M. Abouzaid – 23/1/09 – HMS for T^4

0) $Fuk(T^2) \longleftrightarrow D^b Coh(E)$
Polishchuk–Zaslow

$\int_{T^2} \omega + iB \longleftrightarrow$ modular parameter

1) Fukaya: B real n-torus, $X = T^*B/\mathbb{Z}^n \dashleftarrow\dashrightarrow Y = TB/\mathbb{Z}^n$

working over $\mathbb{C}$ — sympl. — complex

$\left\{\begin{matrix}\text{affine Linear} \\ \text{Lagrangians}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{coherent} \\ \text{sheaf}\end{matrix}\right\}$

$HF^* \longleftrightarrow Ext^*$

+ A_∞-alg. structures for transverse Lagrangians
(but not on $CF(L,L)$...)

2) Kontsevich–Soibelman

considered $\left\{\begin{matrix}\text{unbranched Lag.} \\ \text{multisections}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{"vector} \\ \text{bundle"}\end{matrix}\right\}$

over Novikov ring.

Since $D^b Coh$ is split-generated by an ample line bundle
$\Rightarrow D^b Coh(Y) \hookrightarrow D^\pi Fuk$

Main result: (joint w/ Ivan Smith)

|| For T^4, the embedding above is an equivalence

More precisely:

$E :=$ Take elliptic curve over $\Lambda_{\mathbb{R}} = \left\{ \sum c_i t^{\lambda_i}, \begin{matrix} c_i \in \mathbb{C} \\ \lambda_i \to +\infty \end{matrix} \right\}$

"Analytification": $\Lambda^*_{\mathbb{R}}/\langle t\rangle$

Then $Fuk(T^2) \simeq D^b Coh(E)$

Thm (A.–Smith): || $D^\pi Fuk(T^2 \times T^2) \cong D^b Coh(E \times E)$

Def: $\mathrm{Fuk}^{\text{naive}}(X) = \left\{ \begin{array}{l} \text{Lagrangians } L, \quad \text{Maslov}(L)=0, \text{ spin} \\ + \exists J \text{ s.t. } L \text{ doesn't bound a } J\text{-hol. disc} \end{array} \right\}$

Thm: $D^\pi \mathrm{Fuk}^{\text{naive}}((T^2)^n) \cong D^b\mathrm{Coh}(E^n)$

Main ingredient: generation criterion

Thm: Let $\mathcal{A} \subset \mathrm{Fuk}^{\text{naive}}(X)$ such that

(1) the composition $QH^*(X) \xrightarrow{\text{"bulk deformation" (interior marked pt)}} HH^*(\mathrm{Fuk}^{\text{naive}}(X)) \xrightarrow{\text{restrict}} HH^*(\mathcal{A})$ is an isomorphism

(2) the diagonal of $\mathcal{A}$ ($\mathcal{A}$ as $(\mathcal{A},\mathcal{A})$-bimod.) can be resolved up to idempotent, as a bimodule, by tensor products $\mathcal{Y}_\ell(K_-) \otimes \mathcal{Y}_r(K_+)$ — "smoothness"

where $K_\pm \in \mathcal{A} \rightsquigarrow \mathcal{A}$-modules
$$\mathcal{Y}_\ell(K)(L) := CF^*(K_-, L)$$
$$\mathcal{Y}_r(K_+)(L) := CF^*(L, K_+)$$

Then $\mathcal{A}$ split-generates $D^\pi \mathrm{Fuk}^{\text{naive}}(X)$

Point: the "resol.ⁿ of diagonal" here is carried out staying in X, not going to $X \times X$

Tool: (Máu – Wehrheim – Woodward in progress)

$\mathrm{Fuk}^\#(X)$: • objects $= \quad pt \xrightarrow{L_{01}} X_1 \xrightarrow{L_{12}} \cdots \xrightarrow{L_{k-1,k}} X_k = X$

where $L_{i,i+1} \subset \bar{X}_i \times X_{i+1}$ object of "naive Fukaya cat."

• morphisms: (for simplicity: 2-step case)

$\vec{L} = (\underset{\subset X_1}{L_{01}}, \underset{\subset \bar{X}_1 \times X}{L_{12}})$, $\vec{K} = (\underset{\subset Y_1}{K_{01}}, \underset{\subset \bar{Y}_1 \times X}{K_{12}})$

$\rightsquigarrow CF^{\alpha}(\vec{L},\vec{K}) = \begin{array}{c|c|c|c|c} pt & X_1 & X & Y_1 & pt \\ \end{array}$

$\qquad\qquad\qquad\qquad L_{01} \quad L_{12} \quad K_{12} \quad K_{01}$

complex generated by intersections between

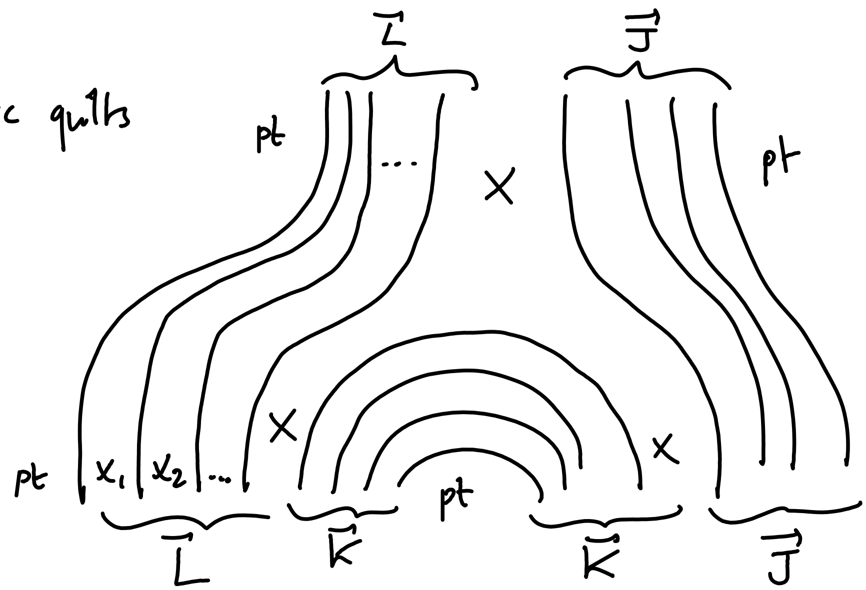
$$(L_{01} \times \overline{K_{12}}) \cap (\overline{L_{12}} \times K_{01}) \subset X_1 \times \overline{X} \times Y_1$$

(= constant quilted strips)

differential counts holom. quilted strips

- <u>composition:</u> $CF^{\alpha}(\vec{L},\vec{K}) \otimes CF(\vec{K},\vec{J}) \to CF^{\alpha}(\vec{L},\vec{J})$

counts holomorphic quilts



ie. collection of holom. strips in X_i, Y_j, Z_k

and one holom. disc in X

s.t. along seams $X_i \,|\, X_{i+1}$, matching condition

$\qquad ev \in L_{i,i+1} \subset X_i \times X_{i+1}$.

<u>Point:</u> Now $\exists$ functor $\mathcal{F}uk^{naive}(X \times \overline{X}) \longrightarrow End(\mathcal{F}uk^{\#}(X))$

$\uparrow$ category of A_∞-functors

on objects: $L \longmapsto (\vec{L}_{in} \longmapsto \vec{L}_{in} \# L)$ concatenation w/ L

Consider $\mathcal{A}^{\#} \subset Fuk^{\#}(X)$: subcat. where

$$\begin{cases} \text{all } X_i \equiv X \\ \text{all } L_{i,i+1} \text{ are either } \Delta \subset \bar{X} \times X \\ \qquad \text{or } \bar{K}_- \times K_+ , \quad K_\pm \in Ob(\mathcal{A}) \end{cases}$$

Lemma:

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathcal{A}^{\#} \\ \downarrow & & \downarrow \\ Tw(\mathcal{A}) & \hookrightarrow & Tw(\mathcal{A}^{\#}) \end{array}$$

↳ this is an equivalence

$\mathcal{A}(X \times \bar{X}) \subset Fuk^{Naive}(X \times \bar{X})$

↳ objects Δ, $\bar{K}_- \times K_+$ $(K_\pm \in \mathcal{A})$

$\Rightarrow \mathcal{A}(X \times \bar{X}) \xhookrightarrow[\text{Mau-Wehrheim-Woodward}]{MWW} End(Tw\,\mathcal{A})$

$$K_- \times K_+ \longmapsto (L \mapsto CF^*(L, K_-) \otimes K_+)$$

$$\Delta \longmapsto id$$

Claim: under the assumptions of the thm, MWW is a fully faithful embedding

PF: for $(K_\pm, \Delta)$ this is obvious (Yoneda)

$(K_\pm, K'_\pm)$

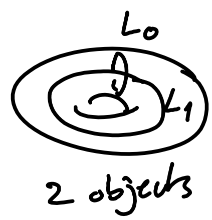
for $(\Delta, K_\pm)$ this follows from Poincaré duality

for (Δ, Δ) this follows from the assumption $QH^* \xrightarrow{\sim} HH^*(\mathcal{A})$. ∎

Assumption (2) (smoothness) for $\mathcal{A}$

$\Rightarrow$ resolution of diagonal in $X \times \overline{X}$

$$\Delta \in \mathrm{Fuk}(X \times \overline{X}) \xmapsto{\ MWW\ } \mathrm{End}(\mathrm{Fuk}^{\text{naive}}(X)) \ni id$$

$\mathcal{A}(T^2) =$ 

$\rightsquigarrow \quad Tw^{\pi}(\mathcal{A}(T^2)) \simeq D^b Coh(E)$.

$\mathcal{A}(T^4) = \{$ cat. with objects $L_i \times L_j \}$

then

$$\begin{array}{ccc} \mathcal{A}(T^4) & \xhookrightarrow[MWW]{} & \mathrm{End}(Tw(\mathcal{A}(T^2))) \\ & & \wr\wr \\ D^b Coh(E^2) & \hookrightarrow & \mathrm{End}(D^b(coh\, E)) \end{array}$$

<u>Point:</u> the image of $\mathcal{A}(T^4)$ under this chain of maps split-generates $D^b Coh(E^2)$.

QED.

<u>Rmk:</u> as a corollary, the Kapustin-Orlov coisotropic branes are not needed, they actually arise as formal direct summands in $D^{\pi}\mathrm{Fuk}$!