Homological mirror symmetry

HMS (Kontsevich 1994, Hori-Vafa 2000, Kapustin-Li 2002, Katzarkov ∼2002, . . .) relates symplectic and algebraic geometry via their categorical structures.

A symplectic manifold M is mirror to a Landau-Ginzburg model, which is given by a regular non-constant function (the so-called superpotential) W on a smooth algebraic variety X. We will also allow orbifold versions, where M and (X, W) carry actions of a finite group G.

Ex: The mirror of $M = S^2$ is $X = \mathbb{C}^*$, with $W(u) = u + u^{-1}$.

Ex: The mirror of the orbifold $M = S^2/G$, $G = \mathbb{Z}/p$, is the p-fold cover of the previously considered Landau-Ginzburg model. Explicitly, $X = \mathbb{C}^*$ with $W(u) = u^p + u^{-p}$.

Categorical setup

Fix $\lambda \in \mathbb{C}$. Let $\mathrm{Fuk}(M, \lambda)$ be the Fukaya category with mass λ. If M is compact, the category will be zero unless λ is an eigenvalue of small quantum multiplication with $c_1(M)$ (Auroux et al.)

Ex: For $M = S^2$, each of the categories $\mathrm{Fuk}(M, \pm 2)$ contains a single object $L_\pm$, whose endomorphism ring is a Clifford algebra $HF^*(L_\pm, L_\pm) \cong \mathbb{C}[t]/t^2 \pm 1$.

For the most part, we'll take $\lambda = 0$, and omit that from the notation. Consider formal completions of the Fukaya category:

- the derived category $D^b\mathrm{Fuk}(M)$; and
- the split-closed derived category $D^\pi\mathrm{Fuk}(M)$;

both of which are triangulated categories over $\mathbb{C}$.

The advantage of passing to the split-closed derived category is that often, it can be proved that this is split-generated by a single object L. In that case, to reconstruct the category, it suffices to know the endomorphism ring $HF^*(L, L)$ together with its higher order products (A_∞-structure).

Given $W : X \to \mathbb{C}$ and $\lambda \in \mathbb{C}$, take $X_\lambda = W^{-1}(\lambda)$. Let $D^b\mathrm{Coh}(X_\lambda)$ be its derived category of coherent sheaves, and $\mathrm{Perf}(X_\lambda)$ the subcategory of perfect complexes. Following Orlov, define

$$D^b\mathrm{Sing}(W, \lambda) = D^b\mathrm{Coh}(X_\lambda)/\mathrm{Perf}(X_\lambda).$$

This is zero whenever X_λ is smooth. In fact, whenever $U \subset X_\lambda$ is a Zariski open subset containing the singularities of X_λ, then (Orlov)

$$D^b\mathrm{Sing}(W, \lambda) \cong D^b\mathrm{Coh}(U)/\mathrm{Perf}(U).$$

As before we omit λ if the choice is $\lambda = 0$. We also consider the split-closure $D^\pi\mathrm{Sing}(W, \lambda)$, which actually depends only on the formal neighbourhood of the singular set (Orlov, unpublished).

If X is affine, $D^b\mathrm{Sing}(W, \lambda)$ is equivalent to the category of matrix factorizations of $W - \lambda$. By definition, a matrix factorization is a $\mathbb{Z}/2$-graded projective $\mathbb{C}[X]$-module E with an odd differential δ_E,

$$\delta_E^2 = (W - \lambda)\,\mathrm{id}.$$

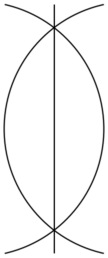
Technical remark. In general, the Fukaya category is not defined over $\mathbb{C}$, but over a field Λ of Laurent series in one variable $\hbar$ (in the most general setting, these are Laurent series with complex coefficients and real exponents). Intuitively, this corresponds to a formal rescaling of the symplectic form, $\log(1/\hbar)\omega$, so $\hbar \to 0$ is the large volume limit. Correspondingly, in the mirror Landau-Ginzburg model, both X and W are defined over Λ, hence are a family of functions.

However, if $[\omega]$ is a multiple of $c_1(M)$, one has a certain homogeneity property, which means that all Laurent series will be Laurent polynomials. In particular, one can then set $\hbar = 1$, and define an actual Fukaya category over $\mathbb{C}$. Correspondingly, the mirror will then be defined over $\mathbb{C}$.

The genus two curve

Let M be a closed genus two curve. Katzarkov's conjecture (slightly modified, but presumably equivalent, version) says that the mirror should be a Landau-Ginzburg model $\tilde{W} : \tilde{X} \to \mathbb{C}$, where:

- $\tilde{X}$ is quasi-projective toric Calabi-Yau threefold;
- $\tilde{W}^{-1}(0)$ is the union of three rational surface;
- $\mathrm{Sing}(\tilde{W}^{-1}(0))$ is the union of three rational curves, intersecting in Θ-shape:



Strong evidence for this conjecture was provided by Abouzaid, Auroux, Gross, Katzarkov, Orlov (2006).

Thm: $D^\pi \mathrm{Fuk}(M) \cong D^\pi \mathrm{Sing}(\tilde{W}, 0)$.

We want to build up gradually to the genus two case, including it in a wider discussion which also mentions the previously known cases of genus zero and genus one (Polishchuk-Zaslow 1998). The plan:

$$\begin{tikzcd}
\text{Pair-of-pants} \arrow[r] \arrow[d] \arrow[dd, bend right=60] & \text{Sphere} \\
\text{Three-punctured torus} \arrow[r] & \text{Torus} \\
\text{Orbifold sphere} \arrow[r] & \text{Genus two curve}
\end{tikzcd}$$

Each arrow is one of two steps: compactification or passage to an unbranched covering. The plan is not particularly systematic. For instance, we could reverse the order, going (Pair-of-pants) → (Three-punctured genus two curve) → (Genus two curve).

Open surfaces

The pair-of-pants and its mirror:

$$\begin{cases} M = \{\text{generic line in } \mathbb{C}P^2\} \cap (\mathbb{C}^*)^2, \\ X = \mathbb{C}^3,\ W(x) = -x_1x_2x_3. \end{cases}$$

Thm: $D^\pi \mathrm{Fuk}(M) \cong D^\pi \mathrm{Sing}^{cpt}(W)$, where *cpt* means with cohomology supported at the origin.

On the algebraic side consider $\mathcal{S}$, the skyscraper sheaf at $0 \in W^{-1}(0)$. As a matrix factorization, this is represented by a deformed version of the Koszul resolution of the origin in $\mathbb{C}^3$:

$$\begin{aligned} &E = \Lambda^*(\mathbb{C}^3) \otimes \mathrm{Sym}^*(\mathbb{C}^3), \\ &\delta_E = \iota_{x_1} \otimes x_1 + \iota_{x_2} \otimes x_2 + \iota_{x_3} \otimes x_3 \\ &\quad - x_1 \otimes x_2x_3/3 - x_2 \otimes x_1x_3/3 - x_3 \otimes x_1x_2/3 \end{aligned}$$

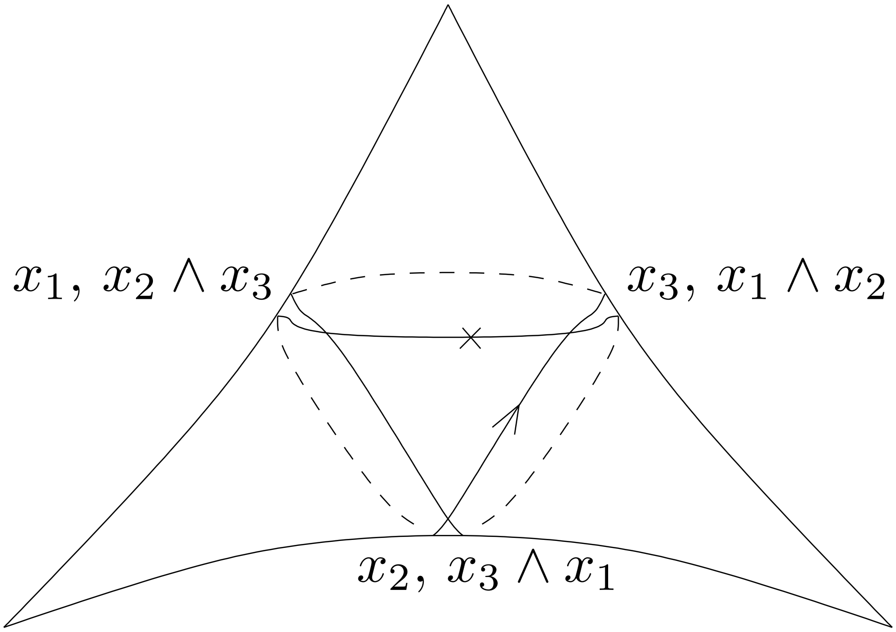
Then

$$\mathrm{Hom}_{D^b\mathrm{Sing}}(\mathcal{S}, \mathcal{S}) \cong \Lambda^*(\mathbb{C}^3).$$

The matrix factorization pictures gives us an underlying dga, which can be used to compute the induced A_∞-structure (Massey products). These are nonzero,

$$\mu^3(x, x, x) = -x_1x_2x_3.$$

On the symplectic side, consider this immersed curve $L \subset M$:



Every selfintersection point contributes two generators to $HF^*(L, L)$, of opposite parity. In addition, there are the two generators arising from the standard cohomology $H^*(L)$. On the whole,

$$HF^*(L, L) \cong \Lambda^*(\mathbb{C}^3).$$

There are two triangles (with their rotated versions) which define the standard exterior product. Again, we have a nontrivial Massey product $\mu^3(x, x, x) = -x_1 x_2 x_3$ (given by counting triangles with an additional marked point).

The punctured torus

Starting from this, more examples can be easily constructed by looking at unbranched covers of the pair-of-pants M. Take a surjection $\pi_1(M) \to \mathbb{Z}^2 \to \Gamma$, and the associated Γ-covering $\tilde{M}$. The dual

$$G \subset \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) = (\mathbb{C}^*)^2 \subset SL_3(\mathbb{C})$$

acts on $\mathrm{Fuk}(M)$, and roughly speaking

$$\mathrm{Fuk}(\tilde{M}) \cong \mathrm{Fuk}(M) \rtimes G.$$

On the other side, we can consider G-equivariant matrix factorizations, which have a corresponding description.

Specifically, introduce (fractionally) graded matrix factorizations, giving elements of $\mathrm{Sym}^k(\mathbb{C}^3)$ degree $2k/3$, so that W has degree 2. This automatically includes symmetry with respect to the central $G = \mathbb{Z}/3 \subset SL_3(\mathbb{C})$. The resulting $D^b\mathrm{Sing}^{gr}(W)$ is a $\mathbb{Z}$-graded lift of $D^b\mathrm{Sing}^G(W)$, and admits a more familiar description. Namely, Orlov constructs an embedding

$$D^b\mathrm{Sing}^{gr}(W) \longrightarrow D^b\mathrm{Coh}(\tilde{X}),$$

where $\tilde{X} = \mathrm{Proj}(\mathbb{C}[x_1, x_2, x_3]/W)$ is the singular elliptic curve in $\mathbb{P}^2$ defined by W. If we pass to idempotent completions,

Fact: $D^\pi\mathrm{Sing}^{gr}(W) \cong \mathrm{Perf}(\tilde{X})$.

On the mirror side, the Γ-covering $\tilde{M} \to M$ is a three-punctured torus. Correspondingly, we can introduce a graded version $\mathrm{Fuk}^{gr}(\tilde{M})$ of the Fukaya category, and then:

Theorem: $\mathrm{Perf}(\tilde{X}) \cong D^\pi\mathrm{Fuk}^{gr}(\tilde{M})$.

This is a "large complex structure" limit version of the standard HMS statement for elliptic curves (on the left $\tilde{X}$ is singular, and on the right $\tilde{M}$ is affine).

Closed surfaces

The "compactification is deformation" slogan. Take M a closed surface, and $D \subset M$ an ample divisor. Then $\mathrm{Fuk}(M \setminus D)$ admits a deformation by counting polygons which pass k times over D with powers $\hbar^k$. Denote the deformed structure by $\mathrm{Fuk}(M, D)$. This is linear over $\mathbb{C}[[\hbar]]$, and

$$\mathrm{Fuk}(M, D)|_{\hbar=0} = \mathrm{Fuk}(M \setminus D),$$
$$\mathrm{Fuk}(M, D) \otimes_{\mathbb{C}[[\hbar]]} \Lambda \hookrightarrow \mathrm{Fuk}(M).$$

When $D \cong K^{\otimes r}$ for some $r \neq 0$ (possibly fractional), all power series in $\mathrm{Fuk}(M, D)$ are polynomials, and one can replace the second part by

$$\mathrm{Fuk}(M, D)|_{\hbar=1} \hookrightarrow \mathrm{Fuk}(M).$$

On the mirror side, one expects a corresponding deformation of the superpotential by $\hbar$ terms.

The sphere

Take $M = S^2$, with $D = 3$ points. Then $W = -x_1x_2x_3 + \hbar(x_1+x_2+x_3)$. Setting $\hbar = 1$, the critical points are at $x_1 = x_2 = x_3 = \pm 1$, and the critical values at $W(x) = \pm 2$. After removing the plane $\{x_1 = 0\}$, make a change of variables

$$\begin{aligned} W &= x_1 + x_1^{-1} - x_1^{-1}(1 - x_1x_2)(1 - x_1x_3) \\ &= u + u^{-1} + vw. \end{aligned}$$

Thm: (Knörrer periodicity; Knörrer, Orlov) Passing from $W(u)$ to $W(u) + vw$ leaves $D^b\mathrm{Sing}$ unchanged.

Hence, one can use the known results to derive:

Cor: $D^\pi \mathrm{Sing}(W, \lambda) \cong D^\pi \mathrm{Fuk}(M, \lambda)$.

The torus

Take $M = T^2$, again with $D = 3$ points. The corresponding deformed potential is

$$W = -x_1x_2x_3 + \hbar(x_1^3 + x_2^3 + x_3^3) + \cdots .$$

The higher order terms are again cubic, hence after a coordinate transform of order $\hbar$, we can write

$$W = -x_1x_2x_3 + \psi(\hbar)(x_1^3 + x_2^3 + x_3^3).$$

ψ is of course explicitly known (mirror map), but not especially relevant for us. Let $\tilde{X} \subset \mathbb{P}^2(\Lambda)$ be the smooth elliptic curve defined by W.

Thm: (Orlov) $D^b\mathrm{Sing}^{gr}(W) \cong D^b\mathrm{Coh}(\tilde{X})$; in particular, it's split-closed.

As a consequence, Polishchuk-Zaslow's result is (essentially) equivalent to:

Cor: $D^\pi\mathrm{Sing}^{gr}(W) \cong D^\pi\mathrm{Fuk}^{gr}(M)$.

The genus two curve

Look at S^2 with three orbifold points of order 5. The orbifold Fukaya category counts holomorphic curves with d-fold ramification at those points. The deformed potential is

$$W = -x_1 x_2 x_3 + x_1^5 + x_2^5 + x_3^5,$$

agreeing with previous mirror symmetry predictions (Rossi, Takahashi). M is a fivefold unbranched orbifold cover of S^2, so:

Thm: $D^\pi \mathrm{Fuk}(M) \cong D^\pi \mathrm{Sing}^G(W, 0)$.

Here $G = \mathbb{Z}/5$, generated by $\mathrm{diag}(e^{2\pi i/5}, e^{2\pi i/5}, e^{6\pi i/5})$. On the other hand, our $\tilde{X}$ was precisely the standard (Nakamura, G-cluster) crepant resolution of the singularity $X = \mathbb{C}^3/G$, and $\tilde{W}$ the pullback of W. There is a version of the McKay correspondence, due to Mehrotra and Quintero-Velez:

Thm: $D^\pi \mathrm{Sing}^G(W, 0) \cong D^\pi \mathrm{Sing}(\tilde{W}, 0)$.

Combining these two facts yields the desired statement. Note that there are also massive modes (5 free G-orbits of singular points of W at each of the two values $W = \pm 2 \cdot 5^{-5/2}$).

Classification theory

All examples considered so far lead to A_∞-structures on $\Lambda(\mathbb{C}^3)$ (or variations thereof). The classification theory is governed by Hochschild cohomology. By the HKR isomorphism,

$$HH^*(\Lambda(\mathbb{C}^3), \Lambda(\mathbb{C}^3)) \cong \Lambda(\mathbb{C}^3) \otimes \mathbb{C}[[x_1, x_2, x_3]]$$

where the right hand side are formal polyvector fields. Kontsevich's formality theorem yields an L_∞ quasi-isomorphism of the underlying dg Lie algebras, hence an equivalence of the associated Maurer-Cartan (deformation) theories.

Generally speaking, Maurer-Cartan theory considers a $\mathbb{Z}/2$-graded dg Lie algebra $\mathfrak{g}$. One studies solutions $\alpha \in \mathfrak{g}^1$ of the Maurer-Cartan equation. For $\mathfrak{g} = \Lambda(\mathbb{C}^3) \otimes \mathbb{C}[[x_1, x_2, x_3]]$ we can write $\alpha = \alpha^0 + \alpha^2$, and the equation splits as

$$\begin{gathered} \tfrac{1}{2}[\alpha^2, \alpha^2] = 0, \\ [\alpha^0, \alpha^2] = 0. \end{gathered}$$

The first part says that $\{f, g\} = \alpha^2(df \wedge dg)$ is a formal Poisson bracket, and the second one that α^2 is a cocycle in the Koszul complex associated to $(\partial_{x_1}\alpha^0, \partial_{x_2}\alpha^0, \partial_{x_3}\alpha^0)$.

Elements of $\mathfrak{g}^1$ act by formal diffeomorphisms of $\mathbb{C}^3$, and also by changing α^2 to a cohomologous Koszul cocycle.

Fact: (Finite determinacy; Tougeron) Let W be any polynomial with an isolated singularity at 0, and Milnor number μ. Then any other polynomial which agrees with W up to order $\mu+1$ can be transformed into W by a change of variables.

Fact: (Standard) For W as before, the Koszul complex is a resolution of $\mathbb{C}[[x_1, x_2, x_3]]/(\partial_{x_i} W)$.

These two properties reduce classification issues to checking finitely many constants in the A_∞-structure. For the application to mirror symmetry for a closed genus two surface, it turns out to be enough to show that $\alpha^0 = W$+terms of order > 5.

More generally, for curves of genus $g \geq 2$, one expects (Rossi) the mirror to have superpotential

$$W = -x_1x_2x_3 + x_1^{2g+1} + x_2^{2g+1} + x_3^{2g+1}.$$

considered equivariantly with respect to $\mathbb{Z}/(2g+1) \subset SL_3(\mathbb{C})$. It is easy to see that the leading order terms are correct. That leaves finitely many other coefficients to check. I have not looked at the details. Alternatively, one could start with the genus two case and use unbranched covers. A particularly easy case:

Thm: Let M be closed of genus six. Then

$$D^\pi \mathrm{Fuk}(M) \cong D^\pi \mathrm{Sing}^G(W),$$

where $W = -x_1x_2x_3 + x_1^5 + x_2^5 + x_3^5$, and G is now $\mathbb{Z}/5 \times \mathbb{Z}/5 \subset SL_3(\mathbb{C})$, the full symmetry group of W.