M. Kontsevich – 22/1/09 – Part 2

**DISCLAIMER: these notes are incomplete & probably full of mistakes.**

$X$ compact CY 3-fold, $\mathcal{M}_X$ moduli of $\mathbb{C}$ structures on $X$

$u \in \mathcal{M}_X \rightsquigarrow \Gamma_u = H_3(X_u, \mathbb{Z}) \simeq \mathbb{Z}^{2g} + \text{torsion}$ with Hodge structure

$$\langle \cdot, \cdot \rangle : \Gamma \otimes \Gamma \longrightarrow \mathbb{Z}$$

**Physics:**

$N=2$ string theory on asymptotically flat spacetime $\underset{\text{at } \infty}{\sim} \mathbb{R}^{3+1} \times N=(2,2)$ CFT

depends on $\mathcal{M}_X \times \mathcal{M}_{X^\vee}$ + string coupling constant.

Scattering theory for particles $\rightsquigarrow \mathcal{H}$ Hilbert space

with action of $N=2$ super-Poincaré group

$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$ charges $\in \Gamma = H_3(X, \mathbb{Z})$

Mass of any particle of charge $\gamma$ must be $\geq \left| \int_\gamma \Omega_u^{3,0} \right|$ ; BPS state : when equality

Trace (expansion in $U(su)$) $\rightsquigarrow$ invariant $\Omega(\gamma) \in \mathbb{Z}$ should not depend on $\mathcal{M}_{X^\vee}$.

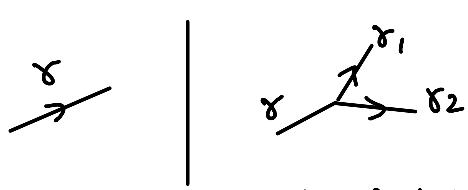
- Depending on $u \in \mathcal{M}_X$, stability changes $\rightsquigarrow$ particles can decompose.

**Math.:** $\Omega_u(\gamma) \in \mathbb{Z}$ change as we cross walls & stability cond changes

ie. when $\exists\, \gamma_1, \gamma_2 \in \Gamma / \gamma_1 + \gamma_2 = \gamma$

$\gamma_1 \not\parallel \gamma_2$, but $\int_{\gamma_1} \Omega^{3,0} \parallel \int_{\gamma_2} \Omega^{3,0}$

**Physics:** Denef-Moore wall-crossing formula:




wall in moduli of stability

$$\Delta \Omega_u(\gamma) = \pm \langle \gamma_1, \gamma_2 \rangle \Omega_u(\gamma_1) \Omega_u(\gamma_2)$$

## Our wall-crossing formula:

choose $\varepsilon : \Gamma \to \pm 1$ s.t. $\dfrac{\varepsilon(\gamma_1+\gamma_2)}{\varepsilon(\gamma_1)\,\varepsilon(\gamma_2)} = (-1)^{\langle \gamma_1, \gamma_2 \rangle}$

Torus $\mathrm{Hom}(\Gamma, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{2g}$ $\quad \gamma \in \Gamma \rightsquigarrow$ monomial $x^\gamma \in$ torus

Define $T_\gamma : x^{\gamma'} \longmapsto (1-\varepsilon(\gamma)x^\gamma)^{\langle \gamma, \gamma' \rangle} x^{\gamma'}$

$V \subset \mathbb{C}$ angular sector $< 180^\circ \longmapsto A_V := \overrightarrow{\prod\limits_{\substack{\gamma\in\Gamma \\ \int_\gamma \Omega^{3,0} \in V}}} T_\gamma^{\Omega_u(\gamma)}$

V 

Wall-crossing formula $\iff$ Axiom: as $u$ varies, $A_V$ does not change as long as no lattice point crosses $\partial V$.

- Think of $\Omega_u(\gamma)$ as an invariant "counting" special Lagr. $S^3$ in class $\gamma$ i.e. counting of stable objects in $Fuk(X)$ with class $\gamma \in H_3(X, \mathbb{Z})$, with a stability cond. determined by the complex str. on $X$.

## Properties of $\Omega_u(\gamma)$:

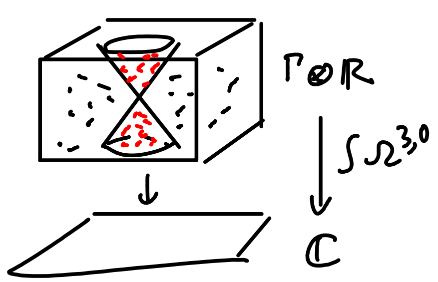
1) satisfies wall-crossing formula

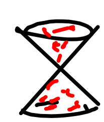
2) support property:

$\forall u \in \mathcal{M}_X$, $\exists$ norm $\|\cdot\|_u$ on $\Gamma \otimes \mathbb{R}$ s.t.

$$\Omega_u(\gamma) \neq 0 \Rightarrow \|\gamma\|_u \leq \underbrace{m(\gamma, u)}_{\text{mass}} := \frac{\left|\int_\gamma \Omega^{3,0}\right|}{\sqrt{\int_X |\Omega^{3,0}|^2}}$$

i.e.




support of $\Omega_u$ is contained in complement of cone 

This ensures the products $A_V$ are well-def$^{d}$ power series ...

Wall-crossing formula $\Rightarrow$ all $\Omega_u(\gamma)$'s, $u \in \mathcal{M}_X$ are determined by the values $\Omega_{u_\gamma}(\gamma)$, $u_\gamma$ = "attractor point" for class $\gamma$

$= \text{s.t. } \exists \lambda \in \mathbb{C}^* / \operatorname{Re}[\lambda \Omega^{3,0}_{u_\gamma}] \underset{\text{P.D.}}{=} \gamma \in H_3(X, \mathbb{R})$

$\widetilde{\mathcal{M}}_X \leftarrow$ choose $u \in \mathcal{M}_X$ and $\Omega^{3,0}$ $\xhookrightarrow{}$ $H^3(X_u, \mathbb{C}) \xrightarrow{\operatorname{Re}} H^3(X, \mathbb{R})$

$\mathbb{C}^* \downarrow$ locally embeds as Lagrangian cone

$\mathcal{M}_X$

so $\bigcup_{\gamma \in \Gamma}$ attractors is dense

- $\Omega_u(\gamma) = ?$ Fix $\gamma$, then $m(\gamma, u) = \frac{1}{\operatorname{vol}(\Omega^{3,0})} \left| \int_\gamma \Omega^{3,0} \right| \geq 0$.

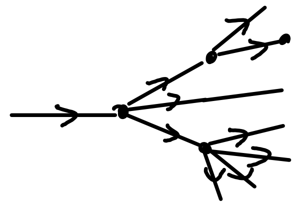
function on domain of $u$

Consider <u>attractor flow</u> = gradient flow line of $m(\gamma, u)$.

☆ all critical pts are local minima $\begin{cases} \text{good case: attractor pt} \\ \text{bad case: } m(\gamma, u) = 0 \text{ @ } \partial\mathcal{M} \end{cases}$

Break up when $m(\gamma, u) = \sum m(\gamma_i, u)$ for $\sum \gamma_i = \gamma$ (destabilization).

$\rightsquigarrow$ <u>attractor trees</u>: = split attractor flow



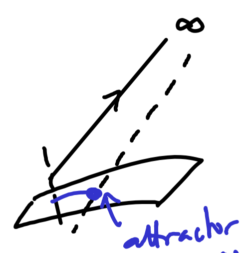
$\leftarrow$ cv to attractor pt or to $\partial\mathcal{M}$

(<u>non</u> trivalent trees !!)

These trees admit canonical lifts to $\widetilde{\mathcal{M}}_X$

Each edge is linear in $H^3(X, \mathbb{C}) \hookleftarrow \widetilde{\mathcal{M}}_X$

& attractor vertices are actually rays $\to \infty$




Then, using WCF along attractor tree, get $\Omega_u(\gamma)$ from

$\begin{cases} \text{values } \Omega_{u_{\gamma_i}}(\gamma_i) \text{ at attractor points} \\ \text{values at boundary of moduli space — however those are constrained} \end{cases}$

because at boundary $m(\gamma, u) \to 0$ and so support condition ......

FROM HERE ONWARDS, THESE NOTES GET SKETCHY ...

- <u>less degenerate picture</u>:

Denef formula for multi-centred black holes on $\mathbb{R}^{3+1}$ <u>asympt. flat</u>

$\mathbb{R}^3 \setminus$  $\longrightarrow \mathcal{M}_X$

black holes

Attractor trees should be limits of harmonic maps $\mathbb{R}^3 \setminus \{x_i\} \longrightarrow \tilde{\mathcal{M}}_X$ , $\sum \frac{\gamma_i \cdot c_i}{|x - x_i|} + \dots$

$x_i \in \mathbb{R}^3$

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$\overrightarrow{\prod} T_\gamma^{\Omega(\gamma)}$ $\rightarrow$ can glue domains into a holomorphic symplectic manifold.

NB: base $\sim \mathrm{Hom}(\Gamma_u, \mathbb{R})$

$\tilde{\mathcal{M}}_X \xrightarrow{\sim}$ Lagr. cone in $H^3(X, \mathbb{R}) \supset H^3(X, \mathbb{Z})$ $\rightsquigarrow$ bundle of tori $\mathrm{Hom}(\Gamma_u, S^1)$.

Introduce walls of second type in $\tilde{\mathcal{M}}_X$ where $\exists \gamma / \int_\gamma \Omega^{3,0} \in \mathbb{R}_{>0}$

start with torus bundle $\mathrm{Hom}(\Gamma_u, \mathbb{C}^*) \longrightarrow \tilde{\mathcal{M}}_X$

& correct using sympl. transformations $\overrightarrow{\prod} T_\gamma^{\Omega(\gamma)}$ ....

<u>Conj</u>: $\rightsquigarrow$ will get a corrected <u>holom. sympl. mfld</u>, <u>smooth</u>, asymptotic at infinity to $\tilde{\mathcal{M}}_X$.

<u>Conj</u>: | we get in this way the moduli space of vector multiplets for string theories.

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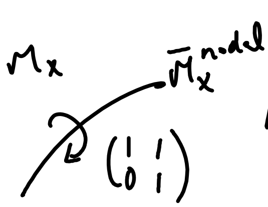
<u>Ex.</u> •  $S^3$ , $\Omega(\gamma) = 1$ $\rightarrow$ $\mathcal{M}_X \subset \overline{\mathcal{M}_X}$

$\sum x_i^2 = \varepsilon$

compactif. of $\mathcal{M}_X$ by adding nodal degenerations.




Nodal sings. determine a divisor in $\overline{\mathcal{M}_X}$, with monodromy of local system $\Gamma_u = H^3(X_u, \mathbb{Z})$ given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

- Conj: Ooguri-Strominger-Vafa: at $u_\gamma$ attractor point, $\Omega_{u_\gamma}(N\gamma) \sim e^{\text{const}\cdot N^2}$ where $\text{const} = m(u,\gamma)^2$

cf Harvey-Moore for K3 fibrations$/\mathbb{P}^1$

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- Near cusp of $\mathcal{M}_X$, as $u \to$ cusp,

$\Omega_u(\gamma)$, $\gamma \in$ affine hyperplane in $\Gamma_u$ should be invariant under monodromy at distance one from 0.

This corresponds to torsion-free sheaves of rk$=1$ on $X^\vee$ i.e. DT-invariants of $X^\vee$.

- Vector multiplets + moduli space $\sim$

$$\begin{array}{c} \text{fibration} \\ \downarrow \ \text{fiber} = \mathrm{Hom}(\Gamma_u, S^1) \\ \tilde{\mathcal{M}}_X \end{array}$$

Claim: $\exists$ refined invariant s.t. $\Omega_u(\gamma) = \sum_{\sigma \text{ integer}}$ refined invts $\in \mathbb{Z}$

Interpretation: $\Omega_u(\gamma) = \#$ stable objects in $D^b(X^\vee)$

$K_0(D^b(X^\vee)) \longrightarrow$ Deligne cohomology of $X^\vee$

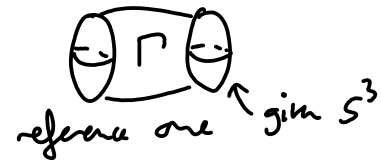
$H^3(X^\vee, \mathbb{C})/F_2H^3 \longrightarrow * \longrightarrow \bigoplus H^{p,p}(X^\vee) \cap H^\bullet(X,\mathbb{Z})$

$$\begin{array}{c} H^{ev}(X^\vee, \mathbb{Z}) \times \text{complex sympl. mfld} \\ \downarrow \ \text{Deligne cohomology} \\ \tilde{\mathcal{M}}_{X^\vee} \end{array}$$

$\rightsquigarrow$ predict refinement of $\Omega_u(\gamma)$ according to this finer decomposition of count of stable objects on $X^\vee$.

(This is mirror to: counting stable objects in $D^b(X^\vee)$ according to their class in <u>Chow ring</u>, which is finer than just $H^{even}$ class.

Here: e.g. if looking at Lagr. $S^3$'s,




$\int_\Gamma \omega \wedge \omega \in \mathbb{R}/\mathbb{Z}$ secondary information about our $S^3$

Count separately SLagr. $S^3$'s according to this)