Licata):

1. If you have a <u>strong</u> categorified $\mathfrak{sl}_2$-action then you can construct the complex $(*)$ and $\mathrm{Cone}(*) : D(\lambda) \to D(-\lambda)$

Thm (cont'd)

2. $E^{(i)}$, $F^{(j)}$ as above form a strong categorified action, and yield $\operatorname{Cone}(*) : D(T^*Gr(k,N)) \xrightarrow{\cong} D(T^*Gr(N-k,N))$

<u>Point:</u> <u>strong</u> sl_2-action has not only $E \circ F \simeq F \circ E \oplus Id^{\lambda}$, but also extra natural transformations

$$X : E[-1] \longrightarrow E[1]$$

$$T : EE[1] \longrightarrow EE[-1]$$

satisfying (1) $T^2 = 0$

(2) on $\underbrace{E\overbrace{EE}^{T_2 \text{ acts here}}}_{T_1 \text{ acts here}}$, $T_1 T_2 T_1 = T_2 T_1 T_2$

(3) on EE, $TX_1 - X_2 T = Id$.

X_1 acts here X_2 acts here