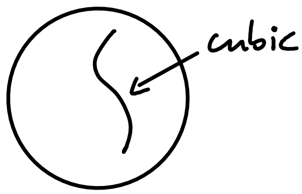
R. Pandharipande – 21/1/09 – The tropical vertex group

(with Gross & Siebert)

## 1) A curve counting problem:

Ex: 


$(\mathbb{P}^2, \text{cubic})$ is a "log-Calabi-Yau"

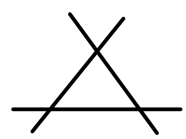
want to count curves $\mathbb{C} \longrightarrow \mathbb{P}^2 \setminus C$ of degree $d$

Note that $\dim \overline{\mathcal{M}}_0(\mathbb{P}^2, d) = 3d-1$

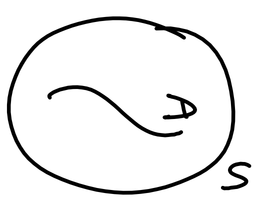
but constraint of having a single intersection point with $C$, of multiplicity $3d$, is also codim. $3d-1$

$\Rightarrow$ get a 0-dim$^{\underline{l}}$ problem, hence can get a number.

Problem studied by N. Takahashi 1999
– related to $g=0$ GW invts of local $\mathbb{CP}^2$.

Here: want to pick the singular cubic  ? can't have an intersection of mult. $3d$ ?

More generally:




$(S, D)$, $\beta \in H_2(S, \mathbb{Z})$

s.t. $\beta \cdot c_1(S) = \beta \cdot D$

(ie. $(S,D)$ is "log CY wrt $\beta$")

$\Rightarrow$ count $g=0$ curves with full contact at a single point of $D$

This is actually def$^{\underline{d}}$ rigorously as a relative Gromov-Witten invariant for $(S,D)$

## 2) The tropical vertex group:

$\mathbb{C}^* \times \mathbb{C}^*$ : $\quad \mathrm{Aut}^{Gr} = GL(2,\mathbb{Z})$ acts $\begin{pmatrix} x \mapsto x^a y^b \\ y \mapsto x^c y^d \end{pmatrix}$

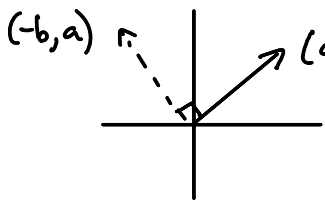
$\mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ = not much more (translate $\begin{matrix} x \mapsto cx \\ y \mapsto cy \end{matrix}$)

want to study formal 1-param. families of Aut:

$\mathbb{C}^* \times \mathbb{C}^* = \mathrm{Spec}\, \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$

$\mathcal{A} = \mathrm{Aut}_{\mathbb{C}[[t]]}\left(\mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]\right)$

This is pretty big. Consider special elements of $\mathcal{A}$:




$(a,b) \in \mathbb{Z}^2 \setminus \{0\}$

$f = 1 + t x^a y^b g$, $\quad g \in \mathbb{C}[x^a y^b][[t]]$ arbitrary

$\rightarrow$ automorphism $\quad \theta_{(a,b),f} = \begin{Bmatrix} x \mapsto x f^{-b} \\ y \mapsto y f^{a} \end{Bmatrix} \in \mathcal{A}$

Inverse: $\theta_{(a,b),f}^{-1} = \theta_{(a,b),f^{-1}}$

(this works because $\theta_{(a,b),f}$ fixes $x^a y^b \mapsto x^a y^b$)

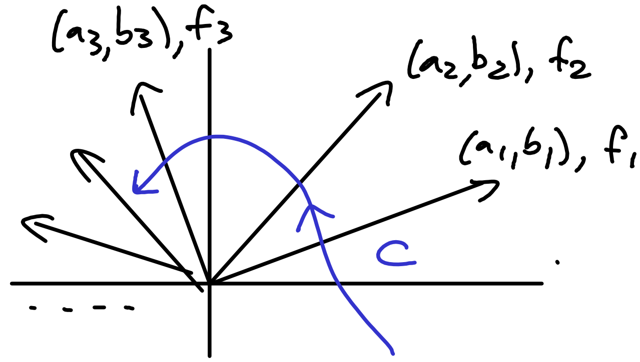
Tropical vertex group: $TVG \subset \mathcal{A}$ generated by all such $\theta_{(a,b),f}$

- Symplectic structure: $\mathbb{C}^* \times \mathbb{C}^*$, $\omega := \frac{dx}{x} \wedge \frac{dy}{y}$. Then $\theta_{(a,b),f}^* \omega = \omega$.

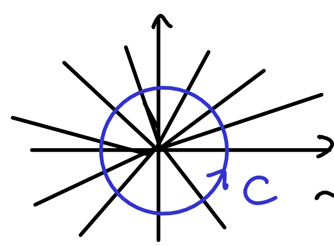
$\Rightarrow TVG \subset \mathrm{Aut}^{symp} \subset \mathcal{A}$

[in fact... "volume preserving" wrt $\prod \frac{dx_i}{x_i}$ would be more appropriate]

- To any diagram + a curve $C$ transverse to rays of diagram, we can associate an element of TVG, namely the ordered product along $C$ of all $\Theta^{\pm 1}_{(a_i,b_i),f_i}$ ← depending on sign of intersection

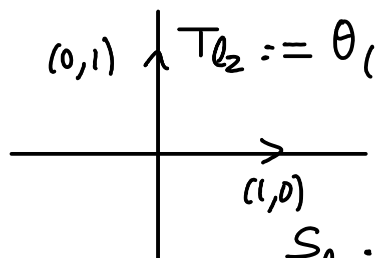



- Lemma (Kontsevich, Soibelman):

> "Given a diagram, can add extra rays (possibly ∞ many) to it (algorithmically) s.t.
>
> 
> 
>
> ⇝ product is 1."

More precisely:

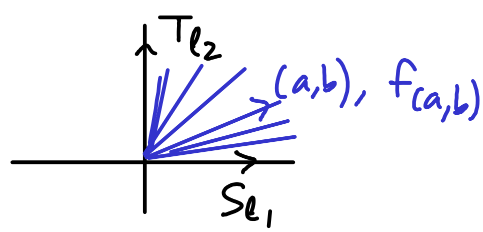
$(0,1)$ $T_{\ell_2} := \Theta_{(0,1),(1+ty)^{\ell_2}}$




$S_{\ell_1} := \Theta_{(1,0),(1+tx)^{\ell_1}}$

$$\Rightarrow \exists! \left(f^{\ell_1,\ell_2}_{(a,b)}\right)_{\substack{(a,b)\in\mathbb{Z}^2_+\\ \text{primitive}}} \text{ s.t. } T_{\ell_2}^{-1} S_{\ell_1}^{-1} T_{\ell_2} S_{\ell_1} = \overleftarrow{\prod_{b/a \uparrow}} \Theta_{(a,b),f_{(a,b)}}$$

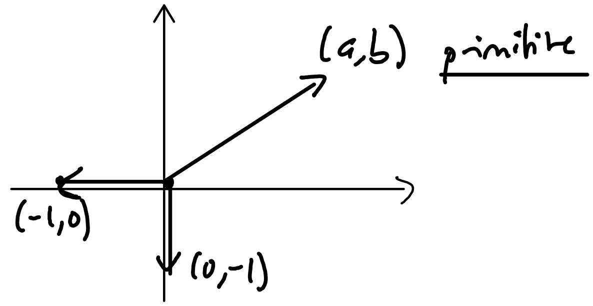
i.e. can complete




uniquely

Q$^{\underline{n}}$: ‖ for given $\ell_1, \ell_2$, what is $f_{(a,b)}$ ?

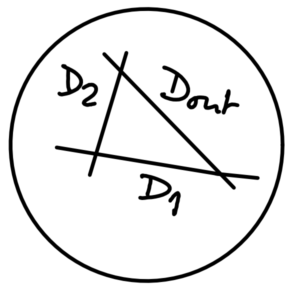
Answer involves curve counting problem in $\mathbb{P}^2_{1,a,b}$ !!




define $\mathbb{P}^2_{1,a,b}$

$\rightsquigarrow f_{a,b}$ = generating series for genus 0 log CY curve counts

in




Consider ordered partitions $P = (p_1, \dots, p_{l_1})$, $\sum p_i = ak$

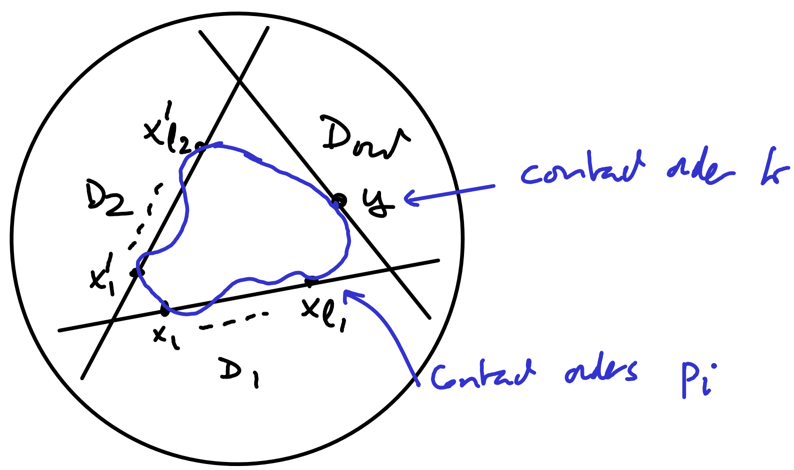
$P' = (p'_1 \dots p'_{l_2})$, $\sum p'_i = bk$

$\exists\ \beta_k \in H_2(\mathbb{P}^2_{1,a,b})$ s.t. $\beta_k \cdot D_{out} = k$

$\beta_k \cdot D_2 = bk$

$\beta_k \cdot D_1 = ak$

Then want to count rational curves in $\left(\mathbb{P}^2_{(1,a,b)},\ D_1 \cup D_2 \cup D_{out}\right)$




i.e. considering $(\tilde{X}, \tilde{D}_{out}) \xrightarrow{\pi} \mathbb{P}^2_{1,a,b}$ blowup at $x_i$ & $x'_i$

let $\tilde{\beta}_k = \pi^* \beta_k - \sum p_i E_i - \sum p'_i E'_i$

$\rightsquigarrow$ $N_{(a,b)}(P,P') :=$ rel. GW invt for $g=0$, $\tilde{\beta}_k$ in $(\tilde{X}, \tilde{D}_{out})$
with a single (order $k$) intersection w/ $\tilde{D}_{out}$
(as in introduction)

Then
$$\log f_{(a,b)} = \sum_{k \geq 1} k\, C^k_{a,b}(l_1, l_2)\,(tx)^{ak}(ty)^{bk}$$
where
$$C^k_{a,b}(l_1,l_2) = \sum_{|P_1| = ak,\ |P'| = bk} N_{(a,b)}(P,P').$$

In fact, could write a "cleaner" formula over $\mathbb{C}[[t_1, \dots t_{l_1}, s_1 \dots s_{l_2}]]$
(1 deformn parameter for <u>each</u> marked pt on $D_1 \cup D_2$)

Then count $N_{(a,b)}(P,P') \prod (t_i x)^{p_i} \prod (s_i y)^{p'_i}$

Also, even better interpretation if we factor not

$f = 1 + a_1 x + a_2 x^2 + \dots$ but

$f = (1+x)^{a_1} (1+x^2)^{a_2} (1+x^3)^{a_3} \dots\dots$