MIRROR SYMMETRY: LECTURE 9

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1. The Quintic (contd.)

To recall where we were, we had

$$X_\psi = \{(x_0 : \cdots : x_4) \in \mathbb{P}^4 \mid f_\psi = \sum_0^4 x_i^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0\} \tag{1}$$

with

$$G = \{(a_0, \dots, a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 \mid \sum a_i = 0\}/\{(a,a,a,a,a)\} \cong (\mathbb{Z}/5\mathbb{Z})^3 \tag{2}$$

acting by diagonal multiplication $x_i \mapsto x_i \xi^{a_i}, \xi = e^{2\pi i/5}$. We obtained a crepant resolution $\check{X}_\psi$ of X_ψ/G. This family has a LCSL point at $z = (5\psi)^{-5} \to 0$. There was a volume form $\check{\Omega}_\psi$ on $\check{X}_\psi$ induced by the G-invariant volume form Ω_ψ on X_ψ by pullback via $\pi : \check{X}_\psi \to X_\psi/G$. We computed its period on the 3-torus

$$T_0 = \{(x_0 : \cdots : x_4) \mid x_4 = 1, |x_0| = |x_1| = |x_2| = \delta, |x_3| \ll 1\} \tag{3}$$

(or, on the mirror, $\check{T}_0 \subset \check{X}_\psi$) to be

$$\int_{T_0} \Omega_\psi = -(2\pi i)^3 \sum_{n=0}^\infty \frac{(5n)!}{(n!)^5 (5\psi)^{5n}} \tag{4}$$

In terms of $z = (5\psi)^{-5}$, the period is proportional to

$$\phi_0(z) = \sum_{n=0}^\infty \frac{(5n)!}{(n!)^5} z^n \tag{5}$$

Setting $\Theta = z\frac{d}{dz} : \Theta(\sum c_n z^n) = \sum n c_n z^n$, we obtained the *Picard-Fuchs equation*

$$\theta^4 \phi_0 = 5z(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\phi_0 \tag{6}$$

Proposition 1. *All periods $\int \check{\Omega}_\psi$ satisfy this equation.*

Note that all period satisfy some 4th order differential equation: $H^3(\check{X}_\psi, \mathbb{C})$ is 4-dimensional, so $[\check{\Omega}_\psi], \frac{d}{d\psi}[\check{\Omega}_\psi], \cdots, \frac{d^4}{d\psi^4}[\check{\Omega}_\psi]$ are linearly related. Thus, so are their integrals over any 3-cycle.

Idea of proof. We view Ω_ψ and its derivatives as residues. Let

$$\overline{\Omega} = \sum_{i=0}^{4} (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4 \tag{7}$$

be a form on $\mathbb{C}^5$. It is homogeneous of degree 5 (not 0), so we need to multiply by something of degree -5 to get a form on $\mathbb{P}^4$. If f, g are homogeneous, deg $f =$ deg $g + 5$, $\frac{g\overline{\Omega}}{f}$ is a meromorphic 4-form on $\mathbb{P}^4$. For instance, $\frac{5\psi\overline{\Omega}}{f_\psi}$ has poles along X_ψ. Now, given a 4-form with poles along some hypersurface X, it has a *residue* on X which is ideally a 3-form on X, but is at least a class in $H^3(X, \mathbb{C})$.

Recall from complex analysis, if $\phi(z)$ has a pole at 0, $\operatorname{res}_0(\phi) = \frac{1}{2\pi i}\int_{S^1} \phi(z)dz$. Now, let's say that we have a 3-cycle C in X: we can associate a "tube" 4-cycle in $\mathbb{P}^4$ which is the preimage of C in the boundary of a tubular neighborhood of X. Then

$$\int_C \operatorname{res}_X \left(\frac{g\overline{\Omega}}{f}\right) := \frac{1}{2\pi i}\int_\Gamma \frac{g\overline{\Omega}}{f} \tag{8}$$

If we only have simple poles along X, we get a 3-form characterized by

$$\operatorname{res}_X \left(\frac{g\overline{\Omega}}{f}\right) \wedge df = g\overline{\Omega} \tag{9}$$

at any point of X.

Now, $\Omega_\psi = \operatorname{res}_{X_\psi}\left(\frac{5\psi\overline{\Omega}}{f_\psi}\right)$, and differentiating k times gives

$$\frac{\partial^k}{\partial \psi^k}[\Omega_\psi] = \operatorname{res}_{X_\psi}\left(\frac{g_k\overline{\Omega}}{f_\psi^{k+1}}\right) \tag{10}$$

Thus we can express

$$\Theta^4[\Omega_\psi] = \operatorname{res}_{X_\psi}\left(\frac{g_\Theta\overline{\Omega}}{f_\psi^5}\right) \tag{11}$$

for some g_Θ, and write $5z(5\Theta+1)\cdots(5\Theta+4)[\Omega_\psi]$ in the same form.

We compare the residues of forms with order 5 poles along X_ψ using Griffiths pole order reduction. Assume that ϕ is a 3-form with poles of order ℓ along X_ψ,

$$\phi = \frac{1}{f_\psi^\ell}\sum_{i<j}(-1)^{i+j}(x_i g_j - x_j g_i) dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_4 \tag{12}$$

with deg $(g_0 \cdots g_4) = 5\ell - 4$, then

$$d\phi = \frac{1}{f_\psi^{\ell+1}}\left(\ell \sum_j g_j \frac{\partial f_\psi}{\partial x_j} - f_\psi \sum_j \frac{\partial g_j}{\partial x_j}\right)\overline{\Omega} \tag{13}$$

In particular, if we have something of the form $(\sum g_j \frac{\partial f_\psi}{\partial x_j})\frac{\overline{\Omega}}{f_\psi^{\ell+1}}$ (the Jacobian ideal is the span of $\{\frac{\partial f_\psi}{\partial x_i}\}$), it can be written as something with a lower order pole plus something exact. We obtain our result iteratively, showing in each stage that the top order term belongs to the Jacobian ideal, and reduce to a lower order term. When we get to order 1, we find that the residue is 0. $\square$

There is a theory of differential equations with regular singular points, i.e. differential equations of the form

$$\Theta^s f + \sum_{j=0}^{s-1} B_j(z)\Theta^j f = 0 \tag{14}$$

where $\Theta = z\frac{d}{dz}$ and $B_j(z)$ are meromorphic functions which are holomorphic at $z=0$. As with solving ordinary differential equations, we reduce to a 1st order system of differential equations $\Theta w(z) = A(z)w(z)$, where

$$A(z) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & \cdots & 0 & 1 \\ -B_0(z) & \cdots & \cdots & \cdots & -B_{s-1}(z) \end{pmatrix}, w(z) = \begin{pmatrix} f(z) \\ \Theta f(z) \\ \vdots \\ \Theta^{s-1} f(z) \end{pmatrix} \tag{15}$$

The fundamental theorem of these differential equations states that there exists a constant $s \times s$ matrix R and an $s \times s$ matrix of holomorphic functions $S(z)$ s.t.

$$\Phi(z) = S(z)\exp((\log z)R) = S(z)(\mathrm{id} + (\log z)R + \frac{\log^2 z}{2}R^2 + \cdots) \tag{16}$$

is a fundamental system of solutions to $\Theta w(z) = A(z)w(z)$, and moreover if $A(0)$ doesn't have distinct eigenvalues differing by an integer, we can take $R = A(0)$. This Φ is multivalued, and $z \mapsto e^{2\pi i}z$ gives $\Phi(z) \mapsto \Phi(z)e^{2\pi i R}$ (where $e^{2\pi i R}$ is the monodromy).

In our case, $\mathcal{D}\phi = \Theta^4\phi - 5z(5\Theta+1)\cdots(5\Theta+4)\phi = 0$, so the coefficient of Θ^4 is $1-5^5z$, and the coefficients of $\Theta^0, \cdots, \Theta^3$ are constant multiples of z. Then

$$\Theta^4\phi - \frac{5z}{1-5^5z}P_3(\Theta)\cdot\phi = 0 \tag{17}$$

where P_3 is independent of z. Then

$$R = A(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

is nilpotent, and our assumption holds. The corresponding monodromy is

$$T = e^{2\pi i R} = \begin{pmatrix} 1 & 2\pi i & \frac{(2\pi i)^2}{2} & \frac{(2\pi i)^3}{6} \\ 0 & 1 & 2\pi i & \frac{(2\pi i)^2}{2} \\ 0 & 0 & 1 & 2\pi i \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

If $\omega(z) = \int_\beta \check{\Omega}_\psi$ is a period, then it is a solution of the Picard-Fuchs equation, and thus a linear combination of $\Phi(z)_{1i}$'s. There exists a basis $b_1, \dots, b_4$ of $H_3(\check{X}, \mathbb{C})$ s.t. $\int_{b_i} \check{\Omega}_\psi = \Phi(z)_{1i}$. The monodromy action in this basis is T (T maximally unipotent implies that 0 is LSCL).

1.1. **More periods of $\check{\Omega}_\psi$.** The first fundamental solution we obtained is $\phi_0 = \Phi(z)_{11}$, which is invariant under monodromy and regular at $z = 0$. Since $\dim \operatorname{Ker}(T - \mathrm{id}) = 1$, it is unique up to scaling, and $\phi_0(z) = \sum_{n=0}^\infty \frac{(5n)! z^n}{(n!)^5}$. We next obtain $\phi_1 = \Phi(z)_{12}$ s.t. $\phi_1(e^{2\pi i} z) = \phi_1(z) + 2\pi i \phi_0(z)$, which is unique up to multiples of ϕ_0. Since $\Phi(z) = S(z)\exp(R\log z)$, $\phi_1(z) = \phi_0(z)\log z + \tilde{\phi}(z)$, with $\tilde{\phi}(z)$ holomorphic. Now

$$\Theta^j(f(z)\log z) = (\Theta^j f)\log z + j(\Theta^{j-1} f) \tag{20}$$

If we write $F(x) = x^4 - 5z\prod_{j=1}^4 (5x + j)$, then

$$\begin{aligned} \mathcal{D}\phi_1(z) &= F(\Theta)(\phi_0(z)\log z + \tilde{\phi}(z)) \\ &= (F(\Theta)\phi_0)\log z + F'(\Theta)\phi_0 + F(\Theta)\tilde{\phi} \end{aligned} \tag{21}$$

Since $0 = \mathcal{D}\phi_0 = \mathcal{D}\phi_1$, we find $\mathcal{D}\tilde{\phi}(z) = -F'(\Theta)\phi_0(z)$. This gives a recurrence relation on the coefficients of $\tilde{\phi}(z)$, and one obtains:

$$\tilde{\phi}(z) = 5\sum_{n=1}^\infty \frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n} \frac{1}{j}\right) z^n \tag{22}$$

We want canonical coordinates on the moduli space of complex structures: there are $\beta_0, \beta_1 \in H_3(\check{X}, \mathbb{Z})$, with monodromy $\beta_0 \mapsto \beta_0, \beta_1 \mapsto \beta_1 + \beta_0$, and

$$\begin{aligned} \int_{\beta_0} \check{\Omega} &= C\phi_0(z) \\ \int_{\beta_1} \check{\Omega} &= C'\phi_0(z) + C''\phi_1(z) \end{aligned} \tag{23}$$

The monodromy acts on the latter by $\int_{\beta_1} \check{\Omega} \mapsto \int_{\beta_1+\beta_0} \check{\Omega}$, implying that $2\pi i C'' = C$. Thus, the canonial coordinates are

$$
\begin{aligned}
w &= \frac{\int_{\beta_1} \check{\Omega}}{\int_{\beta_0} \check{\Omega}} \\
&= \frac{C'}{C} + \frac{1}{2\pi i}\frac{\phi_1}{\phi_0} \\
&= \frac{1}{2\pi i}\log c_2 + \frac{1}{2\pi i}\log z + \frac{1}{2\pi i}\frac{\tilde{\phi}}{\phi_0} \\
q &= \exp(2\pi i w) = c_2 z \exp\left(\frac{\tilde{\phi}(z)}{\phi_0(z)}\right)
\end{aligned}
\tag{24}
$$