1/22/2009

Discussion Session

Auroux: Relation between wall-crossing

{DT is mirror to SLAG}

Formally similar to wall crossing

in moduli of CY

Q: Is there a geometric explanation?

Soibelman. The CY picture has

extra data: "Central charge"

See Gaiotto-Moore-Neitzke

Consider moduli of vector multiplets

$$
\begin{tikzcd}
\mathcal{M}^{VM} \arrow[d] \\
\mathrm{Stab}(\mathcal{C})
\end{tikzcd}
$$

$\leftarrow$ Hyperkähler at ∞, foliated in Abelian varieties

(e.g. $D^b(X)$)

In the base, there are walls (as in other picture)

So formally similar to K3

In GMN story, the total space is not understood mathematically, but is expected to be total space of integrable system.

They construct hyperkähler metric by giving a procedure to produce a twistor family of cplex structures.

<u>Claim:</u> Wall-crossing formula $\Longleftrightarrow$ Continuity of GMN metric.

Note that walls in GKN have the additional info of the central charge Z which defines them.

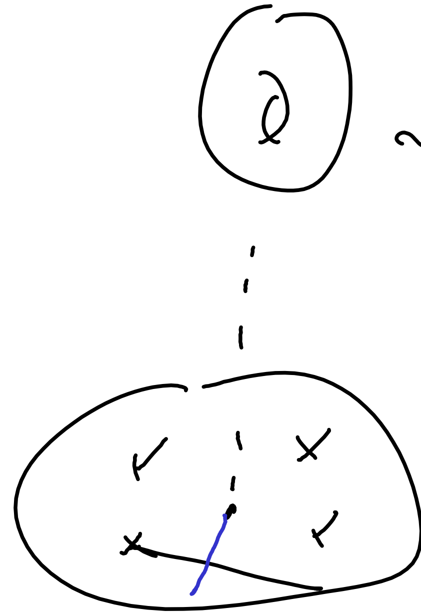
<u>Q:</u> Can one construct such a metric for K3?

Missing piece: Central Charge.

<u>Fukaya:</u>

In both cases, we have A_∞ categories

In SYZ picture,



$\rightsquigarrow CF^{\sim}(L,L)$ an A_∞ alg.

Crossing the wall gives an A_∞ homomorphism between A_∞ alge that are both iso to $\mathbb{Q}$-htpy type of T^n

Q: on Moduli side, do we have an analogous construction replacing the $CF^{\sim}(L,L)$ by $D^b Coh(X)$.

At the level of K-theory, this should already be true (i.e. If we tensor K-theory with Novikov ring, then the Wall-crossing formula should encode continuity of identification).

<u>Caveat:</u> Given M Hyperkähler, with S^2 family of $\mathbb{C}$-structures. If I & J are orthogonal,

A-e

$D^b\mathrm{Coh}(M, I)$ & $\mathrm{Fuk}(M, K)$ related.

True for K3.

<u>Tony</u> Take

$$\begin{array}{ccc} T^n & \longrightarrow & M \\ & & \downarrow \\ & & B \end{array}$$

SLAG for K, $\Rightarrow T^n$ is J-cplex

Now, hyperkähler rotate to make fibe holomorphic, then Fourier-Mukai as abelian fibration wrt J

obtain
$$\begin{tikzcd} M^\vee \arrow[d] \\ B \end{tikzcd}$$

Q: Should there be Quantum correction

Now dualize this fibration to get mirror to original object

Tony: $M^\vee \neq M$ in general
$\simeq$ htpy equiv

e.g. C a curve

$$\begin{tikzcd} \mathcal{M}_{Higgs, C}(Sp(4)) \arrow[rr, leftrightarrow, bend left, "\text{Fourier-Mukai}"] \arrow[dr, "h"] & & \mathcal{M}_{Higgs, C}(SO(5)) \arrow[dl] \\ & B & \end{tikzcd}$$

not htpy equiv

Q: When is Fourier–Mukai defined?

A: For K3, this can be done when we have only nodal singularities.

Denis: Because the SYZ fibration is not preserved by instanton corrections, we might be missing an extra layer in the Fourier–Mukai construction.

Tony: Consider

$$\begin{tikzcd} X \arrow[d] \\ \mathbb{P}^1 \arrow[u, bend left, dashed, "\sigma"] \end{tikzcd}$$

Then $X \times_{\mathbb{P}^1} X$ carries a sheaf which is just

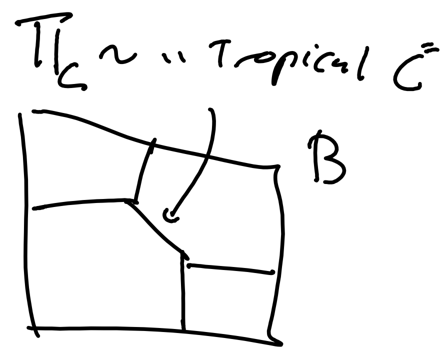
$\mathcal{J}_\Delta \leftarrow$ ideal sheaf of Δ

Bondal–Orlov criterion $\Rightarrow$ This gives an equivalence.

How does the correspondence work on objects?

Consider $C \subset M \leftarrow$ cplex

If M has SYZ fibration, we hope that C degenerates to tropical object in base B

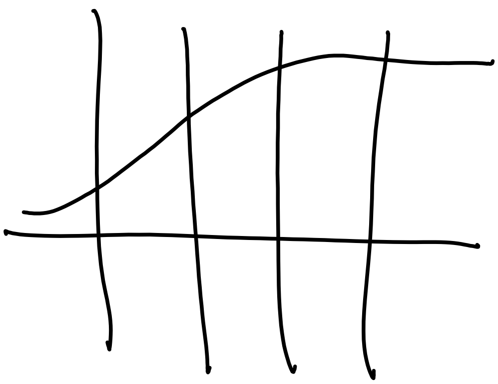
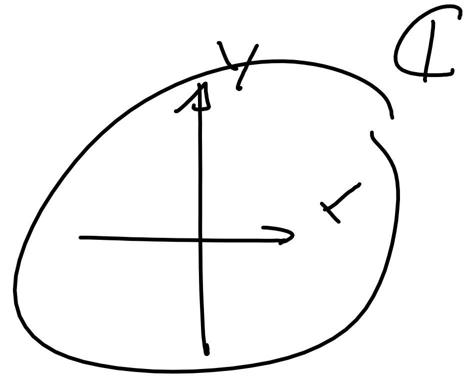


On the other side, the uncorrected picture is passing from TB to T^*B hence from $T\Pi_C/\mathbb{Z}^n \sim C$ to

$T^*\Pi_C/(\mathbb{Z}^n)^\vee$ which gives a Lagrangian

conormal

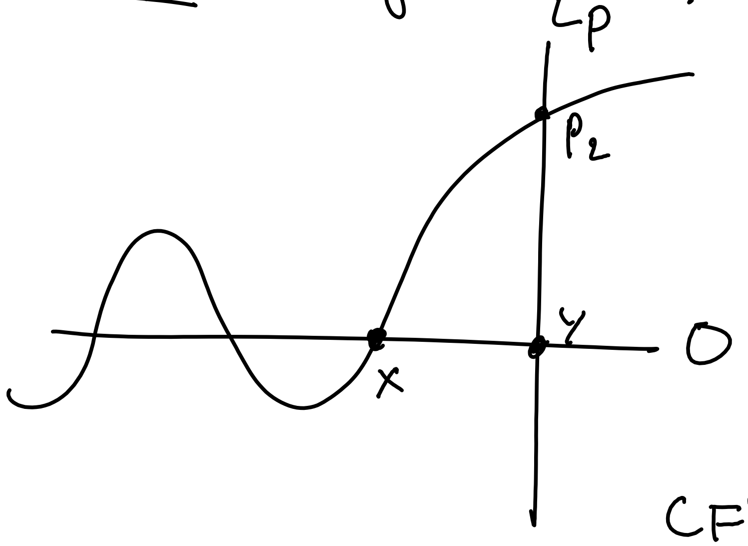
Leung Yau-Zaslow worked out for unbranched Lagrangian sections an equation that is conjecturally mirror to Yang-Mills

 $\longleftrightarrow$ 

$L = gr\left(\sum f_i(x)\,dx_i\right)$ $\qquad$ $\nabla = d + \sum f_i(y_i)\,dy_i$

For morphisms with 0-section, we can relate intersections of L with 0-section with sections of the "mirror" line bundle supported near appropriate points.

<u>Fukaya</u>: Simpler Story.



Then

$\langle \mu_X(Y), P_2 \rangle =$

$\langle m_2(x,y), P_2 \rangle$

<u>Key point</u>:

$CF^*(L, L_p)$ is complex family

When we cross a wall, the change of coordinates in wall-crossing-formula is arranged to be compatible with the identification of Floer hlgy

The main problem is the case where L is not a ~~e~~ section, in which case it's not clear how to extend to the caustics.