I. Zharkov – 24/1/09 – Geometry of numbers & tropical curves

$V \cong \mathbb{R}^g$, $\Gamma_1, \Gamma_2 \subset V$ lattices

$Q : \Gamma_2^* \xrightarrow{\sim} \Gamma_1$ (polarization)

induced by a bilinear form Q on V^*,
assumed Q symmetric definite positive

$\rightarrow$ $X = V/\Gamma_1$ tropical PPAV (principally polarized ab. var.)

(here Γ_2 = integral lattice defining affine structure
Γ_1 = period lattice)

Tropical theta function:

Θ: PL convex function $V \to \mathbb{R}$

$$\Theta(x) := \max_{\lambda \in \Gamma_2^*} \left\{ \lambda \cdot x - \frac{1}{2} \lambda \cdot Q(\lambda) \right\}$$

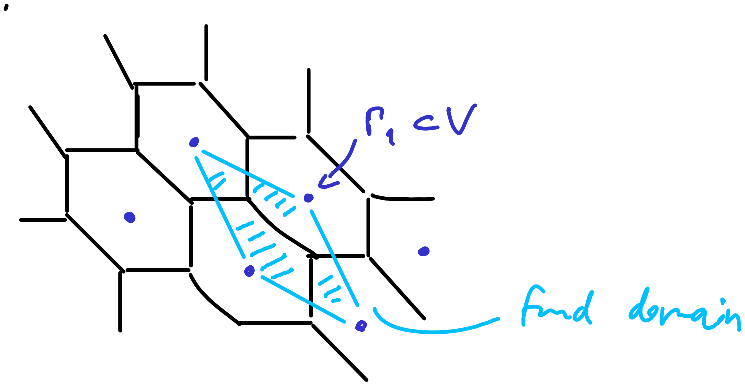
Θ-divisor := corner locus of Θ.

Since $\Theta(x+\gamma) = \Theta(x) + Q^{-1}(\gamma) \cdot x + \frac{1}{2} Q^{-1}(\gamma) \cdot \gamma$ $\forall \gamma \in \Gamma_1$,

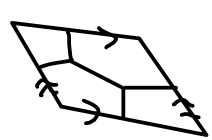
the Θ-divisor is Γ_1-periodic & hence defines a tropical divisor in X.

Ex: $g=2$ theta divisor:

$\equiv$ honeycomb

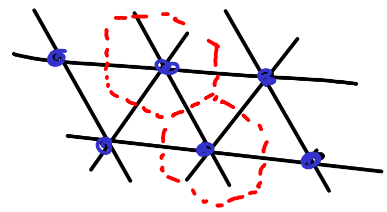


• Cells are Voronoi cells for the lattice $\Gamma_1 \subset V$

(on V/Γ_1,  $\Rightarrow$ genus 2 graph)

- equivalently: cells are dual to Delaunay decomposition of dual lattice

  $\Gamma_2^* \subset V^*$



Def: A *dicing* is a Delaunay decomp. given by families of hyperplanes through lattice points of Γ_2^*

In dims. 2 & 3 every Delaunay decomp. is a dicing.
(false in dim ≥ 4)

Maximal $\iff$ can't add any other hyperplanes without creating new vertices $\notin$ lattice.

- From a maximal dicing to the Voronoi cells:

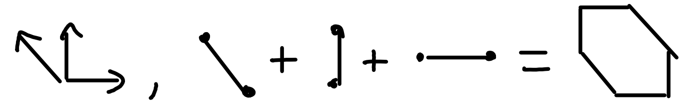
  $\ell_1 \dots \ell_r \in \Gamma_2$ primitive normal vectors to hyperplanes in a maximal dicing. $\Rightarrow$ TFAE:

  (1) $\{\ell_1 \dots \ell_r\}$ totally unimodular $\quad (r \geq g)$

  (2) $Q = \sum \alpha_i (\ell_i)^2$ for $\alpha_i > 0$

  (3) Voronoi cell $\doteq$ zonotope $\sum [\alpha_i \ell_i]$
  ↖ Minkowski sum

  ge. in



→ Q: what are maximal dicings in higher dim's?

Example: $D^\perp(g)$ – first perfect dicing

$D^1(g)$ = max. dicing into $g!$ simplices

Delaunay cells $0 \le x_{\sigma(1)} \le \dots \le x_{\sigma(g)} \le 1$, $\sigma \in \mathfrak{S}_g$

dicing hyperplanes defined by linear functions $l_1, \dots, l_g$ coord. duals

and $l_{ij} = l_i - l_j$

$\rightsquigarrow \frac{g(g+1)}{2}$ normal vectors = "zone vectors"

Zonotope = Minkowski sum of $(l_1, \dots l_g,\ l_{ij} = l_i - l_j)$

= <u>g-permutahedron</u>

<u>Relation to tropical curves:</u>

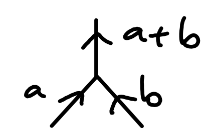
- C metric graph (no 1-valent vertices)

  $g = b_1(C)$

  $E = \{\text{edges + fixed orientation}\}$; $V :=$ functionals on circuits (see below)

- 1-forms = $\Gamma_2^* :=$ lattice of $\mathbb{Z}$-circuits

  ie assign integer currents to each edge satisfying Kirchhoff's law

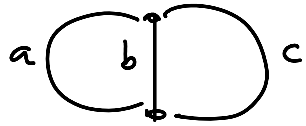
  ie. 

- Γ_2 = integer functionals on circuits

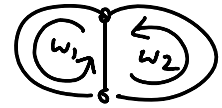
  = lattice gen^{d} by $\{v_e\}$, where $v_e : \{\text{circuits}\} \to \mathbb{Z}$ evaluation of circuit at edge $e \in E$

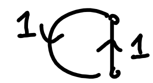
- Γ_1 = lattice of $\mathbb{Z}$-cycles on C

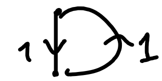
Q: $\Gamma_2^* \xrightarrow{\sim} \Gamma_1$.

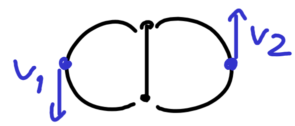
Ex: 

lengths of edges a, b, c

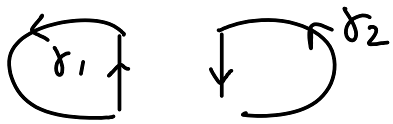
circuits: $\Gamma_2^* = \mathrm{span}\{w_1, w_2\}$ where 

ie. 

and 

$\Gamma_2 = \mathrm{span}(v_1, v_2)$ where 

v_i = circuit $\mapsto$ current in edge ...
(v_i dual to w_i)

cycles: 

, $\Gamma_1 = \mathrm{span}(\gamma_1, \gamma_2)$

Then via Q, $\gamma_1 = \begin{pmatrix} a+b \\ -b \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} -b \\ b+c \end{pmatrix}$

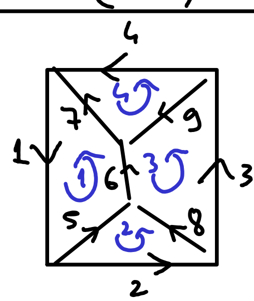
lengths of edges

(integrating circuit w_1 over γ_1 gives $a+b$
w_2 —"— $-b$)

In gal $Q = \sum_{e \in E} (\mathrm{length}(e))\, v_e^2$, $(v_e)_{e \in E}$ zone vectors

(here: $v_1, v_2, v_3 = -v_1 - v_2$)
since related as functionals on circuits

Schottky: <u>construct (max.) dicings from (trivalent) graphs</u>

$g = 4$ case: 

<u>9 edges</u>, 4 cycles

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

<u>9 zone vectors $\subset \mathbb{Z}^4$</u>

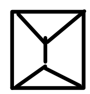
$\subset D^1(g)$

<u>In general</u>: get <u>$3g-3$ zone vectors</u>

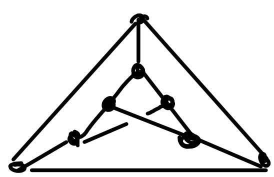
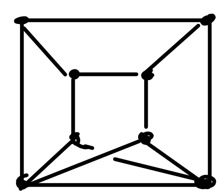
(trivalent genus g graph has $3g-3$ edges)

★ For $g=4$ there are 2 maximal dicings:

→ $D^1(4)$ (not given by a graph)

→ this one 

★ genus 5: more max. dicings from graphs

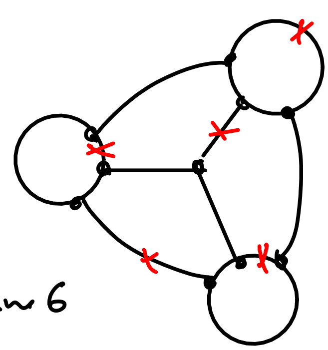
 and 

but there's an extra dicing that doesn't come from a graph:

$$D^3(5) := \begin{pmatrix} 1 & & & & & 1 & 0 & 1 & 0 & 1 \\ & 1 & & & & 0 & 1 & 0 & 1 & 1 \\ & & 1 & & & 1 & 1 & 0 & 0 & 1 \\ & & & 1 & & 0 & 0 & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

<u>10 zone vectors</u>

<u>Hope:</u>
[<u>Ludmil</u>]



genus 6

← double cover of plane quartic

× = where sheets $\asymp$

$\tilde{C}$ has genus 11 + involution τ

Prym variety of this unramified double cover

$$\mathrm{Prym}(\tilde{C}, \tau) := (V^{*-})^* / \Gamma_1^- \quad \text{antiinvariant part of Jacobian of } \tilde{C}$$

is a $g=5$ abelian variety & gives this dicing.

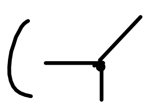
<u>Tropical Hodge:</u>

- A_k = tropical k-cycles

  $Z \in A_k \longrightarrow [Z] \in \Lambda^k \Gamma_2 \otimes \Lambda^k \Gamma_1$

<u>Constr:</u> $Z = \sum w_i P_i$ $\qquad$ P_i polyhedral cell of dim. k

tropical cycle, <u>not</u> an alg. trop cycle!

( $\partial = \bullet$!!)

$\Rightarrow$ build a k-chain with coeffts in $\Lambda^k \Gamma_2$:

$$\tilde{Z} := \sum w_i \underbrace{\det \langle p_i \rangle}_{\in \Lambda^k \Gamma_2} \otimes p_i$$

This is now a <u>cycle</u> in $C_k(X; \Lambda^k \Gamma_2)$

$\Rightarrow$ class $[Z] \in H_k(X; \Lambda^k \Gamma_2) \cong \Lambda^k \Gamma_2 \otimes \Lambda^k \Gamma_1$.

Actually takes values in

- <u>Hodge classes:</u> $\quad \mathrm{Hodge}_k := (\Lambda^k \Gamma_2 \otimes \Lambda^k \Gamma_1) \cap \ker \varphi$

$$\varphi : \Lambda^k V \otimes \Lambda^k V \to \Lambda^{k-1} V \otimes \Lambda^{k+1} V$$

$$u_1 \wedge \dots \wedge u_k \otimes v_1 \wedge \dots \wedge v_k \longmapsto \sum_i (-1)^i u_1 \wedge \dots \hat{u}_i \dots \otimes u_i \wedge v_1 \wedge \dots \wedge v_k$$

Hodge conj: $\left\| \begin{array}{ccc} A_k \text{ tropical } k\text{-cycles} & \longrightarrow & \text{Hodge}_k \\ Z & \longmapsto & [Z] \end{array} \right.$ is onto

How to get \$1,000,000:

→ show trop. Hodge fails (map is not surjective) for some maximal dicing,

e.g. ∃ candidates of type $D^3(5)$
(for suitable moduli parameter)

→ then build $X_\varepsilon = (\mathbb{C}^*)^g / e^{\Gamma, 1/\varepsilon}$

family of honest abelian varieties

should also fail (normal) Hodge conj. as $\varepsilon \to 0$ because otherwise Tropicalization of family of cycles would give tropical cycle representing given Hodge class