MIRROR SYMMETRY: LECTURE 13

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0.1. **Lagrangian Floer Homology (contd).** Let (M, ω) be a symplectic manifold, L_0, L_1 compact Lagrangian submanifolds intersecting transversely. We defined $CF(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ and the differential

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ \phi \in \pi_2(M, L_0 \cup L_1) \\ \mathrm{ind}(\phi) = 1}} \#(\mathcal{M}(p, q, \phi, J)/\mathbb{R}) T^{\omega(\phi)} \cdot q \tag{1}$$

where $\mathcal{M}$ is the set of finite energy J-holomorphic maps $u : \mathbb{R} \times [0,1] \to M, u(s,0) \in L_0, u(s,1) \in L_1, \lim_{s \to +\infty} u = p, \lim_{s \to -\infty} u = q$.

0.2. **Product Structure.** We want to define a map

$$CF^*(L_0, L_1) \otimes CF^*(L_1, L_2) \to CF^*(L_0, L_2) \tag{2}$$

Look at $u : D^2 \to M$ a J-holomorphic disk whose image is a triangle between L_0, L_1, L_2. Mark points $1, j, j^2$ on the boundary, with $u(j) = p \in L_0 \cap L_1, u(j^2) = q \in L_1 \cap L_2, u(1) = r \in L_0 \cap L_2$, and $u([1,j]) \subset L_0, u([j, j^2]) \subset L_1, u([j^2, 1]) \subset L_2$. Removing our three points from the disk gives a space biholomorphic to a pair of pants, i.e. a Riemann surface with boundary with 3 strip-like ends. Now, let $\mathcal{M}(p, q, r, [u], J)$ be the set of such maps: as a moduli space, its expected dimension is $\mathrm{ind}([u]) = \deg(r) - (\deg(p) + \deg(q))$ (trivialize u^*TM, and pick graded lifts of TL_i: $\deg(p)$ is the Maslov index of the path from the reference setup from last time to T_pL_0, T_pL_1).

Definition 1. *(Assuming transversality)*

$$q \circ p = \sum_{\substack{r \in L_0 \cap L_2 \\ \mathrm{ind}([u]) = 0}} (\#\mathcal{M}(p, q, r, [u], J)) T^{\omega(u)} r \tag{3}$$

Note. As usual, this assumes transversality and orientability of moduli spaces. Moreover, $\mathrm{Aut}(D^2)$ acts freely transitively on ordered triples of boundary points, so $(1, j, j^2)$ is arbitrary. Finally, the lack of symmetry of the index formula in p, q, r is because the degree $\deg(r) \in CF^*(L_0, L_2)$ is $n - \deg(r \in CF(L_2, L_0))$. Recall that our reference frame as $R^n, (e^{-i\theta}\mathbb{R})^n \subset \mathbb{C}^n$, which we stated to have index 0 for $\theta > 0$ small: the reversed frame $(e^{-i\theta}\mathbb{R})^n, \mathbb{R}^n$ has index n. In general, we have a "Poincaré duality" $CF^*(L, L') \cong CF^{n-*}(L', L)^\vee$ compatible with our operations (e.g. differentials are dual).

Proposition 1. *If $[\omega] \cdot \pi_2(M, L_i) = 0$, then the product structure defined above satisfies the Leibniz rule w.r.t. ∂, and hence induces a product on HF^*; this product structure will be associative.*

For the Leibniz rule: consider index 1 moduli spaces (triangles with edges segments of L_0, L_1, L_2 and corners p, q, r as above). We compactify by adding limit configurations, specifically bubblings of disks and broken configurations (where we get the same broken strips at our strip-like ends). Our strip may break at p, q, or r, giving us contributions $q \circ (\partial p), (\partial q) \circ p, \partial(q \circ p)$ respectively. Since the number of ends of an oriended 1-manifold with boundary is 0, we have $\partial(q \circ p) = \pm(\partial q) \circ p \pm q \circ (\partial(p))$. Thus, p, q closed implies that

$$\partial(q \circ p) = \pm(\partial q) \circ p \pm q \circ (\partial(p)) = 0 \tag{4}$$

so $q \circ p$ is closed as well. Moreover, if $p = \partial p'$ is exact, so is

$$q \circ p = \pm\partial(q \circ p') \pm (\partial q) \circ p' = \pm\partial(q \circ p') \tag{5}$$

Thus, we have a well-defined product on HF^*.

We also have higher-order operations

$$CF^*(L_0, L_1) \otimes \cdots \otimes CF^*(L_{k-1}, L_k) \stackrel{m_k}{\to} CF^*(L_0, L_k)[2-k] \tag{6}$$

(a grading shift). Note that $m_1 = \partial$, and m_2 is my product above. To obtain these, look at J-holomorphic maps from disks D^2 with $k+1$ marked points $z_0, \ldots, z_k$ on the boundary (cyclically ordered distinct, not fixed in advance), s.t. the image under u is a disk between $L_0, \ldots, L_k$ with $u(z_0) = q \in L_0 \cap L_k, u(z_i) = p_i \in L_{i-1} \cap L_i$. Repeating the above procedure, we obtain a moduli space $\mathcal{M}(p_1, \ldots, p_k, q, [u], J)$ with expected dimension deg $(q) - (\sum \deg p_i) + k - 2$, where the $k-2$ comes from the dimension of the moduli space of disks with $(k+1)$ marked points. Assuming orientability and transversality,

$$m_k(p_k, \ldots, p_1) = \sum_{\substack{q \in L_0 \cap L_k \\ \mathrm{ind}([u]) = 0}} (\#\mathcal{M}(p_1, \ldots, p_k, q, [u], J)) T^{\omega(u)} q \tag{7}$$

Remark. The moduli space $\mathcal{M}_{0,k+1}$ of disks with $(k+1)$ boundary marked points (distinct, in order, modulo $\mathbb{D}^2$ automorphisms) is contractible of dimension $k-2$, and compactifiies to $\overline{\mathcal{M}}_{0,k+1}$, the moduli space of stable genus 0 Riemann surfaces with one boundary component, $k+1$ boundary marked points. That is, they are trees of disks attached at marked nodal points such that each component carries at least 3 special points. For instance, $\overline{\mathcal{M}}_{0,4}$ has general point given by 4 points on the boundary of a unit disk. WLOG, we can set the first three at three fixed points, and let the fourth move between the first and the third. When it hits either of these points, we force bubbling at the boundary and obtain two limiting configurations, and our moduli space is a line segment with two boundary points corresponding to them. In general, the objects we obtain are *associahedra.*

Thus, when considering sequences of $(k+1)$-marked holomorphic disks, the limiting configurations allowed by Gromov compactnness are those with bubbling of spheres or disks, breaking of strips at marked points, and degeneration of the domain to $\partial\overline{\mathcal{M}}_{0,k+1}$. We get relations by considering 1-dimensional moduli spaces.

Proposition 2. *Assuming no bubbling of disks and spheres,* $\forall\, m \geq 1$,$(p_1, \ldots, p_m)$, $p_i \in L_{i-1} \cap L_i$,

$$\sum_{\substack{k,\ell \geq 1\\ k+\ell = m+1\\ 0 \leq j \leq \ell - 1}} (-1)^* m_\ell(p_m, \ldots, p_{j+k+1}, m_i(p_{j+k}, \ldots, p_{j+1}), p_j, \ldots, p_1) = 0 \tag{8}$$

where $* = \deg\,(p_1) + \cdots + \deg\,(p_j) + j$.

For instance, we obtain

$$\begin{aligned} m_1(m_1(p)) &= 0 \\ m_1(m_2(p,q)) + (-1)^{\deg\, q+1} m_2(m_1(p), q) + m_2(p, m_1(q)) &= 0 \\ m_1(m_3(p,q,r)) \pm m_2(m_2(p,q), r) \pm m_2(p, m_2(q,r)) & \\ \pm m_3(m_1(p), q, r) \pm m_3(p, m_1(q), r) \pm m_3(p, q, m_1(r)) &= 0 \end{aligned} \tag{9}$$

which says that m_1 is a differential, m_2 satisfies the Leibniz rule, and m_2 is associative up to homotopy given by m_3 (i.e. it is associative in HF^*).

Proof. Idea: consider a 1-dimensional moduli space $\mathcal{M}(p_1, \ldots, p_m, q, [u], J)$ and its ends. Transversality and no bubbling implies that our limiting configurations come from bubbling on $\mathcal{M}_{0,k+1}$ (i.e. nearby points colliding). Setting the total number of ends (with orientation) to be zero gives us the sum of terms in the proposition. $\square$

Definition 2. *An* A_∞ *category is a linear "category" where morphism spaces are equipped with algebraic operations* $(m_k)_{k\geq 1}$ *satisfying the* A_∞*-relations (those defined above).*

The Fukaya category will be the A_∞-category whose objects are Lagrangian submanifolds (with extra data), the morphisms are Floer complexes, and the algebraic operations are as above.