A. Efimov – 24/1/09 – Noncommutative Grassmannians (w/ Lunts, Orlov)

$\mathcal{A}$ DG-cat./k, $E \in D(\mathcal{A}^{op})$

<u>Def:</u> R artinian DG alg: $R \to k$, $m = \ker(R \to k)$, $m^N > 0$

$$DEF_R(E) = \{(M, \sigma) / M \in D((\mathcal{A} \otimes R)^{op}),\ \sigma : M \overset{L}{\underset{R}{\otimes}} k \xrightarrow{\sim} E\}$$

deformation of E over R

$DEF(E)$: 2-dgart $\to$ Gpd (groupoids)

$$1\text{-}Hom(R, Q) = \left\{ \begin{array}{l} M \in D(R \otimes Q^{op}), \quad M \simeq Q \text{ in } D(Q^{op}) \\ \theta : M \underset{Q}{\otimes} k \to k \text{ in } D(R) \end{array} \right\}$$

Consider E st. (1) $Ext^{<0}(E, E) = 0$

(2) $Ext^0(E, E) = k$

(3) $\dim Ext^*(E, E) < \infty$

$A = Ext^*(E, E)$ A_∞-algebra,

$\bar{A}$ augmentation A_∞-ideal

Koszul dual $\hat{S} = (B\bar{A})^*$, $\hat{S}^i = 0$ for $i > 0$

<u>Thm:</u> $1\text{-}Hom(\hat{S}, -) \cong DEF\_(E)$

<u>Ex:</u> 1. X quasiproj, $x \in X(k)$ smooth

Then in $D(QCoh(X)) \simeq D(A)$, $RHom(\mathcal{O}_x, \mathcal{O}_x)$ is formal

$Ext^\bullet(\mathcal{O}_x, \mathcal{O}_x) \simeq \Lambda^*(T_x X)$

$\leadsto H^i(\hat{S}) = 0$ for $i \neq 0$, $H^0(\hat{S}) = k[[x_1, \dots x_n]]$, $n = \dim X$.

2. (towards noncomm. Grassmannian...)

$\dim V = n$, $W \subset V$ $\dim W = m$, consider $\mathcal{O}_{\mathbb{P}(W)}$ in $D^b Coh\, \mathbb{P}(V)$

$RHom(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is formal,

$$A := \mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)}) \cong \bigoplus_{d=0}^{n-m} \Lambda^d(V/W) \otimes S^d W^*$$

$\rightsquigarrow$ $H^i(\hat{S}) = 0$ for $i \neq 0$

$H^0(\hat{S})$ = algebra ... represents noncomm. Grassmannian $NGr(m,V)$

Namely, subspace $W \longleftrightarrow x_W \in NGr(m,V)(k)$

s.t. $\hat{\mathcal{O}}_{x_W} \cong H^0(\hat{S})$

Properties of $NGr(m,V)$: MAIN RESULTS

1) $W \subset V$ dim. m $\rightsquigarrow$ $x_W \in NGr(m,V)$, $\hat{\mathcal{O}}_{x_W} = H^0(\hat{S})$

2) ϕ: $\mathrm{Perf}(NGr(m,V)) \hookrightarrow D^b\mathrm{Coh}(\mathbb{P}(V))$

$\mathrm{Im}\,\phi = \langle \mathcal{O}(m-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$

3) $\phi(\mathcal{O}_{x_W}) \cong \mathcal{O}_{\mathbb{P}(W)}$

Construction: • $\mathcal{A}$ $\mathbb{Z}$-algebra, ie. category with $\mathrm{Ob}(\mathcal{A}) = \mathbb{Z}$ $\Leftrightarrow$

hom. spaces $\mathcal{A}_{ij}$, $i,j \in \mathbb{Z}$, products $\mathcal{A}_{jk} \otimes \mathcal{A}_{ij} \to \mathcal{A}_{ik}$

units $1_i \in \mathcal{A}_{ii}$

• B $\mathbb{Z}$-graded algebra $\Rightarrow$ define such a cat. $\mathcal{A}_B$ with

$(\mathcal{A}_B)_{ij} = B^{j-i}$, product structure from B

• Now consider $\mathrm{Proj}(\mathcal{A})$, and introduce $\mathrm{QCoh}(\mathrm{Proj}(\mathcal{A}))$, $\mathcal{O}_{\mathrm{Proj}}(\mathcal{A})$:

$\mathrm{Mod}\text{-}\mathcal{A} / \mathrm{Tors}\text{-}\mathcal{A} = \mathrm{QMod}(\mathcal{A})$

$x \in M_i$ torsion if $x\mathcal{A}_{ji} = 0$ for $j \ll i$.

$\mathrm{QCoh}(\mathrm{Proj}(\mathcal{A})) := \mathrm{QMod}(\mathcal{A})$

Also, P_i, $i \in \mathbb{Z}$ ... $\longrightarrow$ $\mathcal{O}_{\mathrm{Proj}(\mathcal{A})} = \pi(P_0)$.

Noncomm. Grassmannian:

$$NGr(m,V) = Proj(\mathcal{A}^{m,V})$$

where $\mathcal{A}^{m,V}$ defined by a quiver with relations:

- $\mathcal{A}^{m,V}_{ij} = 0$ for $i>j$
- $\mathcal{A}^{m,V}_{ii} = k$
- $\mathcal{A}^{m,V}_{i,i+1} = \begin{cases} V^* & \text{if } (n-m+1)\nmid i \\ \Lambda^{n-m}V & \text{if } (n-m+1)\mid i \end{cases}$

ie: $\cdots \underset{m-n-1}{\bullet} \xrightarrow{\Lambda^{n-m}V} \underset{m-n}{\bullet} \xrightarrow{V^*} \cdots \underset{-1}{\bullet} \xrightarrow{V^*} \underset{0}{\bullet} \xrightarrow{\Lambda^{n-m}V} \underset{1}{\bullet} \rightarrow \cdots$

- products: relations def^d by ideal

$$I_{i,i+2} = \begin{cases} \Lambda^2 V^* & \text{for } (n-m+1)\nmid i,\, i-1 \\ \Lambda^{n-m-1}V & \text{otherwise} \end{cases}$$

(note: $\Lambda^{n-m-1}V \underset{\text{canon.}}{\subset} V^*\otimes\Lambda^{n-m}V$)

$\mathcal{T}$ enhanced tri. cat. (Bondal, Polishchuk)

- $\sigma = \{E_1, \dots, E_k\}$ exc. collection
- $S = \{E_i\}_{i\in\mathbb{Z}}$ helix gen^d by σ

  ie. $\{E_i, \dots, E_{i+k-1}\}$ exc. collections

Def: S is geometric if $Ext^{>0}(E_i, E_j)$ ... (for $i \le j$)

σ is geometric if S is geometric

- $\sigma = (E_1 .. E_k)$ geometric exc. coll., S gen^d by σ

$B = End(\sigma)$, $\mathcal{A} = End(S)$

$\Rightarrow$ Thm: $\| \; D(QMod(\mathcal{A})) \simeq D(B^{op})$

In particular: $\{\pi(P_i)\}_{i\in\mathbb{Z}}$ helix

$(\pi(P_i), \dots, \pi(P_{i+k-1}))$ compactly generates $D(QMod(A))$

$\longrightarrow$ $D_{Perf}(Proj(A)) = \langle \pi(P_i), i \in \mathbb{Z} \rangle$

$(\pi(P_i), \dots, \pi(P_{i+k-1}))$ full exc. coll. in $D_{Perf}(Proj(A))$.

<u>In our case:</u> $\mathcal{A}^{m,V}$ as above

$B^{m,V} = End(\mathcal{O}(m-n) \oplus \dots \oplus \mathcal{O}(-1) \oplus \mathcal{O})$

<u>Thm:</u> $D(NGr(m,V)) \simeq D((B^{m,V})^{op})$

#2 in summary

<u>Cor:</u> $\phi: Perf(NGr(m,V)) \hookrightarrow D^b Coh(\mathbb{P}(V))$

$Im\, \phi = \langle \mathcal{O}(m-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$

$f: Sp(B) \to Proj(\mathcal{A})$

$f^\alpha: QMod(\mathcal{A}) \to Mod\text{-}B$ "commutes" with colimits

$+ \ \theta: f^\alpha(\pi(P_0)) \simeq B$

Lf^α: $L^i f^\alpha(\pi(P_i)) = 0 \quad \forall i > 0$

$f^\alpha(\pi(P_i))$ flat B^{op}-modules

<u>Def:</u> $X_{\mathcal{A}}: Alg_k \to Gpd$

$$X_{\mathcal{A}}(B) = \left\{ (f^\alpha, \theta) \middle/ \begin{array}{l} f^\alpha: QMod(A) \to Mod\text{-}B \text{ commuting w/ colimits,} \\ \exists L^i f^\alpha,\ L^i f^\alpha(\pi(P_i)) = 0 \ \forall i > 0 \\ f^\alpha(\pi(P_i)) \text{ are flat} \end{array} \right\}$$

Morally, $X_{\mathcal{A}}(B) = \{f: Sp(B) \to Proj(\mathcal{A})\}$

Indeed: <u>Prop:</u> $Y \subset \mathbb{P}^n$, A homogeneous coord. ring $\rightsquigarrow \mathcal{A}$ $\mathbb{Z}$-algebra

B commutative Noetherian alg.

$\Rightarrow$ then $X_{\mathcal{A}}(B) = \mathrm{Maps}(\mathrm{Spec}(B), Y)$

$\uparrow$ presheaf of sets

<u>In our case</u>: $X_{\mathcal{A}_{m,V}}$ presheaf of sets; k-points $X_{\mathcal{A}_{m,V}}(k)$:

<u>Thm:</u> $NGr(m,V)(k) = \{ W \subset V \,/\, 1 \leq \dim W \leq n \}$

$N(\mathbb{P}(V))(k) = \{ W \subset V \,/\, 0 \leq \dim W \leq \dim V - 1 \}$

Moreover $\phi(\mathcal{O}_{x_W}) = pr_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})$

where $pr_r^{m,V} : D^b(\mathbb{P}(V)) \to \mathrm{Im}\,\phi$

If $\dim W = m$, $\mathcal{O}_{\mathbb{P}(W)} \simeq \phi(\mathcal{O}_{x_W})$.

#3 in summary

- Now look at <u>completions of local rings</u>:

$X : \mathrm{Alg}_k \to \mathrm{Sets}$, $\quad x \in X(k)$

$F_{X,x} : \mathrm{art} \to \mathrm{Sets}$

$F_{X,x}(R) = \{ f \in X(R) \,/\, X(i)(f) = x \}$ (deform^ns of $x \in X(k)$ over R)

$i : R \to k$ induces $X(i) : X(R) \to X(k)$

(think of f as $f : \mathrm{Sp}(R) \to X$)

<u>Def:</u> $\hat{\mathcal{O}}_x$ pro artinian algebra, prorepresenting $F_{X,x}$

i.e. $F_{X,x}(-) \cong \mathrm{Hom}(\hat{\mathcal{O}}_x, -)$

Thm: $x_W \in NGr(m,V)$, $\dim W = m \Rightarrow \hat{\mathcal{O}}_{x_W} \simeq H^0(\hat{S})$

where $\hat{S}$ = completion of Koszul dual $A = \mathrm{Ext}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$

$\hat{A}^! = H^0(\hat{S})$.

#1 in summary

$Sp(H^0(\hat{S})) \to NGr(m,V)$

Rmk: compare with Kontsevich-Rosenberg:

$\dim V = n$	vs.	Kontsevich-Rosenberg
$m = n-1$		$N(\mathbb{P}(V^\vee))$
$NGr(n-1, V)$		

$$D(QCoh(NGr(n-1,V))) \simeq D(QCoh(N\mathbb{P}(V^\vee)))$$

$$\| $$

$$D(Rep(Q_n))$$

n arrows

$$\begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix}$$

but $QCoh(NGr(n-1,V)) \not\simeq QCoh(N\mathbb{P}(V^\vee))$

$$\mathcal{O}_{NGr} \longrightarrow \mathcal{O}_{N\mathbb{P}(V^\vee)}$$

NB: so... we have morphisms $Gr(\ell, V) \hookrightarrow NGr(m,V)$

$\forall\, \ell \leq m$!!