1. Degenerations and Monodromy (contd.)

Last time, we considered families $\mathcal{X} \xrightarrow{\pi} D^2$ where for $t \neq 0$, $X_t \cong X$ (with varying $J$) and for $t = 0$, $X_0$ is typically singular. We saw that monodromy around $t = 0$ induces $\phi^* \in \text{Aut}(H^n(X_0, \mathbb{Z}))$.

**Theorem 1.** All eigenvalues of $\phi_*$ are roots of unity: thus $\exists N, k$ s.t. $(\phi_*^N - \text{id})^k = 0$. Moreover, $k \leq n + 1$.

Replacing $\phi$ by $\phi^N$ (the “base change” $X'_t = X_{t^N}$), we can assume that $\phi^*$ is unipotent, i.e. $(\phi_* - \text{id})^k = 0$. It is maximally unipotent if $k = n + 1$. We can further define a weight filtration associated to a unipotent $\phi^*$ coming from the Jordan block decomposition of $\phi^*$: letting

$$N = \log(\phi_*) = (\phi_* - \text{id}) - \frac{(\phi_* - \text{id})^2}{2} + \cdots + (-1)^{n+1}(\phi_* - \text{id})^n$$

act on $V = H^n(X, \mathbb{Q})$, we obtain a filtration $0 \subseteq W_0 \subseteq \cdots \subseteq W_{2n} = V$ s.t. $N(W_i) \subseteq W_{i-2}$ and $N^k : W_{n+k}/W_{n+k-1} \to W_{n-k}/W_{n-k-1}$. We construct this as follows:

- First, $N^n : W_{2n}/W_{2n-1} \to W_0$ so $W_0 = \text{im } (N^n), W_{2n-1} = \text{Ker } (N^n)$.
- Then let $V' = W_{2n-1}/W_0$, so $N$ induces $N' \in \text{End}(V')$ (since $W_{2n-1} = \text{Ker } N^n \supseteq \text{im } N$ and $W_0 = \text{im } (N^n) \subseteq \text{Ker } N$) with $(N')^n = 0$. By induction, we obtain

$$0 \subseteq W'_0 \cong W_1/W_0 \subseteq \cdots \subseteq W'_{2n-2} \cong W_{2n-1}/W_0 = V'$$

and

$$W_{2n-2} = \{ v \mid N^{n-1}(v) \in W_0 = \text{im } N^n \} \supseteq \text{im } N$$

so $W_{2n} \xrightarrow{N} W_{2n-2}$. Finally, $W_1 = \{ N^{n-1}(v) \mid N^n(v) = 0 \} \subseteq \text{Ker } N$ so $W_1 \xrightarrow{N} 0$, and we obtain $W_k \to W_{k-2}$ by induction.

**Example.** For the elliptic curves from last time, with $\phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $0 \subseteq W_0 \subseteq W_1 \subseteq W_2 = H^1(C, \mathbb{Q}) \cong \mathbb{Q}^2$, with $W_0 = W_1 = \text{im } N = \text{Ker } N = \text{Span}(a)$ being the direction invariant by monodromy.
Note that if $N$ is the $(n + 1) \times (n + 1)$ Jordan block with 0’s on the diagonal and 1s above (with columns $e_j$), then $W_0 = \text{Span}(e_1), W_{2n-1} = \text{Span}(e_1 \cdots e_n)$, and we can reduce to the equivalent $(n - 1) \times (n - 1)$ Jordan block and reapeat the process with $W_1 = W_0, W_{2n-2} = W_{2n-1}, \cdots, W_{2k-2} = W_{2k-1} = \text{Span}(e_1 \cdots e_k)$. There is a similar story if $N$ is a sum of such Jordan blocks.

**Remark.** In fact, the interplay of weight filtration with Hodge filtration

$$F^p = H^{n,0} \oplus \cdots \oplus H^{p,n-p} \quad (H^n = F^0 \supseteq F^1 \supseteq \cdots, F^p/F^{p+1} \cong H^{p,n-p})$$

(with Griffiths transversality giving $\nabla F^p \subseteq F^{p-1}$ under deformations) gives a notion of “mixed Hodge structure”. By [Schmid], there exists a limiting Hodge filtration as $t \to 0$, but we won’t say any more about those.

Now consider a multidimensional family $\mathcal{X} \to (D^2)^s$ smooth over $(D^*)^S$ where $D^* = D^2 \setminus \{0\}$. Then we have $s$ monodromies $\phi_1, \ldots, \phi_s \in \text{Aut}H_n(X), [\phi_i, \phi_j] = 0$ (since $\pi_1((D^*)^s) = \mathbb{Z}^s$ is abelian), so $N_i = \log \phi_i$ also commute.

**Theorem 2** (Cattani-Kaplan). *All the elements of the form $\sum \lambda_i N_i, \lambda_i > 0$ have the same monodromy weight filtration.*

We want to consider a “universal family” of Calabi-Yau manifolds near a “deepest corner”, called a “large complex structure limit point” in the moduli space.

**Definition 1** (Morrison). *Given a family of Calabi-Yau n-folds $\mathcal{X} \to (D^*)^S \subset (D^2)^s, s = h^{n-1,1}(X),$ s.t. the Kodaira-Spencer map $T_1(D^*)^s \to H^1(TX)$ is an isomorphism at every point of $(D^*)^s$, we say that $0 \in (D^2)^s$ is a large complex structure limit (LCSL) point if

1. The monodromies $\phi_j$ around each factor are all unipotent.
2. Let $N_j = \log \phi_j, N = \sum \lambda_j N_j$ for $\lambda_j > 0$ arbitrary. Then the weight filtration $0 \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{2n} = H^n(X, \mathbb{Q})$ has dim $W_0 = \dim W_1 = 1$, dim $W_2 = \dim W_3 = s + 1$.
3. Let $\alpha_0^*$ be the generator of $W_0$, $\alpha_1^*, \cdots, \alpha_s^*$ the rest of a basis for $W_2$. Then $\exists m_{jk} \in \mathbb{Q}$ s.t. $N_j(\alpha_k^*) = m_{jk} \alpha_0^*, i.e. \phi_j(\alpha_k^*) = \alpha_k^* + m_{jk} \alpha_0^*$. We further require that $(m_{jk})$ is an invertible matrix.

This essentially says that the family is locally a “full deformation”, that we single out a one-dimensional subspace Span($\alpha_0^*$) of $H^n(X)$ preserved by the monodromy, and that, for each factor $D^2$, we get a class $\tilde{\alpha}_j^*$ s.t. $\phi_j(\tilde{\alpha}_j^*) = \tilde{\alpha}_j^* + \alpha_0^*$ and $\tilde{\alpha}_j^*$ is invariant under the other $\phi_i$.

**Remark.** If $h^{n-1,1} = s = 1$, then this is equivalent to the statement that the monodromy around zero is maximally unipotent. For instance, the family of elliptic curves seen last time is an LCSL point.
Now, for a family of Calabi-Yau 3-folds, we have by definition

\[
0 \subset W_0 = W_1 \subset W_2 = W_3 \subset W_4 = W_5 \subset W_6 = H^3(X; \mathbb{Q})
\]

where we use \(N^k : W_{n+k}/W_{n+k-1} \sim W_{n-k}/W_{n-k-1}\) to get the dimensions of \(W_3, W_4, W_5\). Now, \(H^3(X)\) carries an intersection pairing preserved by \(\phi_*\), so \(N = \log \phi_*\) is in the Lie algebra, i.e. \((x, Ny) + (Nx, y) = 0\).

**Lemma 1.** \(W_{4-2i} = W_{2i}^\perp\).

*Proof.* Since \(W_0 = \text{im } N^3, W_4 = W_5 = \text{Ker } N^3, (x, N^3y) = -(N^3x, y) = 0\) for \(x \in W_4, N^3y \in W_0\) and the dimensions match. Furthermore, \(N(W_4) = W_2\) (it is onto since \(N : W_4/W_3 \sim W_2/W_1\) and \(W_0 = \text{im } N^3 = N(\text{im } N^2)\)): thus, for \(x, Ny \in W_2, (x, Ny) = -(Nx, y) = 0\) (since \(W_0 \perp W_4\)) and the dimensions match. \(\square\)

Finally, passing to \(H_3(X, \mathbb{Q})\) by Poincaré duality, let \(S_i = PD(W_i)\) (or equivalently, viewing \(H_3 = (H^3)^*, S_i\) is the annihilator of \(W_{4-2i}\)).

**Proposition 1.** Given an LCSL point in the moduli space of Calabi-Yau 3 folds with \(h^{2,1} = s\), \exists a \(\mathbb{Z}\)-basis \((\alpha_0, \ldots, \alpha_s, \beta_0, \ldots, \beta_s)\) of \(H_3(X, \mathbb{Z})\) s.t. \(\beta_0 \in S_0, \beta_1, \ldots, \beta_s \in S_2, \alpha_1, \ldots, \alpha_s \in S_4, \alpha_0 \in S_6 = H_3(X)\) s.t. \((\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0, (\alpha_i, \beta_j) = \delta_{ij}\).

*Proof.* Let \(\beta_0\) be the \(\mathbb{Z}\) generator of \(S_0\) (unique up to sign), which we extend to a \(\mathbb{Z}\)-basis \(\beta_i\) of \(S_2\). By the lemma, \(S_2\) is Lagrangian w.r.t. the intersection product, so \((\beta_i, \beta_j) = 0\). Let \(\beta_i^*\) be the dual basis of \(S_2^* = H^3/W_2\), i.e. \(\beta_i^* \beta_j = \delta_{ij}\), and let \(\alpha_i \in H_3\) be the Poincaré dual of some lift of \(\beta_i^*\) to \(H^3\). Then \((\alpha_i, \beta_j) = \delta_{ij}\). We can make \((\alpha_i, \alpha_j) = 0\) inductively by replacing \(\alpha_i\) with \(\alpha_i - \sum (\alpha_i, \alpha_j) \beta_j\). Finally, \(\alpha_1, \ldots, \alpha_s \in S_4\) since \((\alpha_i, \beta_0) = 0\) and \(S_4 = S_0^\perp\). \(\square\)

We now define canonical coordinates on our moduli space. Given \(X \to (D^*)^s\) LCSL, let \(\Omega(t_1, \ldots, t_s)\) be the holomorphic volume form on \(X_{(t_1, \ldots, t_s)}\), normalized so that \(\int_{\beta_0} \Omega(t_1, \ldots, t_s) = 1\). Set \(w_i(t_1, \ldots, t_s) = \int_{\beta_i} \Omega(t_1, \ldots, t_s)\). This is not quite a coordinate because of monodromy: as \(t_j\) goes around the origin, \(\beta_i \mapsto \phi_j(\beta_i) = \beta_i - m_{ji} \beta_0\) for some \(m_{ji} \in \mathbb{Z}\) (an integer since these are integer classes). In fact, these are the \(m_{ji}\) from the definition of LCSL. Instead, we set \(q_i = \exp(2\pi i w_i)\): these are well-defined functions on \((D^*)^s\), and are canonical once the basis \(\{\beta_i\}\) is chosen. Note that \(q_i\) is a zero of order \(-m_{ji}\) (i.e. a pole of order \(m_{ji}\)) along \(t_j = 0\); if the \(m_{ji}\)'s are nonpositive, then we get coordinates on \((D^2)^s\), and can choose a basis of \(S_2\) appropriately.

**Example.** For our elliptic curves from last time, \(q = \exp(2\pi i \tau(t))\), \(\tau(t) = \int_{a} \Omega\) where \(\int_a \Omega = 1\).
If $e_i$ is a basis of $H^2(\tilde{X},\mathbb{Z}), e_i$ in the Kähler cone, we obtain coordinates on the complexified Kähler moduli space: if $[B + i\omega] = \sum t_i e_i$, let $\tilde{q}_i = \exp(2\pi i t_i), \tilde{t}_i = \int_{e_i}^* B + i\omega$.

Example. In example above, we have $\tilde{q} = \exp(2\pi i \int_{T^2} B + i\omega)$.

**Conjecture 1** (Mirror Symmetry). Let $f : \mathcal{X} \rightarrow (D^*)^S$ be a family of Calabi-Yau 3-folds with LCSL at 0. Then $\exists$ a Calabi-Yau 3-fold $\tilde{X}$ and choices of bases $\alpha_0, \ldots, \alpha_S, \beta_0, \ldots, \beta_S$ of $H_3(\mathcal{X},\mathbb{Z}), e_1, \ldots, e_S$ of $H^2(\mathcal{X},\mathbb{Z})$ s.t. under the map $m : (D^*)^S \rightarrow \mathcal{M}_{Kah}(\tilde{X}), (q_1, \ldots, q_S) \mapsto (\tilde{q}_1, \ldots, \tilde{q}_S), \tilde{q}_i = q_i$, we have a coincidence of Yukawa couplings

$$
\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \rangle^{\mathcal{X}}_p = \langle \frac{\partial}{\partial \tilde{q}_i}, \frac{\partial}{\partial \tilde{q}_j}, \frac{\partial}{\partial \tilde{q}_k} \rangle^{\mathcal{X}}_{m(p)}
$$

where the LHS corresponds to $\int_{\mathcal{X}} \Omega \wedge (\frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} \Omega)$ and the RHS to a $(1,1)$-coupling, i.e. the Gromov-Witten invariants $\langle e_i, e_j, e_k \rangle^{\tilde{X}}_{0,\beta}$ (since $2\pi i \tilde{q}_i \frac{\partial}{\partial \tilde{q}_i} = \frac{\partial}{\partial \tilde{t}_i} = e_i \in H^{1,1}$).