Homological Mirror Symmetry
for
Isolated Hypersurface Singularities

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at MIAMI 2009 Jan 23

1 Introduction

Our aim:

Understand mysterious correpondence among

isolated singularities

root systems, Weyl groups, Lie algebras

polyhedral groups, Fuchsian groups

finite dimensional algebras

and etc.

by Homological Mirror Symmetry.

This is necessary to study certain global structure of the base space of the universal unfolding.

It is $\mathfrak{h}/\mathcal{W}$ for an ADE singularity.

$\mathfrak{h}$: Cartan subalgebra, $\mathcal{W}$: Weyl group

$^\exists$ 2 different constructions of Frobenius structures (K. Saito's flat structures), deformation theory and Weyl group invariant theory.

Problem: the former is only defined as a germ.

2 Categories of singularities

$f \in S := k[x_1, \dots, x_n]$:

weighted homogeneous isolated singularity

$(\deg(x_i) =: r_i \in \mathbb{Z}_{>0}, \ \deg(f) =: h \in \mathbb{Z}_{>0})$.

$$\dim_k k[x_1, \dots, x_n] \Big/ (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) < \infty.$$

For $f = \sum a_{k_1,\dots,k_n} x_1^{k_1} \dots x_n^{k_n}$,

define an abelian group L_f by

$$L_f := \oplus_{i=1}^{n} \mathbb{Z}\vec{x}_i \oplus \mathbb{Z}\vec{f}/I,$$

where I is the subgroup generated by

$$\vec{f} - \sum_{i=1}^{n} k_i \vec{x}_i, \quad \text{for} \quad a_{k_1,\dots,k_n} \neq 0.$$

There exists a degree map

$$\deg : L_f \to \mathbb{Z}, \quad \vec{x}_i \mapsto r_i.$$

$R_f := S/(f)$ is L_f-graded.

gr^{L_f}-R_f: category of f.g. L_f-graded R_f-modules

proj^{L_f}-$R_f \subset \mathrm{gr}^{L_f}$-$R_f$: subcat. of proj. modules

Consider the triangulated category

$$D_{Sg}^{L_f}(R_f) := D^b(\mathrm{gr}^{L_f}\text{-}R_f)/K^b(\mathrm{proj}^{L_f}\text{-}R_f).$$

Remark 1

$$k(\vec{l}) := (R_f/\mathfrak{m})(\vec{l}) \in D_{Sg}^{L_f}(R_f), \quad \vec{l} \in L_f.$$

It is too difficult to study $D^{L_f}_{Sg}(R_f)$ itself.

$\Longrightarrow$ Replace it by the equivalent category

(and by a “natural” one)

Definition 2

$M \in \mathrm{gr}^{L_f}\text{-}R_f$ is a Cohen-Macauley module if

$$\mathrm{Ext}^i_{R_f}(R_f/\mathfrak{m}, M) = 0, \quad i < \dim R_f.$$

R_f is L_f-graded Gorenstein:

$$K_{R_f} \simeq R_f(-\vec{\epsilon}_f), \quad \vec{\epsilon}_f := \sum_{i=1}^{n} \vec{x}_i - \vec{f},$$

where $(\vec{l})$ is the grading shift by $\vec{l} \in L_f$.

Lemma 3 (Auslander)

A category of Cohen-Macauley R_f-modules $\mathrm{CM}^{L_f}(R_f) \subset \mathrm{gr}^{L_f}\text{-}R_f$ is a Frobenius category.

A Frobenius category is an exact category with enough injectives and projectives and its class of injectives coincides with that of projectives.

Definition 4 (stable category of $\mathrm{CM}^{L_f}(R_f)$)

Define a category $\underline{\mathrm{CM}}^{L_f}(R_f)$ as follows:

$$Ob(\underline{\mathrm{CM}}^{L_f}(R_f)) = Ob(\mathrm{CM}^{L_f}(R_f)).$$

$$\underline{\mathrm{CM}}^{L_f}(R_f)(M,N) := \mathrm{Hom}_{\mathrm{gr}^{L_f}\text{-}R_f}(M,N)/\mathcal{P}(M,N)$$

($g \in \mathcal{P}(M,N)$ iff there exist a projective object P and homomorphisms $g' : M \to P$ and $g'' : P \to N$ such that $g = g'' \circ g'$.)

Proposition 5 (Happel)

$\underline{\mathrm{CM}}^{L_f}(R_f)$ is a triangulated category.

Proposition 6

$\underline{\mathrm{CM}}^{L_f}(R_f)$ is finite

$$\sum_i \dim_k \underline{\mathrm{CM}}^{gr}(R_{f_W})(E, T^i F) < \infty,$$

and Krull-Schmidt, i.e., any object is

a finite direct sum of indecomposable objects.

Proposition 7 (Auslander-Reiten)

There exists the Serre functor $\mathcal{S}$ on $\underline{\mathrm{CM}}^{L_f}(R_f)$.

$\mathcal{S}$ is given by

$$\mathcal{S} := T^{n-2} \circ (-\vec{\epsilon}_f).$$

For $M \in \mathrm{CM}^{L_f}(R_f)$,

$^{\exists}$ **L_f-graded free resolution of M in gr^{L_f}-S**

$$0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow M \longrightarrow 0.$$

$^{\exists}f_0 : F_0 \longrightarrow F_1$ **(homotopy) such that**

$$f_1 f_0 = f \cdot \mathrm{id}_{F_0}, \quad f_0 f_1 = f \cdot \mathrm{id}_{F_1}.$$

Definition 8 (Eisenbud)

$$\overline{F} := \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right)$$

is called a L_f-graded matrix factorization of f.

Remark 9

$$Q := \begin{pmatrix} 0 & f_1 \\ f_0 & 0 \end{pmatrix}, \quad Q^2 = f \cdot \mathrm{Id}.$$

Example 10 (zero objects)

$$Q := \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix}, \quad Q^2 = f.$$

Example 11

$$Q := \begin{pmatrix} 0 & 0 & g_1 & f_1 \\ 0 & 0 & -f_0 & g_0 \\ g_0 & -f_1 & 0 & 0 \\ f_0 & g_1 & 0 & 0 \end{pmatrix}, \quad Q^2 = f + g.$$

Example 12

There exist $f_i \in \mathfrak{m}$, $i = 1, \ldots, n$ such that

$$f = x_1 f_1 + x_2 f_2 + \cdots + x_n f_n.$$

This decomposition defines matrix factorizations which will be isomorphic to $k(\vec{l})$ in $D^{L_f}_{Sg}(R_f)$.

Lemma 13

The category $\mathrm{MF}_S^{L_f}(f)$ of graded matrix factorizations of f is a Frobenius category.

Therefore, its stable category

$$\mathrm{HMF}_S^{L_f}(f) := \underline{\mathrm{MF}}_S^{L_f}(f)$$

is triangulated.

Lemma 14 (HMF$_S^{L_f}(f)$ is fractional CY)

On HMF$_S^{L_f}(f)$, $T^2 = (\vec{f})$. In particular, HMF$_S^{L_f}(f)$ is fractional CY of dimension $(n-2) - 2\frac{\epsilon_f}{h_f}$, where $\epsilon_f := \deg(\vec{\epsilon_f})$ and $h_f := \deg(\vec{f})$.

Remark 15

$$\overline{F} = \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right) \mapsto \mathrm{Coker}(f_1) \in \mathrm{CM}^{L_f}(R_f).$$

Proposition 16 (Buchweitz, Orlov)

There exists a triangulated equivalence

$$\mathrm{HMF}_S^{L_f}(f) \simeq \underline{\mathrm{CM}^{L_f}}(R_f) \simeq D_{Sg}^{L_f}(R_f).$$

Lemma 17 (Category Generating Lemma)

Suppose $\mathcal{T}' := \langle E_1, \ldots, E_n \rangle \overset{\text{full sub tri.}}{\subset} \mathrm{HMF}_S^{L_f}(f)$ generated by an exceptional collection $(E_1, \ldots, E_n)$ satisfies the following:

1. $\mathcal{T}'$ is closed under the shift $(\vec{l})$ for all $\vec{l} \in L_f$,

2. ${}^{\exists}E \in \mathcal{T}'$ isomorphic to $k(\vec{0})$ in $D_{Sg}^{L_f}(R_f)$.

Then $\mathcal{T}' \simeq \mathrm{HMF}_S^{L_f}(f)$.

(c.f. Kajiura-Saito-T, arXiv:0708.0210)

Proof of the Lemma:

Claim 18 ($\mathcal{T}'$ is right admissible)

For any $X \in \mathrm{HMF}^{L_f}_S(f)$ there is an exact triangle

$$N \to X \to M \to TN$$

where $N \in \mathcal{T}'$ and $\mathrm{Hom}(N, M) = 0$.

Claim 19 (The right orthogonal is zero)

$\mathrm{HMF}_S^{L_f}(f)(E(\vec{l}), T^i M) = 0$ for ${}^\forall \vec{l} \in L_f, {}^\forall i \in \mathbb{Z}$

$\iff \mathrm{Ext}^i_{R_f}(R_f/\mathfrak{m}, M) = 0$ for $i \neq d$

($M \in \mathrm{CM}^{L_f}(R_f)$ is a Gorenstein module)

$\iff M \in \mathrm{CM}^{L_f}(R_f)$ is free.

$\iff M \simeq 0$ in $\underline{\mathrm{CM}^{L_f}}(R_f)$.

Therefore, $\mathcal{T}' \simeq \mathrm{HMF}_S^{L_f}(f)$.

Consider the quotient stack

$$X_{L_f} := \left[\mathrm{Spec}(R_f)\backslash\{0\}\,/\mathrm{Spec}(k\cdot L_f)\right].$$

Then,

$$D^b\mathrm{coh}(X_{L_f}) \simeq D^b(\mathrm{gr}^{L_f}\text{-}R_f)/D^b(\mathrm{tor}^{L_f}\text{-}R_f).$$

We have an L_f-graded generalization of Orlov's semi-orthogonal decomposition.

Proposition 20

1. If $\epsilon_f > 0$,

$$D^b\mathrm{coh}(X_{L_f}) \simeq \left\langle D_{Sg}^{L_f}(R_f), \mathcal{A}(0), \ldots, \mathcal{A}(\epsilon_f - 1) \right\rangle,$$

where $\mathcal{A}(i) := \left\langle \mathcal{O}_{X_{L_f}}(\vec{l}) \right\rangle_{\deg(\vec{l})=i}$.

2. If $\epsilon_f = 0$, $D^b\mathrm{coh}(X_{L_f}) \simeq D_{Sg}^{L_f}(R_f)$.

3. If $\epsilon_f < 0$,

$$D_{Sg}^{L_f}(R_f) \simeq \left\langle D^b\mathrm{coh}(X_{L_f}), \mathcal{K}(0), \ldots, \mathcal{K}(-\epsilon_f + 1) \right\rangle,$$

where $\mathcal{K}(i) := \left\langle k(\vec{l}) \right\rangle_{\deg(\vec{l})=i}$.

3 Homological Mirror Symmetry

${}^t f$: transposition of f

Fermat type

$${}^t f := f = x_1^{p_1} + x_2^{p_2} + \cdots + x_n^{p_n}.$$

Chain

$$f = x_1^{p_1} + x_1 x_2^{p_2} + \cdots + x_{n-1} x_n^{p_n}$$

$${}^t f := x_1^{p_1} x_2 + x_2^{p_2} x_3 + \cdots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n}.$$

Loop

$$f = x_n x_1^{p_1} + x_1 x_2^{p_2} + \cdots + x_{n-1} x_n^{p_n}$$

$${}^t f := x_1^{p_1} x_2 + x_2^{p_2} x_3 + \cdots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n} x_1.$$

Conjecture 21 (c.f. T: arXiv:0711.3907)

$$D^b\mathrm{Fuk}^{\rightarrow}({}^t f) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta}/I) \simeq \mathrm{HMF}_S^{L_f}(f),$$

for some quiver $\vec{\Delta}$.

4 Covering ($n = 1$)

Theorem 22 For $f = {}^t f = x^{l+1}$,

there exist triangulated equivalences

$$D^b\mathrm{Fuk}^{\to}(f) \overset{\text{[Seidel]}}{\simeq} D^b(\text{mod-}\mathbb{C}\vec{\Delta}_{A_l}) \simeq \mathrm{HMF}_S^{L_f}(f),$$

where $\vec{\Delta}_{A_l}$ is the Dynkin quiver of type A_l.

5 Curve singularities ($n = 2$)

Consider polynomials of the following types:

Type I: $f = x^p + y^q$ $({}^t f = f)$

Type II: $f = x^p + xy^q$ $({}^t f = x^q + xy^p)$

Type III: $f = yx^p + xy^q$ $({}^t f = f)$

Theorem 23

For any f of type I, II and III,

$\exists$ **a quiver $\vec{\Delta}$ and relations I such that**

$$\mathrm{HMF}_S^{L_f}(f) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta}/I)$$

6 Sketch of Proof

1. **Find enough "good" matrix factorizations.**
2. **Show that these matrix factorizations form a strongly exceptional collection.**
3. **Use the "category generating lemma" to prove the above strongly exceptional collection is full.**

Type I

$$f(x,y) := x^p + y^q$$

Quiver with relations for $(p,q) = (6,7)$:

Order of vertices: lexicographical order

Dotted edges: commutative relations

$$Q_{ab} := \begin{pmatrix} 0 & 0 & y^{q-b} & x^{p-a} \\ 0 & 0 & -x^a & y^b \\ y^b & -x^{p-a} & 0 & 0 \\ x^a & y^{q-b} & 0 & 0 \end{pmatrix}$$

Type II

$$f(x,y) := x^p + xy^q$$

Quiver with relations for $(p,q) = (6,7)$:

Order of vertices: lexicographical order

Dotted edges: commutative or zero relations

$$Q_{ab} := \begin{pmatrix} 0 & 0 & y^{q-b} & x^{p-a} \\ 0 & 0 & -x^a & xy^b \\ xy^b & -x^{p-a} & 0 & 0 \\ x^a & y^{q-b} & 0 & 0 \end{pmatrix}$$

Type III

$$f(x,y) := yx^p + xy^q$$

Quiver with relations for $(p,q) = (5,6)$:

Order of vertices: lexicographical order

Dotted edges: commutative or zero relations

$$Q_{ab} := \begin{pmatrix} 0 & 0 & y^{q-b} & x^{p-a} \\ 0 & 0 & -yx^a & xy^b \\ xy^b & -x^{p-a} & 0 & 0 \\ yx^a & y^{q-b} & 0 & 0 \end{pmatrix}$$

Corollary 24 (ADE singularity)

f	${}^t f$	Type
$x^{l+1}+y^2$	$x^{l+1}+y^2$	A_l
$yx^{l-1}+y^2$	$x^{l-1}+xy^2$	D_l
x^4+y^3	x^4+y^3	E_6
yx^3+y^3	x^3+xy^3	E_7
x^5+y^3	x^5+y^3	E_8.

$$D^b\mathrm{Fuk}^{\to}({}^t f) \overset{\text{[Seidel]}}{\simeq} D^b(\text{mod-}\mathbb{C}\vec{\Delta}) \simeq \mathrm{HMF}^{L_f}_S(f),$$

where $\vec{\Delta}$ is the Dynkin quiver of corresp. type.

There is an easy way to describe $D^b\mathrm{Fuk}^{\to}({}^t f)$.

Proposition 25 (Seidel)

$\exists$ a distinguished basis of vanishing cycles $\mathcal{L}_1, \ldots, \mathcal{L}_\mu$ and a choice of gradings on $\mathcal{L}_i$ such that $\mathrm{Fuk}^{\to}(\mathcal{L}_i, \mathcal{L}_j)$ is at most one dimensional complex concentrated on degree 0.

Hence, $D^b\mathrm{Fuk}^{\to}({}^t f) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta}'/I)$.

(Proof: Use A'Campo's "devide".)

A recipe to get $\vec{\Delta}'$ and I:

1. **Consider a real Morsification g of f.**
2. **Put a vertex $\bullet$ to ODP.**
3. **Put a vertex with a sign $\oplus$ ($\ominus$) into each compact connected component of $\mathbb{R}^2 \backslash g^{-1}(0)$ if g is positive (negative) on the component.**
4. **Draw 1 arrow $\oplus \longrightarrow \bullet$ ($\bullet \longrightarrow \ominus$) if $\bullet$ is on the boundary of the component for $\oplus$ ($\ominus$).**
5. **Draw 1 dotted line $\oplus$ - - - $\ominus$ if there are 2 paths from $\oplus$ to $\ominus$. ($\Rightarrow$ a commutative relation.)**

Corollary 26 (c.f. Auroux-Katzarkov-Orlov)

For any f of type I, the HMS conjecture holds.

There are many f of Type II and Type III for which we can check the HMS conjecture, however, we have not yet suceeded to prove it in general.

(Problem: find “good” real morsifications.)

However, we have

Corollary 27 For any f of type II,

the HMS conjecture holds at "homology level":

$H_1(X_{t_f}, \mathbb{Z}) \simeq K_0(\mathrm{HMF}_S^{L_f}(f))$ and $S_{ij} = \chi_{ij}$

where

S_{ij}: Seifert matrix of a distinguished basis

of vanishing cycles in the Milnor fiber X_{t_f}

("an upper half" of the intersection matrix)

χ_{ij}: $= (\chi(E_i, E_j))_{ij}$ for a full s.e.c. $(E_1, \ldots, E_\mu)$

7 Surface singularities ($n = 3$)

Theorem 28 (DE singularity ($L_f \overset{\deg}{\simeq} \mathbb{Z}, \epsilon_f > 0$))

f	${}^t f$	Type
$yx^m + xy^2 + z^2$	$yx^m + xy^2 + z^2$	D_{2m} ($m \geq 2$)
$yx^m + y^2 + xz^2$	$zx^m + xy^2 + z^2$	D_{2m+1} ($m \geq 2$)
$zx^2 + y^3 + z^2$	$x^2 + y^3 + xz^2$	E_6
$yx^3 + y^3 + z^2$	$x^3 + xy^3 + z^2$	E_7
$x^5 + y^3 + z^2$	$x^5 + y^3 + z^2$	E_8.

$$D^b\mathrm{Fuk}^{\to}({}^t f) \overset{\text{[Seidel]}}{\simeq} D^b(\text{mod-}\mathbb{C}\vec{\Delta}) \overset{\text{[KST]}}{\simeq} \mathrm{HMF}^{\mathbb{Z}}_S(f),$$

where $\vec{\Delta}$ is the Dynkin quiver of corresp. type.

[KST]: H. Kajiura, K. Saito and T.

math.AG/0511155.

Consider polynomials of the following types:

Type I: $f = x^{p_1} + y^{p_2} + z^{p_3}$

Type II: $f = x^{p_1} + y^{p_2} + yz^{\frac{p_3}{p_2}}$

Type III: $f = x^{p_1} + y^{q_3+1}z + yz^{q_2+1}$

Type IV: $f = x^{p_1} + xy^{\frac{p_2}{p_1}} + yz^{\frac{p_3}{p_2}}$

Type V: $f = zx^k + xy^l + yz^m$

Theorem 29

For any f of type I, II, III, IV and V,

$\exists$ a full exceptional collection in $\mathrm{HMF}_S^{L_f}(f)$.

Show the following and use semi-orthogonal decomposition of $\mathrm{HMF}_S^{L_f}(f)$.

Proposition 30

X_{L_f} is isomorphic to $\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}$, an orbifold $\mathbb{P}^1$ with 3-isotropic points of order $\alpha_1, \alpha_2, \alpha_3$:

Type	$A_f := (\alpha_1, \alpha_2, \alpha_3)$
I	(p_1, p_2, p_3)
II	$(p_1, \frac{p_3}{p_2}, (p_2 - 1)p_1)$
III	$(p_1, p_1 q_2, p_1 q_3)$
IV	$(p_1, (\frac{p_3}{p_2} - 1)p_1, \frac{p_3}{p_1} - \frac{p_3}{p_2} + 1)$
V	$(lm - m + 1, lk - k + 1, km - m + 1).$

Remark 31

If $L_f \simeq \mathbb{Z}$, A_f is called the Dolgachev number.

Remark 32

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}>1 \Leftrightarrow \epsilon_f>0 \Leftrightarrow f : \text{ADE sing.}$$

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}=1 \Leftrightarrow \epsilon_f=0 \Leftrightarrow f : \text{Elliptic sing.}$$

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}<1 \Leftrightarrow \epsilon_f<0.$$

Remark 33 If $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}>1$,

then, $\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3} \simeq [\mathbb{P}^1/G_{\alpha_1,\alpha_2,\alpha_3}]$, where

$$G_{\alpha_1,\alpha_2,\alpha_3} :=< g_1, g_2, g_3 \mid g_1^{\alpha_1} = g_2^{\alpha_2} = g_3^{\alpha_3} >$$

binary polyhedral group.

Sketch of the proof (of Proposition):

Set $R_{A_f} := k[X_1, X_2, X_3]/(X_1^{\alpha_1} + X_2^{\alpha_2} + X_3^{\alpha_3})$ and $L_{A_f} := \bigoplus_{i=1}^{3} \mathbb{Z}\vec{X}_i \Big/ \Big(\alpha_i \vec{X}_i - \alpha_j \vec{X}_j; 1 \le i < j \le 3\Big)$.

Note that

$$\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}) \simeq \mathrm{gr}^{L_{A_f}}\text{-}R_{A_f}/\mathrm{tor}^{L_{A_f}}\text{-}R_{A_f}.$$

We can show that $^\exists$ a natural embedding

$$R_f \hookrightarrow R_{A_f}, \quad L_f \hookrightarrow L_{A_f},$$

which induces an equivalence

$$\mathrm{gr}^{L_f}\text{-}R_f/\mathrm{tor}^{L_f}\text{-}R_f \simeq \mathrm{gr}^{L_{A_f}}\text{-}R_{A_f}/\mathrm{tor}^{L_{A_f}}\text{-}R_{A_f}.$$

Proposition 34 (Geigle–Lenzing)

$D^b\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3})$ has a full exceptional collection.

Proposition 35 (Geigle–Lenzing)

If $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}>1$,

$D^b\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}) \simeq D^b(\mathrm{mod}\text{-}\vec{\Delta})$, where $\vec{\Delta}$ is the extended Dynkin quiver of corresponding type.

Theorem 36

Let $\boldsymbol{f}$ be an Arnold's 14 exceptional singularity ($\boldsymbol{L_f \simeq \mathbb{Z}, \epsilon_f = -1}$). Then,

HMF$_{\boldsymbol{S}}^{\mathbb{Z}}(\boldsymbol{f}) \simeq \boldsymbol{D^b}$(mod-$\mathbb{C}\vec{\boldsymbol{\Delta}}_{\boldsymbol{A_f}}/\boldsymbol{I}$), where ($\vec{\boldsymbol{\Delta}}_{\boldsymbol{A_f}}, \boldsymbol{I}$) is

Diagram for $\boldsymbol{A_f = (3,3,4)}$.

$\boldsymbol{I}$: two relations along the double dotted line.

H. Kajiura, K. Saito and T. arXiv:0708.0210

Arnold's 14 exceptional singularities

f	A_f	HMS
$x^7+y^3+z^2$	(2,3,7)	OK
$x^5+xy^3+z^2$	(2,3,8)	OK
$x^5y+y^3+z^2$	(2,4,5)	OK
$x^4+y^3+xz^2$	(2,3,9)	OK
$zx^4+y^3+z^2$	(3,3,4)	$(K_0,\chi+{}^t\chi)\simeq(H_2,-I)$
$yx^4+xy^3+z^2$	(2,4,6)	OK
$x^5+y^2z+z^2$	(2,5,5)	OK
$zx^3+xy^3+z^2$	(3,3,5)	$(K_0,\chi+{}^t\chi)\simeq(H_2,-I)$
$x^3y+y^3+xz^2$	(2,4,7)	OK
$x^4y+y^2z+z^2$	(3,4,4)	$(K_0,\chi+{}^t\chi)\simeq(H_2,-I)$
$x^4+zy^2+z^2$	(2,5,6)	OK
$zx^3+y^3+xz^2$	(3,3,6)	$(K_0,\chi+{}^t\chi)\simeq(H_2,-I)$
$x^3y+y^2z+z^2x$	(3,4,5)	$(K_0,\chi+{}^t\chi)\simeq(H_2,-I)$
$x^4+y^2z+yz^2$	(4,4,4)	$(K_0,\chi+{}^t\chi)\simeq(H_2,-I)$

8 HMS for Cusp Singularities

If $\mathrm{HMF}_S^{L_f}(f) \simeq D^b\mathrm{Fuk}^{\to}({}^t f)$ holds,

then there should exist a category $\mathcal{F}$ and a semi-orthogonal decompotion in Fukaya side:

1. **If $\epsilon_f > 0$,**

$$\mathcal{F} \simeq \left\langle D^b\mathrm{Fuk}^{\to}({}^t f), \mathcal{A}(0), \ldots, \mathcal{A}(\epsilon_f - 1)\right\rangle .$$

2. **If $\epsilon_f = 0$, $\mathcal{F} \simeq D^b\mathrm{Fuk}^{\to}({}^t f)$.**

3. **If $\epsilon_f < 0$,**

$$D^b\mathrm{Fuk}^{\to}({}^t f) \simeq \langle \mathcal{F}, \mathcal{K}(0), \ldots, \mathcal{K}(-\epsilon_f + 1)\rangle .$$

Is $\mathcal{F}$ the derived Fukaya category of a singularity?

$f \in$ **Type I-V (surface sing.)**

Conjecture 37 **(c.f., T: arXiv:0711.3907)**

$$D^b\mathrm{coh}(X_{L_f}) \simeq D^b\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}) \simeq D^b\mathrm{Fuk}^{\rightarrow}(T_{\alpha_1,\alpha_2,\alpha_3}),$$

where $T_{\alpha_1,\alpha_2,\alpha_3} := x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^{\alpha_3} + x_1x_2x_3$.

($(\alpha_1, \alpha_2, \alpha_3)$ **in Fuk. is Gablielov number**)

HMS description of Arnold's strange duality.

(Dolgachev # of f **=Gabrielov # of** ${}^t f$**)**

Remark 38 If $\alpha_3 = 1$, then the conj. is true.

(Auroux-Katzarkov-Orlov, Seidel)

Remark 39 $^\exists$ an isom. of lattices

$$\left(K_0(D^b\mathrm{coh}(X_{L_f})), \chi + {}^t\chi\right) \simeq (H_2(T_B, \mathbb{Z}), -I)\,.$$

Theorem 40 (Conj. is true if $\alpha_3 = 2$)

$D^b\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,2}) \simeq D^b\mathrm{Fuk}^{\to}(T_{\alpha_1,\alpha_2,2})$.

In particular, if $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{2} > 1$ (f:ADE sing.), then we can choose a v.c. $\mathcal{L} \in D^b\mathrm{Fuk}^{\to}(T_{\alpha_1,\alpha_2,2})$ such that

$$\begin{aligned}\left\langle \mathrm{HMF}^{L_f}_S(f), \mathcal{O}(0) \right\rangle &\simeq D^b\mathrm{coh}(\mathbb{P}^1_{\alpha_1,\alpha_2,2}) \\ &\simeq D^b\mathrm{Fuk}^{\to}(T_{\alpha_1,\alpha_2,2}) \\ &\simeq \left\langle D^b\mathrm{Fuk}^{\to}(f), \mathcal{L} \right\rangle .\end{aligned}$$

Remark 41

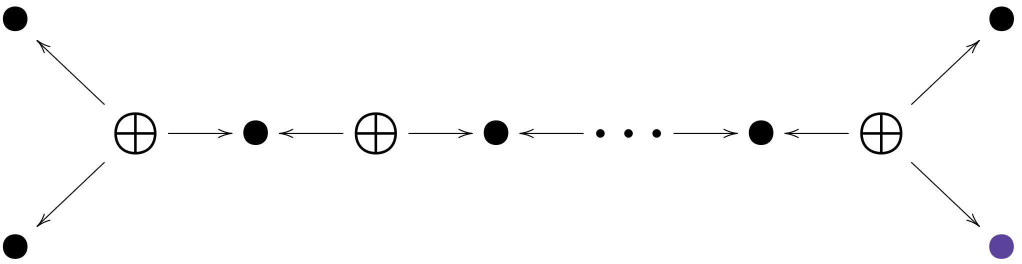
Extended Dynkin = Dynkin + 1 vertex

Quivers with relations for $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{2}>1$:

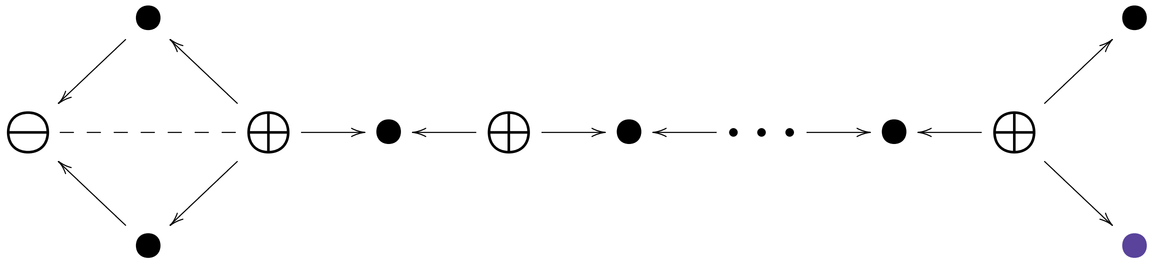
(● $= \mathcal{L}$: vertex to remove)

$x_1^{\alpha_1}+x_2^2+x_3^2+x_1x_2x_3$ (α_1:even ($\tilde{D_{2l}}$)):

($\Rightarrow g = x_1^{\alpha_1}+x_2^2-\frac{1}{4}x_1^2x_2^2$)

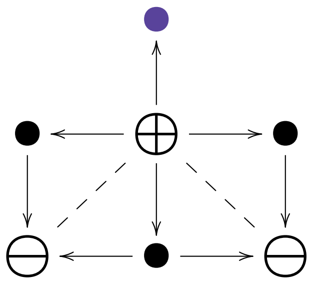


$x_1^{\alpha_1}+x_2^2+x_3^2+x_1x_2x_3$ (α_1:odd ($\tilde{D_{2l+1}}$)):



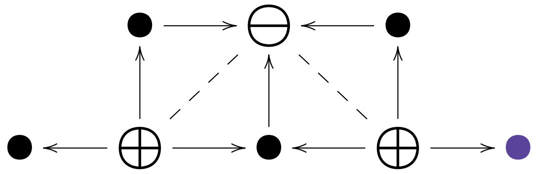
$$x_1^3 + x_2^3 + x_3^2 + x_1x_2x_3 \ (\tilde{E}_6):$$

$$(\Rightarrow g = x_1^3 + x_2^3 - \tfrac{1}{4}x_1^2x_2^2)$$



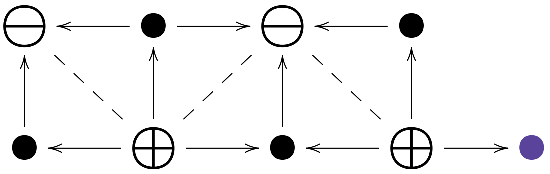
$$x_1^4 + x_2^3 + x_3^2 - x_1x_2x_3 \ (\tilde{E}_7):$$

$$(\Rightarrow g = x_1^4 + x_2^3 - \tfrac{1}{4}x_1^2x_2^2)$$



$$x_1^5 + x_2^3 + x_3^2 - x_1x_2x_3 \ (\tilde{E}_8):$$

$$(\Rightarrow g = x_1^5 + x_2^3 - \tfrac{1}{4}x_1^2x_2^2)$$

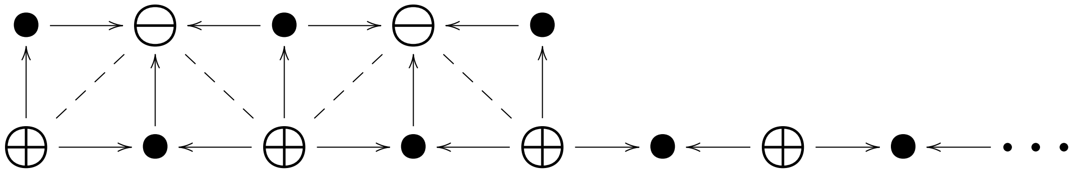


Quivers with relations for $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{2}\leq 1$:

(number of vertices $=\alpha_1+\alpha_2+\alpha_3-1$)

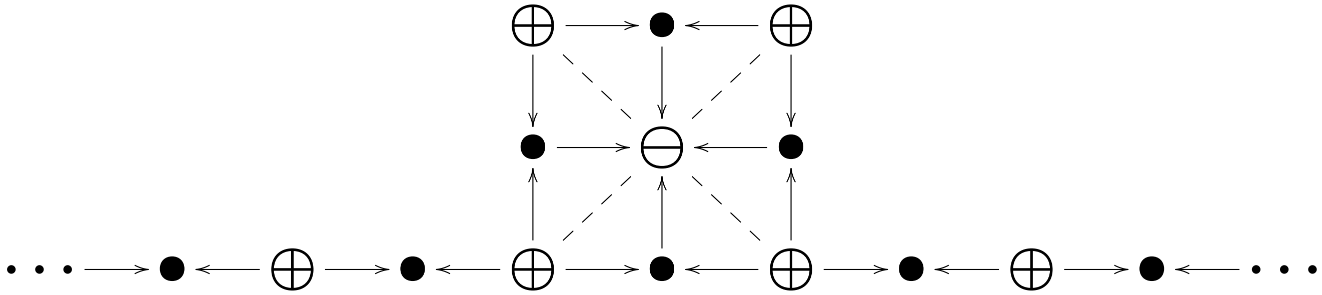
$x_1^{\alpha_1}+x_2^3+x_3^2+x_1x_2x_3$ **(**$\alpha_1\geq 6$**):**

($\Rightarrow g=x_1^{\alpha_1}+x_2^3-\frac{1}{4}x_1^2x_2^2$**)**



$x_1^{\alpha_1}+x_2^{\alpha_2}+x_3^2+x_1x_2x_3$ **(**$\alpha_1,\alpha_2\geq 4$**):**

($\Rightarrow g=x_1^{\alpha_1}+x_2^{\alpha_2}-\frac{1}{4}x_1^2x_2^2$**)**



Can also prove for $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 3)$.

Sorry, no picture

(3-dimensional version of A'Campo's method)

General case, we are now checking details.

(idea: reduction to $(3, 3, 3)$ case)

End.

Thank you very much.