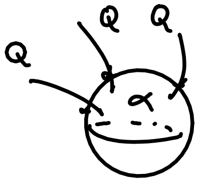
K. Fukaya – 19/1/09 – Mirror symmetry for toric manifolds (with Oh, Ohta, Ono)

<u>Closed GW theory</u>: $\Lambda_0 = \{ \sum a_i T^{\lambda_i},\ \lambda_i \geq 0,\ \lambda_i \to \infty,\ a_i \in \mathbb{Q} \}$

$Q \in H^*(X, \mathbb{Q})$, $\alpha \in H_2(X, \mathbb{Z})$, $\ell \geq 0$



$$\Rightarrow n(\alpha, \ell, Q) = \# \left\{ (u, z_1 \dots z_\ell) \Big/ \begin{array}{l} u: S^2 \to X \text{ holom.} \\ [u] = \alpha,\ u(z_i) \in Q \end{array} \right\} \Big/ \mathrm{Aut} \quad \in \mathbb{Q}.$$

$$\Rightarrow \Psi(Q) = \sum_{\alpha, \ell} \frac{T^{\omega \cdot \alpha}}{\ell !} n(\alpha, \ell, Q) \quad : \quad H^*(X, \Lambda_0) \to \Lambda_0$$

$Q = \sum w_i f_i$, f_i basis of $H^*(X, \mathbb{Q})$ $\to$ Ψ = function of w_i's

$\rightsquigarrow$ define $f_i \cup_Q f_j$ by $\langle f_i \cup_Q f_j, f_k \rangle = \dfrac{\partial^3 \Psi}{\partial w_i \partial w_j \partial w_k}(Q)$

big quantum product at $Q \in H^*(X)$

($\rightsquigarrow$ Frobenius mfld structure on $H^*(X)$)

<u>Open-closed GW theory</u>: $L \subset X$ Lagr. submfld,

$\phi : \underset{\substack{\Psi \\ Q}}{H(X, \Lambda_0)} \times \underset{\substack{\Psi \\ P}}{H(L, \Lambda_0)} \to \Lambda_0$ defined by

- $$n(\beta, k, \ell, P, Q) := \# \left\{ (u, z_1 \dots z_\ell, z'_1 \dots z'_k) \Big/ \begin{array}{l} u: (D^2, \partial D^2) \xrightarrow{\text{holom.}} (X, L),\ [u] = \beta \\ z_i \in \operatorname{int} D^2,\ u(z_i) \in Q \\ z'_i \in \partial D^2,\ u(z'_i) \in P \end{array} \right\} \Big/ \mathrm{Aut}$$
  $\beta \in \pi_2(X, L)$

$$\phi(Q, P) := \sum_{\beta, k, \ell} \frac{T^{\omega \cdot \beta}}{k! \, \ell !} n(\beta, k, \ell, P, Q)$$

(here: z'_i <u>not</u> assumed in cyclic order, hence divide by $k!$)

Then define family of products $q^{Q,P}_{\mathrm{sym}} : \underset{\substack{\Psi \\ \sum w_i f_i}}{H(X, \Lambda_0)^{\otimes \ell}} \otimes \underset{\substack{\Psi \\ \sum x_i e_i}}{H(L, \Lambda_0)^{\otimes k}} \to H(L, \Lambda_0)$

f_i basis of $H(X, \mathbb{Q})$
e_i —"— $H(L, \mathbb{Q})$

by $\langle q^{Q,P}_{sym}(f_{i_1}\otimes\cdots\otimes f_{i_l}, e_{j_1}\otimes\cdots\otimes e_{j_k}), e_{j_0}\rangle_{PD \text{ on } H^*(L)}$

$$:= \frac{\partial^{k+l+1}\phi}{\partial w_{i_1}\cdots\partial w_{i_l}\partial x_{j_1}\cdots\partial x_{j_k}\partial x_{j_0}}(P,Q).$$

This is a symmetrization of an operator

$q^{Q,P}: \underset{\text{symm.}}{\underset{\uparrow}{H(X,\Lambda_0)^{\otimes l}}} \otimes \underset{\text{not symm.}}{\underset{\uparrow}{H(L,\Lambda_0)^{\otimes k}}} \to H(L,\Lambda_0)$ which specializes to

$q^{Q,P}(1, e_{i_1}\otimes\cdots\otimes e_{i_k}) = m_k^{Q,P}(e_{i_1}\cdots e_{i_k})$ "bulk deform^{n}" of A_∞-str. in Floer theory

⚠ in general, definition issues!

- X toric manifold, L = orbit of T^n action
  $X \supset T^n$, $Q = T^n$-equivariant cycle. bulk deform^{n}
  $P = b + x_0[L]^\#$ where $b \in H^1(L)\otimes\Lambda_0$ weak bounding cochain
  $[L]^\# \in H^n(L)$

  In this case, $n(\beta,k,l,P,Q)\in\mathbb{Q}$ is well-def^{d}! (cf. Katz-Liu)

  [Fano case: easy; non-Fano case still ok using toric structure]

- Q_1, Q_2 T^n-equiv^t cycles,

  Then $\phi(Q_1,P)\neq\phi(Q_2,P)$ diff^t bulk deform^{n}...

  however, $P \mapsto \psi^{Q_1}(P) := \phi(Q_1,P)$
  $P \mapsto \psi^{Q_2}(P) := \phi(Q_2,P)$

  coincide up to a change of variables on $H^1(L)\otimes\Lambda_0$!

- **Note:** $P = b + x_0 [L]^{\#} \Rightarrow \Phi(Q,P) = W(Q,b)\, x_0$.

  $W(Q,b) : H^*(X,\Lambda_0) \otimes H^1(L,\Lambda_0) \to \Lambda_0$

  LG superpotential with bulk.

- Quantum cap product (at Floer chain level, bulk defined by Q, boundary defined by P)

$$q^{Q,P} : QH^*(X,\Lambda)^{\otimes l} \longrightarrow CH_*(H(L,\Lambda_0)) := \bigoplus_k \mathrm{Hom}(H(L,\Lambda_0)^{\otimes k}, H(L,\Lambda_0))$$

Hochschild chain complex

Induces in homology: $I_{\#}^{Q,P} : H(X,\Lambda) \longrightarrow HH(HF^{Q,P}(L,\Lambda_0))$

$\hookrightarrow$ Lie + A_∞-alg.

$I_{\#}^{Q,P}$ $\begin{cases} L_\infty\text{-homomorphisms from } (H(X,\Lambda), \text{ trivial Lie bracket}) \\ A_\infty\text{-homomorphisms from } (H(X,\Lambda), \cup_Q + \text{quantum} \dots) \end{cases}$

- consider all $L = T^n$-orbits (fibers of moment map)

  $\Rightarrow$ category Lagfiber(X), objects $= T^n$-orbit.

get: $H(X,\Lambda_0) \longrightarrow \bigoplus_{L \in \mathrm{Lagfiber}(X)} HH(HF(L))$ ring homomorphism (quantum cap action)

Conj. This is an isomorphism under some assumptions.

- $HH(\mathrm{Lagfiber}(X)) \xrightarrow{\text{iso?}} \mathrm{Jac}(\Psi^Q)$

  (Jacobian ring of Ψ^Q: Λ_0-module)

  where $\Psi^Q : H^1(L,\Lambda_0) \to \Lambda_0$

$$\mathrm{Jac}\,\Psi^Q = \bigoplus_{y \in \mathrm{crit}\,\Psi^Q} \frac{\mathcal{O}_y}{\left(\frac{\partial \Psi^Q}{\partial y_i}\right)}$$

Ψ^Q : deformation of q product, $\Psi^Q(P) = \sum q(Q^{\otimes l}, P^{\otimes k})$

Main results:

Thm (FO^3):

$\forall Q \in H(X, \Lambda_0)$, $\exists$ ring isomorphism $\left(H(X,\Lambda_0), \cup_Q\right) \cong \mathrm{Jac}(\psi^Q)$

f_i basis of $H(X, Q)$: $\quad f_i \longmapsto \left[\dfrac{\partial \psi^Q}{\partial w_i}\right]$

where $Q = \sum w_i f_i$

Moreover, PD pairing on $H(X, \Lambda_0) \longleftrightarrow$ residue pairing on Jac:

Thm 2 (FO^3):

$$\langle I_\#(\nu_1), I_\#(\nu_2)\rangle_{Res} = \langle \nu_1, \nu_2\rangle_{PD_X} \qquad \forall \nu_i \in H(X, \Lambda_0)$$

Residue pairing: assume $\psi^Q : H^1(L, \Lambda_0) \to \Lambda_0$ is Morse

Then $\mathrm{Jac}\,\psi^Q \otimes \Lambda = \bigoplus_{y \in \mathrm{crit}\,\psi^Q} \Lambda \cdot 1_y$

pairing: $\langle 1_y, 1_{y'}\rangle_{Res} = \begin{cases} 0 & \text{if } y \neq y' \\ (\det \mathrm{Hess}_y \psi^Q)^{-1} & \text{if } y = y' \end{cases}$

NB: Jacobian ring & residue pairing come up in K. Saito's work on singularities $\rightsquigarrow$ define a Frobenius structure

The thms say the Frobenius structures on $QH(X) \leftrightarrow \mathrm{Jac}\,\psi^Q$ are isomorphic.

- More about the relation b/w Poincaré duality & Hessian of ψ^Q: split Thm 2 into 2 statements:

$I_\# : \underset{\langle,\rangle_{PD_X}}{H(X, \Lambda_0)} \longrightarrow \underset{\langle,\rangle_{PD_L}}{H(L, \Lambda_0)}$ as above (part of map to HH...)

dualize to $\quad I^\# : H(L, \Lambda_0) \to H(X, \Lambda_0)$

$$\langle I_\#(\nu), w\rangle_L = \langle \nu, I^\#(w)\rangle_X$$

Thm 3: $$< I^{\#}(v_1), I^{\#}(v_2) >_{PD_X} = \sum_{e_I, e_J} g^{IJ} < m_2(v_1, e_I), m_2(v_2, e_J) >_{PD_L}$$

where e_I basis of $H(L)$, $g_{IJ} = < e_I, e_J >_L$

Thm 4: (Cho + ε)

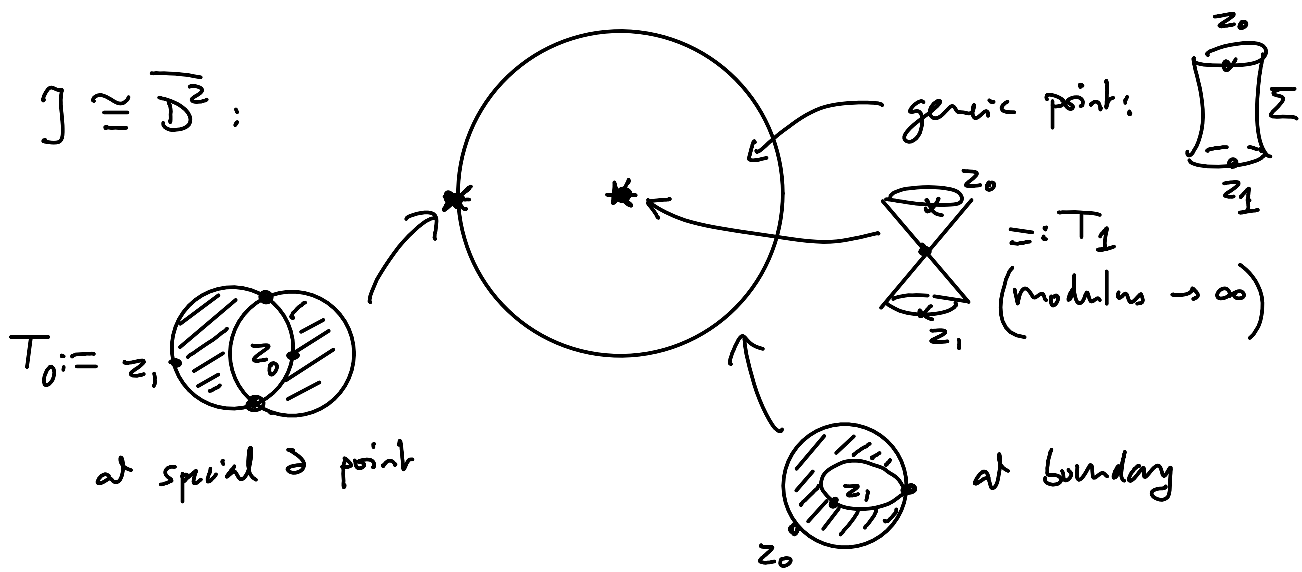
$(H(L, \Lambda_0), m_2) \cong$ Clifford algebra of Hess ψ^Q
and isomorphism respects PD pairing.

Thm 3 + 4 $\Rightarrow$ Thm 2.

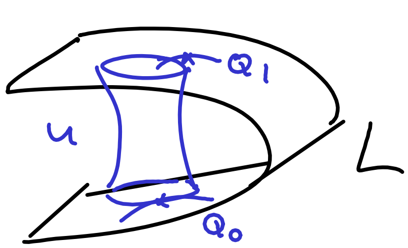
focus on thm. 3:

$$\mathcal{J} = \{ (\Sigma, z_0, z_1) \,/\, \Sigma \text{ genus } 0,\ \partial\Sigma = \underset{z_0}{\underset{\Psi}{S^1}} \sqcup \underset{z_1}{\underset{\Psi}{S^1}} \}$$

$\mathcal{J} \cong \overline{D^2}$:



$$\mathcal{M} = \left\{ (\Sigma, z_0, z_1, u) \,/\, \begin{array}{l} (\Sigma, z_0, z_1) \in \mathcal{J} \\ u : (\Sigma, \partial) \to (X, L) \\ u(z_0) \in Q_0 \in L, \quad u(z_1) \in Q_1 \in L \text{ fixed} \end{array} \right\}$$

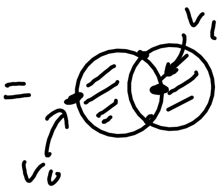


$$\mathcal{M} \xrightarrow{\pi} \mathcal{J}$$

forget map u.

$T_0, T_1 \in \mathcal{J}$ special points:

Then $\#\pi^{-1}(T_0) = \sum g^{IJ} \langle m_2(v_0, e_I), m_2(v_1, e_J)\rangle_{PD(L)}$

$T_0 =$ 

while $\#\pi^{-1}(T_1) = \langle I^{\#}(v_0), I^{\#}(v_1)\rangle_X$

$T_1 =$