A. Kuznetsov – 19/1/09 – Fractional CY categories

$\mathcal{T}$ triangulated cat. with finite dim'l hom spaces

Def: A <u>Serre functor</u> for $\mathcal{T}$ is an exact autoequivalence $S_{\mathcal{T}}: \mathcal{T} \to \mathcal{T}$ s.t. $\exists$ bifunctorial isomorphism $\mathrm{Hom}(F,G)^* \xrightarrow{\cong} \mathrm{Hom}(G, S_{\mathcal{T}}F)$

Ex: $\mathcal{T} = D^b(X)$, X smooth proj. $\Rightarrow S_X(F) = F \otimes \omega_X[\dim X]$ is a Serre functor

Def: A fractional CY category is a triangulated cat. $\mathcal{T}$ s.t. $S_{\mathcal{T}}$ exists and $S_{\mathcal{T}}^b \cong [a]$ for some $a, b \in \mathbb{Z}$, $b \neq 0$
(say $\mathcal{T}$ has fractional dimension a/b).

Ex: 1) X CY variety $\Rightarrow S_X \cong [\dim X]$

2) X = Enriques surface: $\omega_X \neq \mathcal{O}_X$ but $\omega_X^2 \cong \mathcal{O}_X$, so $S_X^2 \cong [4]$.

3) X CY, G acts on X freely $\rightsquigarrow D^b(X/G)$

Def: A semiorthogonal decomp. of $\mathcal{T}$ is a collection $\mathcal{T}_1, \dots, \mathcal{T}_n$ of str. full triangulated subcats. of $\mathcal{T}$ s.t.
1) $\mathrm{Hom}(\mathcal{T}_i, \mathcal{T}_j) = 0 \ \forall i > j$
2) $\forall F \in \mathcal{T} \ \exists F = F_n \to F_{n-1} \to \dots \to F_1 \to F_0 = 0$ st. $\mathrm{Cone}(F_i \to F_{i-1}) \in \mathcal{T}_i$

Lemma: $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ semiorth. decomp., $\mathcal{T}$ has a Serre functor $\Rightarrow \mathcal{A}$ and $\mathcal{B}$ also have Serre functors.

Pf: Construct using <u>mutation functors</u>:
$F \in \mathcal{T} \Rightarrow \exists$ exact triangle $\underset{\mathcal{B}}{\underset{\cap}{F_B}} \to F \to \underset{\mathcal{A}}{\underset{\cap}{F_A}} \to F_B[1]$

Define mutation functors by $L_{\mathcal{B}}(F) := F_A$
$R_{\mathcal{A}}(F) := F_B$

Then $S_{\mathcal{B}} = R_{\mathcal{A}} \circ S_{\mathcal{T}}$, and $S_{\mathcal{A}}^{-1} = L_{\mathcal{B}} \circ S_{\mathcal{T}}^{-1}$ ▲

Thm: $X \subset \mathbb{P}^N$ hypersurface of degree $d \leq N$. Then $\exists$ s.o.d.
$\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(N-d) \rangle$, where
$\mathcal{A}$ is fractional CY of frac! dim. $\frac{(N+1)(d-2)}{d}$.

Def: Let Y be a smooth alg. var., $\mathcal{O}_Y(1)$ a line bundle
A __Lefschetz decomposition__ for $\mathcal{D}^b(Y)$ is a chain of tri. subcats.
$0 \subset \mathcal{A}_{n-1} \subset \mathcal{A}_{n-2} \subset \ldots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ s.t.
$\mathcal{D}^b(Y) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2), \ldots, \mathcal{A}_{n-1}(n-1) \rangle$ is a semiorth. decomp.
$\hookrightarrow$ twist by $\mathcal{O}_Y(1)$

Ex.:
1) $\mathcal{D}^b(\mathbb{P}^N) = \langle \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(N) \rangle \Rightarrow \mathcal{A}_0 = \mathcal{A}_1 = \ldots = \mathcal{A}_N = \langle \mathcal{O}_X \rangle$
2) $\mathcal{D}^b(\mathbb{P}(w_0, \ldots, w_N)) = \langle \mathcal{O}_X, \ldots, \mathcal{O}_X(\Sigma w_i - 1) \rangle \Rightarrow$ similarly.
3) $Y = Gr(2, m)$, $m = 2k+1$: $\mathcal{A}_0 = \ldots = \mathcal{A}_{m-1} = \langle \mathcal{O}, \mathcal{U}^*, \ldots, S^{k-1}\mathcal{U}^* \rangle$
where $\mathcal{U}$ = tautological bundle
4) $Y = Gr(2, m)$, $m = 2k$: $\mathcal{A}_0 = \ldots = \mathcal{A}_{k-1} = \langle \mathcal{O}, \mathcal{U}^*, \ldots, S^{k-1}\mathcal{U}^* \rangle$
$\mathcal{A}_k = \ldots = \mathcal{A}_{m-1} = \langle \mathcal{O}, \mathcal{U}^*, \ldots, S^{k-2}\mathcal{U}^* \rangle$

- If $\mathcal{A}_0 = \ldots = \mathcal{A}_{n-1}$, call the Lefschetz decomp. "__rectangular__"
- __Rmk:__ a Lefschetz decomp. is uniquely determined by $\mathcal{A}_0$:
$\mathcal{A}_{k+1} = \mathcal{A}_k \cap {}^{\perp}(\mathcal{A}_0(-k-1))$

__Main Thm:__ Y smooth projective, $\dim Y = N$, s.t. $\omega_Y = \mathcal{O}_Y(-m)$ for some $m > 0$
and $\mathcal{D}^b(Y) = \langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(m-1) \rangle$.
(i) Let $X = X_d \xrightarrow{f} Y$ smooth divisor in $|dH| = |\mathcal{O}_Y(d)|$, $d \leq m$
Then $\mathcal{D}^b(X) = \langle \mathcal{B}_d, f^*\mathcal{A}, f^*\mathcal{A}(1), \ldots, f^*\mathcal{A}(m-1-d) \rangle$
and $\mathcal{B}_d$ is frac! CY of dim. $N+1 - 2m/d$

(ii) let $X = X'_d \longrightarrow Y$ double cover ramified along a divisor $\in |2dH|$

Then $D^b(X) = \langle \mathcal{B}'_d, f^* \mathcal{A}, \ldots, f^* \mathcal{A}(m-1-d) \rangle$

and $\mathcal{B}'_d$ fract$^{\underline{l}}$ CY of dim. $N+1-m/d$

Sketch proof: $D^b(X) = \langle \mathcal{B}, f^* \mathcal{A}, \ldots, f^* \mathcal{A}(m-1-d) \rangle \Rightarrow$

$$S_{\mathcal{B}}^{-1} = \mathbb{L}_{\langle f^*\mathcal{A}, \ldots, f^*\mathcal{A}(m-1-d) \rangle} \circ \underbrace{\mathcal{O}_X(m-d)[N-1]}$$

note: $\omega_Y \simeq \mathcal{O}_Y(-m) \Rightarrow \omega_X \simeq \mathcal{O}_X(d-m)$

$$= (\underbrace{\mathbb{L}_{f^*\mathcal{A}} \circ \mathcal{O}_X(1)}_{\text{call this } \mathcal{O}})^{m-d} \circ [N-1]$$

since $\begin{cases} \to \mathbb{L}_{\langle \ldots, \ldots, \ldots \rangle} = \mathbb{L}_{\ldots} \circ \mathbb{L}_{\ldots} \circ \cdots \\ \to \mathbb{L}_{\mathcal{A}} \circ \phi = \phi \circ \mathbb{L}_{\phi^{-1}(\mathcal{A})} \end{cases}$

Then $S_{\mathcal{B}}^{-1} \cong \mathcal{O}^{m-d} \circ [N-1]$, and we claim $\mathcal{O}^d \cong [2]$.

(details not given). QED