R. Thomas – 19/01/09 – Curve counting & derived categories I

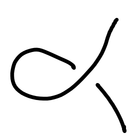
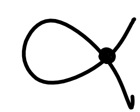
<u>Plan:</u> 1) GW, MNOP, virtual cycles, stable pairs
2) wall-crossing
if enough time, 3) Gopakumar-Vafa BPS count

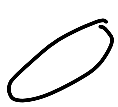
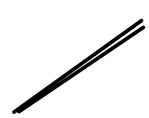
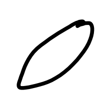
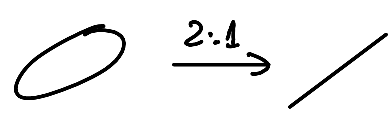
X smooth proj. CY 3-fold (can extend to all 3-folds ...)
$\rightsquigarrow$ all "things" live in families of virt. dim. 0, in fact they're crit pts of various functionals.
$\hookrightarrow$ SLag S^3's, curves, surfaces, sheaves

Hope to define invariants by "counting" them.
Issues: – compactness of moduli space: $\to$ finite invariants?
$\to$ deformation invariance?
– transversality / virtual cycles

In these lectures, focus on counting <u>holom. curves in X</u>. Two approaches:

1) <u>GW theory</u> counts <u>stable maps</u> nodal curve $\to X$
ie keep curve nice, map can become bad.
stable curves, ie. $\#\mathrm{Aut} < \infty$ $\rightsquigarrow$ invts $\in \mathbb{Q}$

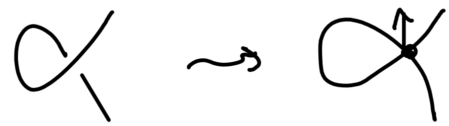
<u>Ex:</u> A) if  degenerates in a family to a nodal curve 
then the domain of the limit stable map is the <u>normalization</u>,
and the limit map is not an embedding at the double point.

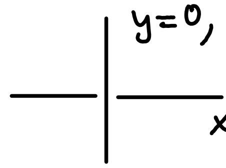
<u>Ex.</u> B) if  degenerates to  (e.g. conics $\to$ line)
then the domain of the limit stable map is the 2:1 cover 
and the limit map is 

(with $\mathbb{Z}/2$-automorphisms ... and excess dim. - choice of branch pts)

2) <u>Subschemes</u>: (MNOP theory)   Hilbert scheme in X

ie. keep map = embedding (inclusion), but curve becomes bad.

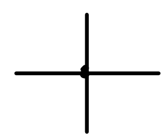
<u>Ex</u>: A)  limit = embedded, but "remembers" <u>direction of deform^n</u>

model: 

Ideal $(x,z)\cdot(y,z-t) = (xy,\ yz,\ x(z-t),\ z(z-t))$

$$\Big\downarrow t\to 0$$

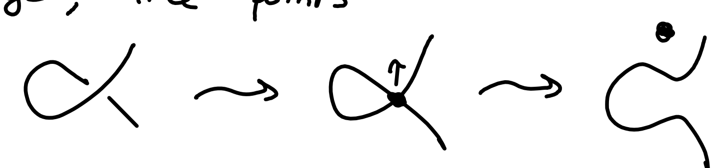
$$(xy,\ yz,\ xz,\ z^2) \subsetneq (xy,\ z)$$

This is <u>not</u> the same as 

because $z \notin$ ideal, instead only $z\cdot\begin{pmatrix}\text{anything}\\ \text{in max. ideal}\end{pmatrix} \in$ ideal.

z remembers infinitesimal dir^n of deformation

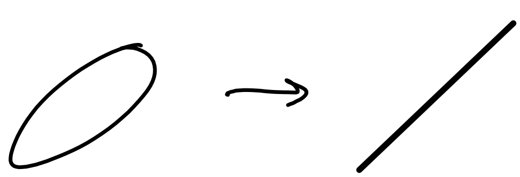
- Can have genus change, free points!

in family, can have 

(genus ↑ by 1 and # free points ↑ by 1)

Ie. our subschemes can contain 0-dim^l subschemes.

On the other hand, embedding ⇒ no autom's. ⇒ <u>Count $\in \mathbb{Z}$</u>

<u>Ex. B)</u>  thickened line

$x^2=0,\ z=0.$

MNOP conjecture: these 2 theories are equivalent.

Define generating functions: given $\beta \in H_2(X)$,

- $$Z_{GW,\beta}(u) = \sum_g N^{\bullet}_{g,\beta}\, u^{2g-2}$$

  disconnected GW theory

- $$Z_{MNOP,\beta}(t) = \sum_n I_{n,\beta}\, t^n$$

  where param. equiv to genus is now

  $$n = \chi(\mathcal{O}_Z) = 1 - g(C) + \#(\text{free points})$$

  subscheme $Z = C \cup (\text{free points})$, $[C] = \beta$

To get rid of free point contributions,

$$Z_{MNOP,0}(t) = \sum_n I_{n,0}\, t^n, \quad \text{where } I_{n,0} = \#^{\mathrm{virt}}(\mathrm{Hilb}^n X)$$

MNOP Conj.: $\forall \beta$,

$$\underbrace{\frac{Z_{MNOP,\beta}(t)}{Z_{MNOP,0}(t)}}_{\text{rational function in } t\text{, invt under } t \leftrightarrow \frac{1}{t}} = Z_{GW,\beta}(u) \quad \text{for } -e^{iu} = t$$

⚠ equality only makes sense via analytic continuation, can't compare term-by-term.

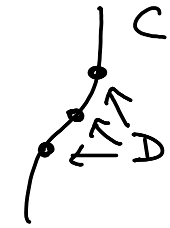
E.g.: l.h.s. might be $\underbrace{t(1+t)^{-2}}_{\text{invt by } t \leftrightarrow \frac{1}{t}} = t \underbrace{-2t^2 + 3t^3 - 4t^4 + \dots}_{\text{not invt by } t \leftrightarrow t^{-1}}$

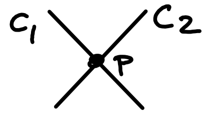
Alternative approach to MNOP theory:

Stable pairs: (F, s)
- F sheaf, with support dim 1, $[F] = \beta$
- $s \in H^0(F)$

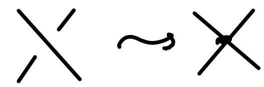
where
- F is pure, i.e. no 0-dim'l subscheme
- s has 0-dim'l coker.

e.g.:
- $(\mathcal{O}_C, 1)$
- $(\mathcal{O}_C(D), s_D)$ $\Leftrightarrow$ curve + free points on C



- 

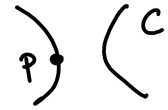
$(\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}, (1,1))$ w/ coker $= \mathcal{O}_p$.

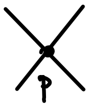
is the stable pair limit of 

(cokernel $\mathcal{O}_p$ of s as section $\mathcal{O}_{C_1 \cup C_2} \xrightarrow[(1,1)]{} \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}$)

(in irreducible case, similarly via normalization...) [ex. A].

This is also the stable pair limit of smoothings



$(\mathcal{O}_{C'}(p), s_p) \longrightarrow$ 

$(\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}, (1,1))$

- $(\mathcal{O}_{2C}, 1)$ limit of example B

Pure $\Rightarrow$ C is Cohen-Macaulay
- no embedded pts
- no free points

NB: when C Gorenstein,
stable pair $\Leftrightarrow$ (C, D), D point of $\mathrm{Hilb}(C)$

(deformation given by $\mathrm{Ext}^1_C(\underset{\substack{\uparrow \\ \text{coker(pair)}}}{Q}, \mathcal{O}_C) \underset{\text{Serre duality}}{\simeq} H^0(C, Q \otimes \omega_C)^*$)

generating function: $Z_{P,\beta}(t) = \sum_n P_{n,\beta} t^n$ $\quad n = \chi(F) = 1 - g(C) + \#\text{free pts}$

Now: $$Z_{P,\beta}(t) = \frac{Z_{DT,\beta}(t)}{Z_{DT,0}(t)} = Z_{GW,\beta}(u)$$