M. Kontsevich – 20/1/09 – Geometric & algebraic aspects of wall-crossing I

Today: hyperkähler geometry & wall-crossing

next talks: algebra (derived categories)

Wall-crossing: family of lattices $\Gamma_u \simeq \mathbb{Z}^d$, $d = 2r$, depending on param. u

+ for generic u, $\Omega : \Gamma_u \to \mathbb{Z}$

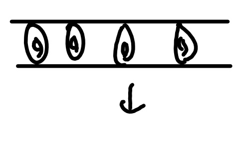
This occurs in
1) geometry of affine structures & asymptotics of SYZ collapse [K.-Soibelman: K3 example]
2) hyperkähler case, integrable systems (gauge theory)
3) susy string theory, moduli of CY 3-fold, attractor flow, black holes
4) algebraic theory of DT-invariants for CY 3-folds.

Ex: SYZ collapse:

$X_\varepsilon, g_\varepsilon, J_\varepsilon$ family of CY manifolds

As $\varepsilon \to 0$, $\begin{cases} \text{maximal degeneration of complex moduli} \\ \mathrm{diam}(X_\varepsilon, g_\varepsilon) = 1 \end{cases}$

$\Rightarrow$ expect $\begin{array}{c} X_\varepsilon \\ \downarrow \\ B \end{array}$



$\dim_{\mathbb{R}} B = n = \dim_{\mathbb{C}} X$

X_ε "converges" to a tropical CY :=

$\begin{cases} \bullet\ B,\ B \supset B^{sing} \text{ codim. } 2 \\ \bullet\ B - B^{sing}\ \ \mathbb{Z}\text{-affine structure, ie. } SL(n,\mathbb{Z}) \ltimes \mathbb{R}^n \text{ structure} \\ \bullet\ g_B \text{ metric on } B - B^{sing},\ (g_B)_{ij} = \partial_i \partial_j H,\ \det(g_B) = \text{const.} \end{cases}$

Relation to CY metric: get a family of semiflat noncomplete CYs by considering the canonical torus fibration (ε) $= \begin{array}{c} TB/\varepsilon TB_{\mathbb{Z}} \\ \downarrow \pi \\ B - B_{sing} \end{array}$

+ semiflat metric $\partial\bar\partial\, \pi^* H$

- <u>local model:</u> $(\mathbb{C}^*)^n \to \mathbb{R}^n$ + glue these
$z_1 \cdots z_n \mapsto \left(\log |z_i| / \log \frac{1}{\varepsilon} \right)$ using $SL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$

- <u>MS:</u> same B, same g, but take dual lattice wrt g

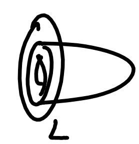
Parameters for $\mathbb{Z}$-aff str. on $B - B_{sing}$ = $\pi_1(B - B_{sing}) \to SL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$
$\Rightarrow$ deform^n given by $H^1(B - B^{sing}$, local system of $\mathbb{R}^n)$

<u>Problem:</u> this constr. of CYs doesn't extend over B^{sing}.

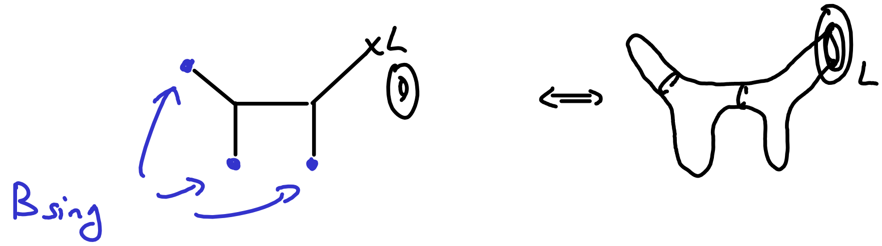
<u>Recall HMS picture:</u> points of X_ε are objects of $D^bCoh(X_\varepsilon)$
$\longleftrightarrow$ objects of $Fuk(X_\varepsilon^\vee)$

On $X_\varepsilon^\vee$, can use limiting almost $\mathbb{C}$ structure
Objects of Fuk = Lagrangian submanifolds with $U(1)$ local systems
Consider Lagrangian tori = fibers of $\pi^\vee$

<u>Problem:</u> some of these fibers of $\pi^\vee$ bound Maslov index 0 discs
$\to m_0$ obstruction



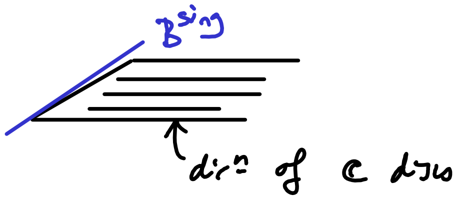
On B, These discs look like tropical trees



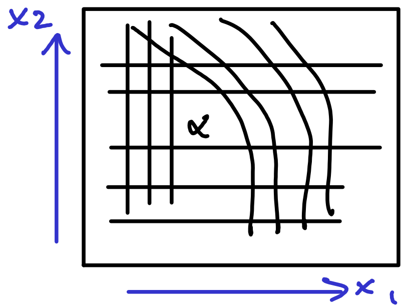
$\bigcup$ all trees forms (infinitely many) codim. 1 walls.

$\Rightarrow$ consider $B \setminus \bigcup$(all trees), then can use naive constr^n
in all chambers.

- <u>Elementary wall-crossing:</u>
since codim $B^{sing} = 2$, each wall is a <u>$\mathbb{Z}$-hyperplane</u> <u>foliated by $\mathbb{Z}$-lines</u>



In normal slice



monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

The naive affine structure: $|x_2| \gg 1 \quad x_1 \longleftrightarrow -x_1 x_2$

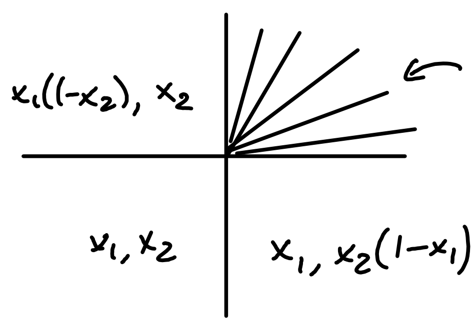
$|x_2| \ll 1 \quad x_1 \longleftrightarrow x_1$

does not yield a well-def^d $\mathbb{C}$ manifold.

Instead, glue by: $x_1 \dots x_n \in \mathbb{C}^*$

$$x_1 \longleftrightarrow x_1(1-x_2)$$
$$x_i \longleftrightarrow x_i \quad \text{for } i \geq 2$$

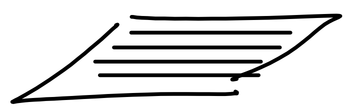
- <u>Scattering</u>: when 2 walls hit each other, generate more walls!!



← 1 wall in easiest case

∞ly many walls in all other cases

- General rule: $\forall$ component of wall,

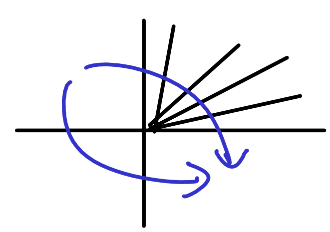


$\mathbb{Z}$-hyperplane $\perp$ x_1-axis

foliated by lines // x_2-axis

$$\Rightarrow \quad x_1 \to x_1 F(x_2), \qquad F(x_2) = 1 + \text{higher order terms}$$
$$x_i \to x_i \text{ for } i > 1$$

Then scattering is governed by requirement that compositions of changes of coords. are <u>compatible</u>



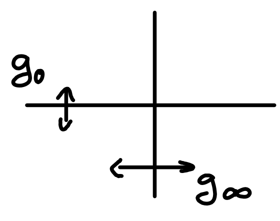
Let $\mathfrak{g} = \prod_{\substack{a,b\geq 0 \\ a+b>0}} \mathbb{C}\, x^a y^b = \prod_{\lambda \in [0,+\infty] \cap \mathbb{Q}\mathbb{P}^1} \mathfrak{g}_\lambda$

$\rightsquigarrow$ group G = symplectic transformation of $\mathbb{C}[[x,y]]$ of the form $x \mapsto x + \dots$, $y \mapsto y + \dots$

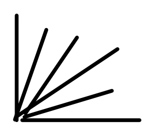
& s.t. preserves $\frac{dx \wedge dy}{xy}$

Then $\forall g \in G$, $\exists!$ decomposition $g = \prod_{\lambda \uparrow} g_\lambda$, $g_\lambda \in G_\lambda$

So: given 2 walls



the scattering



is given by decomposing

$$g_\infty g_0 = \underbrace{g_0 \cdots\cdots g_\infty}_{\text{expansion in } \lambda\uparrow \text{ order}}.$$

NB: in $K3 \to S^2$ case, can prove convergence by constructing an ad hoc deform^n.

- To build CY metric, need to solve real Monge-Ampère eq^n.

No explicit formula in general, but if $X_\varepsilon \downarrow B$ are hyperkähler

then there is a well-known ansatz coming from integrable systems.

(Y,ω) holomorphic symplectic mfld, $\omega = \omega^{2,0}$

$\downarrow \pi$ holom.

B

generic fiber = complex Lagrangian abelian variety

$\dim_{\mathbb{R}} B = 2n$.

$\mathbb{Z}$-affine structure on B: $\mathrm{Re}\Big(\underbrace{d^{-1}\int_{\gamma_i} \omega^{2,0}}_{\alpha_i}\Big)$ $\gamma_i \in H_1(\text{fiber}, \mathbb{Z})$ basis

(α_i = flux of $\omega^{2,0}$ on basis of $H_1(\text{fiber})$ = 1-form on B)

NB: The affine str. on B has $Sp(2m,\mathbb{Z}) \ltimes \mathbb{R}^{2m}$ holonomy

- The fibers are princ. polarized abelian varieties. Choose (γ_i) symplectic basis of $H_1(\text{fiber})$ wrt polarization. Then we take

$$\omega_B = \frac{1}{2\pi i} \sum_{i=1}^{m} \alpha_i \wedge \bar{\alpha}_{m+i} \quad \text{Kähler form on } B$$

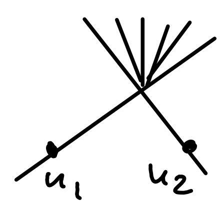
defines the metric ✓

- In fact we have a 1-param. family of affine str. on B! by replacing $\omega^{2,0} \to e^{i\varphi}\omega^{2,0}$ (rotates the $\mathbb{Z}$-affine str.).

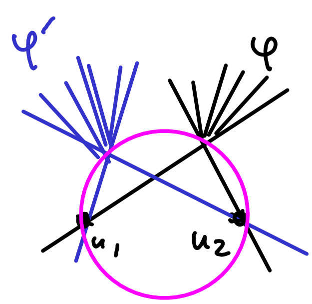
Noncompact example: Seiberg-Witten curve

$$E_u:\quad y + \frac{1}{y} = x^2 - u, \quad u \in \mathbb{C} \qquad \omega^{2,0} = \frac{dx\,dy}{y}$$

walls for given φ:

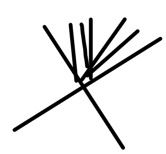


change $\varphi \Rightarrow$ rotate ...



As φ varies, vertices in graphs together form a real hypersurface (here: $\bigcirc$)

- For given φ, scattering diagram



given by identity in symplectomorphism group:

let $T_{a,b} = \begin{cases} x \longmapsto x\,(1-x^a y^b)^{-2b} \\ y \longmapsto y\,(1-x^a y^b)^{+2a} \end{cases}$

Then $T_{1,0}\,T_{0,1} = T_{0,1}\,T_{1,2}\,T_{2,3} \cdots T_{1,1}^{-2} \cdots T_{3,2}\,T_{2,1}\,T_{1,0}$

Another example (next time):

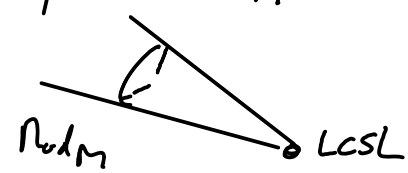
given a 3d CY mfld M,

Mod_M moduli space of $\mathbb{C}$ structures, equipped with

- bundle of holom vol. forms $\mathbb{C}^* \simeq H^{3,0}(M) \setminus \{0\} \to \mathrm{Mod}_M$
- bundle of intermediate Jacobians $H^3(M, \mathbb{C}) \big/ F^2H^3 + H^3(M, \mathbb{Z}) \to \mathrm{Mod}_M$

Then these bundles form an integrable system with base Mod_M

However, things get very singular at LCSL point $\in \mathrm{Mod}_M$.



String theory predicts: $\exists$ "corrected" smooth mfld which "at infinity" looks like Mod_M with this integrable system.