Self dual obstruction theory (Behrend)

- Model: $f: \mathcal{A} \to \mathbb{C}$, $\mathcal{M} = (df)^{-1}(0)$

  Assume $\dim \mathcal{M} > 0$, $\mathcal{M}$ smooth

$$0 \to T\mathcal{M} \to T\mathcal{A}_{|\mathcal{M}} \xrightarrow{D(df)} T^*\mathcal{A}_{|\mathcal{M}} \to \text{coker} \to 0$$

This is the Hessian $(\partial^2_{ij} f)$, symmetric

$\Rightarrow$ The dual map $T^*\mathcal{A}_{|\mathcal{M}} \leftarrow T\mathcal{A}_{|\mathcal{M}}$ is also $D(df)$

and hence $\text{coker} = T^*\mathcal{M}$.

So: Def & Obs are dual to each other
(definition of self-dual obs. theory)

Perturb f by a function g on $\mathcal{M}$: $d(f+\varepsilon g)^{-1}(0) =$ zeros of dg on $\mathcal{M}$

$\rightsquigarrow$ the virtual cycle should be $c_{top}(T^*\mathcal{M}) = (-1)^{\dim \mathcal{M}} e(\mathcal{M})$

- In general: ($\mathcal{M}$ self dual obstruction theory, not necess. smooth)

  Behrend shows: $[\mathcal{M}]^{virt} \in H_0(\mathcal{M})$ is $e(\mathcal{M}, \chi^B) := \sum_{n \in \mathbb{Z}} n \cdot e(\{\chi^B = n\})$

  for some constructible function $\chi^B : \mathcal{M} \to \mathbb{Z}$ which depends only on the local analytic structure of $\mathcal{M}$
  (on smooth part, $\chi^B = (-1)^{\dim \mathcal{M}}$)

In what follows, for simplicity we ignore χ^B & signs, just work with the Euler class (but one should really use χ^B, since $e(\mathcal{M})$ isn't deform^n invariant unlike $[\mathcal{M}]^{virt}$ ...)

★ <u>MNOP theory</u>: do deformation theory not for the subscheme $Z \subset X$
not for the subsheaf $\mathcal{I}_Z \subset \mathcal{O}_X$
but for the sheaf $\mathcal{I}_Z$ as a sheaf w/ fixed trivial determinant:

- $T_Z I_n(X,\beta) = \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{I}_Z)_0$
- $\mathrm{Obs}_Z = \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z)_0$

Serre dual in CY 3-fold case.

★ <u>Stable pairs.</u> do deform^n theory not as a pair, but for the 2-term complex with trivial det. $I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$

- $T_{I^\bullet} P_n(X,\beta) = \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$
- $\mathrm{Obs}_{I^\bullet} = \mathrm{Ext}^2(I^\bullet, I^\bullet)_0$

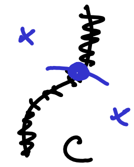
Serre dual

<u>NB:</u> the 2 theories agree when $\mathrm{coker}(s) = 0$:
then $I^\bullet = \{\mathcal{O}_X \overset{s}{\twoheadrightarrow} F\} \cong$ ideal sheaf $\mathcal{I}_Z = \ker(s)$
and deform^n theories match. The two theories differ in their treatment of free points / cokernels. Then want to compare $e(\mathcal{M})$'s (or rather, $e(\mathcal{M}, \chi^B)$'s).

- Work with a fixed Cohen-Macaulay curve $C \subset X$, $\chi(\mathcal{O}_C) = 1-g$.

$\rightarrow$ $I_n(X,C) \subset I_{1-g+n}(X,\beta)$
locus of subschemes whose largest CM subscheme is C.

$Z = C \cup$ embedded pts
free pts



$\rightarrow$ $P_n(X,C) \subset P_{1-g+n}(X,\beta)$
pairs supported in C

Then: the formula $\dfrac{Z_{MNOP,\beta}}{Z_{MNOP,0}} = Z_{P,\beta}$

Corresponds to:

$$I_{n,C} = \overset{n \text{ points on } C}{P_{n,C}} + \overset{n-1 \text{ pts on } C,\ 1 \text{ in } X}{P_{n-1,C}\, e(X)} + P_{n-2,C}\, e(\mathrm{Hilb}^2 X) + \dots + \overset{n \text{ pts in } X}{e(\mathrm{Hilb}^n X)}$$

where $I_{n,C} = e(I_n(X,C), \chi^B)$

$P_{n,C} = e(P_n(X,C), \chi^B)$ etc...

<u>Wall-crossing</u>: If $Z = C \cup \{p_i\}$ then $\mathcal{I}_Z \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_{p_i}$

$$(*)\quad \mathcal{O}_{p_i}[-1] \to \mathcal{I}_Z \to \mathcal{I}_C$$

Move across a wall in space of stability cond$^{\underline{n}}$s so that the slope (phase) of $\mathcal{O}_{p_i}[-1]$ crosses that of $\mathcal{I}_C$:

- When bigger, $(*)$ destabilizes $\mathcal{I}_Z$, so $\mathcal{I}_Z$ not stable, so it disappears from our counts.

However, extensions of the form $\mathcal{I}_C \to\ ??\ \to \mathcal{O}_{p_i}[-1]$ are now stable.

These extensions are of the form

$$I^\bullet = \{\mathcal{O}_X \to F\}$$

$$\mathcal{I}_C \xrightarrow{\ker} \qquad \xrightarrow{\text{coker}} \mathcal{O}_{p_i}[-1]$$

$\mathcal{O}_C \to F \to \mathcal{O}_{p_i}$ governed by $\mathrm{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C) \cong \mathrm{Ext}^1(\mathcal{O}_{p_i}[-1], \mathcal{I}_C)$

$= \mathrm{Hom}(\mathcal{O}_{p_i}, \mathcal{I}_C)$

- Want to compare (weighted) <u>Euler classes across the wall</u>:

<u>Toy Model</u>: sheaves E which can become unstable as we cross a wall in Kähler moduli space, $0 \to A \to E \to B \to 0$

where A & B don't have subsheaves & hence stable.

$\rightsquigarrow$ $\mathrm{Hom}^{\leq 0}(A,B) = 0 = \mathrm{Hom}^{\leq 0}(B,A)$

By Serre duality, $\mathrm{Hom}^{\geq 3}$ are also zero.

$\mathrm{Ext}^1(B,A) = \mathrm{Ext}^2(A,B)^\vee$ governs $\quad 0 \to A \to E \to B \to 0 \quad (1)$

$\mathrm{Ext}^1(A,B)$ governs extensions $\quad 0 \to B \to F \to A \to 0 \quad (2)$

Cross wall $\rightsquigarrow$ lose all extensions (1) i.e. $\mathbb{P}(\mathrm{Ext}^1(B,A))$ $\quad e = \mathrm{ext}^1(B,A)$

gain all extensions (2) i.e. $\mathbb{P}(\mathrm{Ext}^1(A,B))$ $\quad e = \mathrm{ext}^2(B,A)$

$\to$ the difference, by Riemann-Roch, is $\quad \chi(B,A) = -\chi(A,B)$

<u>Hence:</u> the difference in invariants as we cross wall is

$$\chi(B,A)\, e(\mathcal{M}_A)\, e(\mathcal{M}_B).$$

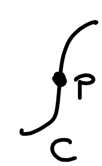
- <u>Our case:</u> $A = \mathcal{O}_p[-1]$ $\longleftarrow$ split up, semistable, has autom's

  $B = \mathcal{I}_C$ $\longleftarrow$ stable

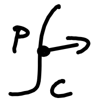
<u>Start with a simple case:</u> C smooth, 1 free point.

<u>want:</u> $I_{1,C} = P_{1,C} + e(X)$

C smooth, $\begin{cases} p \in X \setminus C & \rightsquigarrow \text{ contributes } 1 \text{ to } I_{1,C} \\ & \phantom{\rightsquigarrow \text{ contributes }} 0 \text{ to } P_{1,C} \text{ since } p \notin C \\ p \in C & \rightsquigarrow \text{ contributes } 2 = e(\mathbb{P}^1) \text{ to } I_{1,C} \\ & \phantom{\rightsquigarrow \text{ contributes }} 1 \text{ to } P_{1,C} \end{cases}$

Indeed: in $P_{1,C}$,  is a single point of moduli space

(consider $\mathcal{O} \xrightarrow[s_p]{} F$, coker $= \mathcal{O}_p$)

in $I_{1,C}$, consider  with a normal direction $\Rightarrow$ $\mathbb{P}^1$-worth.

Hence $I_{1,C} = e(X \setminus C) + \overset{e(\mathbb{P}^1)}{2}\, e(C)$ vs. $P_{1,C} = e(C)$

$\Rightarrow I_{1,C} = P_{1,C} + e(X)$ ✓

- For general C (not smooth),

  $p \in X$ contributes

  $$e(\mathbb{P}\mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_p)) = \hom(\mathcal{I}_C, \mathcal{O}_p) \quad \text{to } I_{C,1}$$

  $$e(\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_C)) = \mathrm{ext}^1(\mathcal{I}_C, \mathcal{O}_p) \quad \text{to } P_{C,1}$$

  By RR, $\hom(\mathcal{I}_C, \mathcal{O}_p) - \mathrm{ext}^1(\mathcal{I}_C, \mathcal{O}_p) = 1$

  hence difference in invts is $+1$ $\forall p \in X$

  $\int_X \Rightarrow$ get difference in invariants is $e(X)$.