R. Thomas – 22/1/09 – Part III

Recall:

$C \subset X^3$ Cohen-Macaulay curve in a CY3 $+ T$ 0-dim^l sheaf

$\rightsquigarrow$ we compared contributions of (C,T)'s to $\begin{cases} \text{MNOP invts} \\ \text{stable pairs} \end{cases}$

Recall: in case where $|T| = 1$: only one point

$\mathcal{O}_p$, $p \in X$ contributes

$$e(\mathbb{P}Hom(J_C, \mathcal{O}_p)) = hom(I_C, \mathcal{O}_p) =: h_p \quad \text{to } I_{C,1}$$

$$e(\mathbb{P}Ext^1(\mathcal{O}_p, \mathcal{O}_C)) = ext^1(I_C, \mathcal{O}_p) =: e_p \quad \text{to } P_{C,1}$$

By RR, $h_p - e_p = 1$

Integrating over X, get $I_{C,1} - P_{C,1} = e(X)$

- Look now at case $|T| = 2$.

$\to$ easiest case: $\mathcal{O}_p \oplus \mathcal{O}_q$: contribution to MNOP invt I_C :

$$\Big(Hom(I_C, \mathcal{O}_p \oplus \mathcal{O}_q) - Hom(I_C, \mathcal{O}_p) - Hom(I_C, \mathcal{O}_q) + \{0\}\Big) \Big/ Aut(\mathcal{O}_p \oplus \mathcal{O}_q)$$

$$= Hom^*(I_C, \mathcal{O}_p) \times Hom^*(I_C, \mathcal{O}_q) \Big/ \mathbb{C}^* \times \mathbb{C}^*$$

$$= \mathbb{P}Hom(I_C, \mathcal{O}_p) \times \mathbb{P}Hom(I_C, \mathcal{O}_q)$$

contributes $e(\dots) = h_p h_q$, while for pairs, similar calc$^n \Rightarrow e_p e_q$

Difference: $e_p + e_q + 1$. (recall $RR \Rightarrow h_p = e_p + 1$)

$\to$ for $\mathcal{O}_p \oplus \mathcal{O}_p$: $\Big(Hom(I_C, \mathcal{O}_p)^{\otimes 2} \setminus \{\text{rank 1 homs}\}\Big) \Big/ Aut(\mathcal{O}_p \oplus \mathcal{O}_p)$

$$Hom(I_C, \mathcal{O}_p)^{\otimes 2} = Hom(\mathbb{C}^2, Hom(I_C, \mathcal{O}_p))$$

$$Aut(\mathcal{O}_p \oplus \mathcal{O}_p) = GL(2, \mathbb{C})$$

$\Rightarrow$ get $Gr(2, Hom(I_C, \mathcal{O}_p))$

has Euler char. $e = \binom{h_p}{2}$

vs. for pairs, $\binom{e_p}{2}$ $\Rightarrow$ <u>difference</u>: e_p.

$\to$ for <u>$\mathcal{O}_{2p}$</u>: $\left(Hom(I_C, \mathcal{O}_{2p}) - Hom(I_C, m_p)\right)/Aut$

$$\left.\begin{array}{l} e = h_{2p} - h_{m_p} \\ \text{vs for pairs, } e_{2p} - e_p \end{array}\right\} \text{ difference } 1.$$

So we have $\begin{cases} e_p + e_q + 1 & \text{over } X \times X - \Delta \\ e_p & \text{for } \mathcal{O}_p \oplus \mathcal{O}_p \\ 1 & \text{for } \mathcal{O}_{2p} \end{cases}$

- $\int_{X \times X} e_p = P_{1,C} \cdot e(X)$

  pairs invt for 1 pt on C

- $\int_{Hilb^2 X} 1 = e(Hilb^2 X)$

Hence: $I_{2,C} - P_{2,C} = P_{1,C}\, e(X) + e(Hilb^2 X)$ ✓

2 pts on C — 1 pt on C, 1 free pt — 2 free pts.

<u>NB:</u> if $e(G) = 0$ then can't use division formula $e(X/G) = \dfrac{e(X)}{e(G)}$

However, can use <u>virtual Poincaré polynomials</u> P_q $\left(\lim_{q \to 1} P_q = e\right)$

Those divide well. E.g.: $\mathbb{P}^n = \dfrac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*}$:

$$P_q(\mathbb{C}) = q \quad \rightsquigarrow P_q(\mathbb{P}^n) = \frac{q^{n+1} - 1}{q - 1} = q^n + \dots + 1 \xrightarrow[q \to 1]{\lim} e(\mathbb{P}^n) = n+1$$

similarly, $P_q(Gr(k,n)) \xrightarrow[q\to 1]{} \binom{n}{k}$

$P_q(GL(n,\mathbb{C})) \underset{q\to 1}{\sim} n!\,(q-1)^n.$

Now on to <u>general case</u>:

- $T \in \mathcal{T}$: $\quad Surj(I_C, T) = Hom(I_C, T) - \bigcup_{T_1 \subset T} Hom(I_C, T_1)$

$$+ \bigcup_{T_2 \subset T_1 \subset T} Hom(I_C, T_2) - \dots$$

divide by $Aut(T)$, take Poincaré polynomials, $q \to 1$

(for pairs, similarly $PureExt = Ext^1(I_C, T) - \bigcup_{T_1 \subset T} Ext^1(I_C, T) + \dots$)

<u>Reorder sum over all T</u>:

$$\bigcup_{T \in \mathcal{T}} Hom(I_C, T)\, t^{[T]} - \bigcup_{T_1, T_2 \in \mathcal{T}} Hom(I_C, T_1)\, Ext^1(T_2, T_1)\, t^{[T_1 + T_2]}$$

$Aut(T)$ $\quad$ $Aut(T_1).\,Hom^{*}(T_2, T_1)$

for extensions $T_1 \to T \to T_2$

This is good because get $q^{ext^1(T_2,T_1) - hom(T_2,T_1)}$

which by RR + Serre duality $\equiv q^{ext^1(T_1,T_2) - hom(T_1,T_2)}$

<u>Tool</u>: <u>Hall algebra</u> (Ringel, Joyce, Toen, Kontsevich-Soibelman, Bridgeland, Toda, Nagao ...)

$$1_{\mathcal{T}} * 1_{\mathcal{T}} = \left\{ \begin{matrix} \text{extensions} \\ \text{filtrations} \end{matrix} \; T_1 \to T \to T_2 \right\} \qquad \text{sheaf} / \mathcal{T}$$

Extend to any $\begin{matrix} A, B \\ \downarrow \\ \mathcal{T} \end{matrix}$ sheaf over $\mathcal{T}$ e.g. $(Hom(I_C, T))_{T \in \mathcal{T}}$

$\left[\xrightarrow[\text{Poincaré poly.}]{P_q} q^{hom(I_C, T)} \text{ function on } \mathcal{T} \right]$

given $\begin{matrix} A, B \\ \downarrow \\ \mathcal{T} \end{matrix}$,

$$\begin{tikzcd} A * B \arrow[r] \arrow[d] \arrow[rr, bend left] & 1_{\mathcal{T}} * 1_{\mathcal{T}} \arrow[r, "T"] \arrow[d, "(T_1, T_2)"] & \mathcal{T} \\ A \times B \arrow[r] & \mathcal{T} \times \mathcal{T} & \end{tikzcd}$$

ie. $A * B \to \mathcal{T}$ = at T st. $T_1 \to T \to T_2$, get $A_{T_1} \times B_{T_2}$

- given $T_0 \in \mathcal{T}$, can form $1_{T_0} \to \mathcal{T}$ $\left(\begin{matrix} \mathbb{C} \text{ over } T \neq T_0 \\ \text{pt over } T_0 \end{matrix} \right)$

$$1_{T_0} * 1_{\mathcal{T}} = \{ \text{extensions } T_0 \to T \to T' \}$$
$$\downarrow$$
$$\mathcal{T}$$

In this language: $\mathrm{Surj}(I_C, \cdot) * 1_{\mathcal{T}} \longrightarrow \mathcal{T}$

is exactly $\mathrm{Hom}(I_C, \cdot)$

($\mathrm{Surj}(I_C, \cdot) * 1_{\mathcal{T}}$ gives, at T st. $T_1 \to T \to T_2$,

$\mathrm{Surj}(I_C, T_1)$ ...... and $\bigcup_{T_1 \subset T} \mathrm{Surj}(I_C, T_1) = \mathrm{Hom}(I_C, T)$)

Invert this identity?

Use: $1_{\mathcal{T}} = 1_0 + 1_{\mathcal{T}^*}$

0-sheaf
(identity for $*$)

nonzero sheaves

$$\Rightarrow 1_{\mathcal{T}}^{-1} = 1_0 - 1_{\mathcal{T}^*} + 1_{\mathcal{T}^*} * 1_{\mathcal{T}^*} - \cdots$$

And now: $\mathrm{Surj}(I_C, \cdot) = \mathrm{Hom}(I_C, \cdot) * 1_{\mathcal{T}}^{-1}$

$(= \mathrm{Hom}(I_C, \cdot) - \mathrm{Hom}(I_C, \cdot)\, \mathrm{Ext}^1(*, \cdot) + \cdots)$

(NB: This is again a clever reordering trick ...)

Now: $Z^{C}_{MNOP}(q,t) = \int_{\mathcal{T}} \mathrm{Surj}(I_C, \cdot)\, t^{[\cdot]}$

$$\int_{\mathcal{T}} := P_q\left(\mathcal{T}, \frac{P_q(\text{fiber})}{P_q(\mathrm{Aut})}\right)$$

i.e. sum Poincaré polynomials of fibers mod Aut according to base $\mathcal{T}$ & its own Poincaré poly.

⚠ need to first integrate then $q \to 1$ to get $e(\ldots)$ otherwise things diverge at $q \to 1$ due to the sum rearrangements we've made...

key: $\int_{\mathcal{T}} A * B = \int_{\mathcal{T}} B * A$

Indeed, $A * B$ = take all A's, all B's & take extensions of underlying sheaves $T \in \mathcal{T}$

$\Rightarrow$ LHS = integrate over $A \times B$, $q^{\mathrm{ext}^1(T_2,T_1) - \hom(T_2,T_1)}$

\# extensions $T_1 \to T \to T_2$

autom's of T vs. of $T_1 \times T_2$

RHS = integrate over $A \times B$, $q^{\mathrm{ext}^1(T_1,T_2) - \hom(T_1,T_2)}$

Those are equal by RR + Serre duality ◀

Now: $Z^{C}_{MNOP}(q,t) = \int_{\mathcal{T}} \mathrm{Hom}(I_C, \cdot)(t) * 1_{\mathcal{T}}^{-1}(t)$

$$= \int_{\mathcal{T}} q^{h_T} t^{[T]} * 1_{\mathcal{T}}^{-1}(t)$$

whereas $Z_P^C(q,t) = \int_T 1_T^{-1}(t) * \mathrm{Ext}^1(I_C, \cdot)(t)$

(relation between PureExt & Ext^1 same as b/w Surj & Hom except 1_T comes on other side

$$= \int_T 1_T^{-1}(t) * q^{e_T} t^{[T]}$$

$$= \int_T q^{e_T} t^{[T]} * 1_T^{-1}(t)$$

commutativity

Now RR $\Rightarrow$ $h_T - e_T = [T]$

So: $Z_P^C(q, \underline{qt}) = \int_T q^{e_T + [T]} t^{[T]} * 1_T^{-1}(\underline{qt})$

$$= \int q^{h_T} t^{[T]} * 1_T^{-1}(\underline{qt})$$

while $Z^C_{MNOP}(q,t) = \int \underbrace{q^{h_T} t^{[T]} * 1_T^{-1}(qt)}_{\substack{\text{pairs integrand} \\ \mathrm{Pairs}(q,qt)}} * \underbrace{1_T(qt) * 1_T^{-1}(t)}_{\substack{\text{MNOP integrand} \\ \text{for } C=\emptyset}}$

since when $C = \emptyset$, $\hom(\mathcal{O}_X, T) = [T]$

$\Rightarrow Z^\emptyset_{MNOP} = \int q^{[T]} t^{[T]} * 1_T^{-1}(t)$

Now, the $\left\{\begin{matrix}\text{pairs} \\ \text{MNOP}^0\end{matrix}\right\}$ integrands each have $\lim_{q \to 1}$

* At $T \in \mathcal{T}$, MNOP^0 integrand = surjections $\mathcal{O}_X \to T$

$$= \begin{cases} 1 & \text{if } T = \mathcal{O}_Z \in \mathrm{Hilb}^\alpha(X) \\ 0 & \text{otherwise.} \end{cases}$$

Hence: $Z^{c}_{\Pi NOP}(q,t) = \int_{T \times \mathrm{Hilb}^{\alpha}(X)} \mathrm{Pairs}\,(q, qt)\; q^{\mathrm{ext}^1 - \mathrm{hom}}$

$$\begin{array}{ccc} q \to 1 \downarrow & & \downarrow q \to 1 \\ Z^{c}_{\Pi NOP}(t) & = & Z^{c}_{P}(t) \cdot Z^{0}_{\Pi NOP}(t). \end{array}$$

QED ▲

- In this simplified version (without χ^B Behrend function) things work on the nose, and in fact don't need to specialize Poincaré poly. at $q=1$ ....

  [However "(Poincaré polynomial)$^{\text{virt}}$" isn't deformation invt, only the Euler class].