Last time, we were considering $\mathbb{CP}^1$ mirror to $\mathbb{C}^*, W = z + \frac{e^{-\Lambda}}{z}$ for $\Lambda = 2\pi \int_{\mathbb{CP}^1} \omega$: the latter object is a Landau-Ginzburg model, i.e. a Kähler manifold with a holomorphic function called the “superpotential”. Homological mirror symmetry gave

$$D^\infty \text{Fuk}(\mathbb{CP}^1) \cong H^0 \text{MF}(W)$$
$$D^b \text{Coh}(\mathbb{CP}^1) \cong D^b \text{Fuk}(\mathbb{C}^*, W)$$

We stated that the Fukaya category of $\mathbb{CP}^1$ was a collection indexed by “charge” $\lambda \in \mathbb{C}$, and defined $\text{Fuk}(\mathbb{CP}^1, \lambda)$ to be the set of weakly unobstructed Lagrangians with $m_0 = \lambda \cdot [L]$. This is an honest $A_\infty$-category, as the $m_0$’s cancel and the Floer differential squares to zero, whereas from $\lambda$ to $\lambda'$ we’d have $\partial^2 = \lambda' - \lambda$. For instance, for $L \cong S^1$, $(L, \nabla)$ is weakly unobstructed, with $m_0 = W(L, \nabla) \cdot [L]$. However, $HF(L, L) = 0$ unless $L$ is the equator and $\text{hol}(\nabla) = \pm \text{id}$. Then $L_{\pm}$ has $HF \cong H^*(S^1, \mathbb{C})$ with deformed multiplicative structure, $HF^*(L, L) \cong \mathbb{C}[t]/t^2 = \pm e^{-\Lambda/2}$.

We now look at the matrix factorizations of $W - \lambda, \lambda \in \mathbb{C}$. These are $\mathbb{Z}/2\mathbb{Z}$-graded projective modules $Q$ over the ring of Laurent polynomials $R = \mathbb{C}[\mathbb{C}^*] \cong \mathbb{C}[z^{\pm 1}]$ equipped with $\delta \in \text{End}^1(Q)$ s.t. $\delta^2 = (W - \lambda) \cdot \text{id}_Q$. That is, we have maps $\delta_0 : Q_0 \to Q_1, \delta_1 : Q_1 \to Q_0$ given by matrices with entries in the space of Laurent polynomials s.t. $\delta_0 \circ \delta_1 = (W - \lambda) \cdot \text{id}_{Q_1}, \delta_1 \circ \delta_0 = (W - \lambda) \cdot \text{id}_{Q_0}$. Now $\text{Hom}(Q, Q')$ is $\mathbb{Z}/2\mathbb{Z}$ graded, with

$$\text{Hom}^0 = \left\{ Q_0 \xrightarrow{\delta_0} Q_1 \right\}$$

This has a differential $\partial$ s.t. $\partial(f) = \delta' \cdot f \pm f \cdot \delta$ and $\partial^2 = 0$. We obtain a homology category $H^0 \text{MF}(W - \lambda): \text{hom} = H^0(\text{Hom}, \partial)$, i.e. “chain maps” up to “homotopy”.

**Theorem 1.** $H^0(\text{MF}(W - \lambda)) = 0$, i.e. all matrix factorizations are nullhomotopic, unless $\lambda$ is a critical value of $W$. 
Warning: again, looking at homomorphisms from $\text{MF}(W - \lambda)$ to $\text{MF}(W - \lambda')$, then $\partial^2 \neq 0$, $\partial^2(f) = \partial^2 \cdot f - f \cdot \partial^2 = (\lambda - \lambda')f$.

Example. $W = z + \frac{e^{-\Lambda}}{z}$ has critical points $\pm e^{-\Lambda/2}$ with critical values $\pm 2e^{-\Lambda/2}$. Then

$$W \pm 2e^{-\Lambda/2} = z \pm 2e^{-\Lambda/2} + \frac{e^{-\Lambda}}{z} = (z \pm e^{-\Lambda/2})(1 \pm \frac{e^{-\Lambda/2}}{z})$$

(3)

$$Q_\pm = \{ \mathbb{C}[z^{\pm 1}] \xrightarrow{\frac{z \pm e^{-\Lambda/2}}{1 \pm e^{-\Lambda/2} z^{-1}}} \mathbb{C}[z^{\pm 1}] \}$$

Then

$$\text{End}_{\text{H}^0\text{MF}}(Q_\pm) = \{ \begin{array}{c} R \xrightarrow{f} R \\ f \end{array} \} / \text{homotopy}$$

(4)

is multiplication by $f \in \mathbb{C}[z^{\pm 1}]$. The maps $\partial$ sends

$$\begin{array}{c} R \xrightarrow{h} R \\ (x \pm e^{-\Lambda/2})h \end{array} \quad \begin{array}{c} R \xleftarrow{h} R \\ (x \pm e^{-\Lambda/2})h \end{array}$$

(5)

and similarly on the other side, so

$$\text{End}(Q_\pm) = \mathbb{C}[z^{\pm 1}]/(z \pm e^{-\Lambda/2}, 1 + e^{-\Lambda/2} z^{-1}) \cong (\mathbb{C}[z^{\pm 1}]/z \pm e^{-\Lambda/2}) \cong \mathbb{C}$$

(6)

Similarly $\text{Hom}_{\text{H}^0\text{MF}}(Q_\pm, Q_\pm[1]) \cong \mathbb{C}$.

Indeed, in the case of the two maps $z - c, 1 - cz^{-1}$, we take vertical maps $z, 1$, so

$$\begin{array}{c} R \xrightarrow{z - c} R \\ z \\ 1 - cz^{-1} \end{array} \quad \begin{array}{c} R \xleftarrow{z - c} R \\ 1 \\ 1 - cz^{-1} \end{array}$$

(7)

giving us $\mathbb{C}[z^{\pm 1}]/(z - c)$.

Next, $D^b\text{Coh}(\mathbb{C}P^1)$ is generated by $\mathcal{O}(-1)$ and $\mathcal{O}$, i.e. the smallest full subcategory containing $\mathcal{O}, \mathcal{O}(-1)$ and closed under shifts and cones contains all of $D^b$. More generally, via Beilinson we have that

$$D^b\text{Coh}(\mathbb{C}P^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O} \rangle$$

(8)

The idea is the the diagonal $\Delta \subset \mathbb{C}P^n \times \mathbb{C}P^n$ is the (transverse) zero set of $s = \sum_{i=0}^n \frac{\partial}{\partial x_i} \otimes y_i$, which is a section of $E = T(-1) \boxtimes \mathcal{O}(1) = \pi_1^*(T\mathbb{C}P^n \otimes$
\( \mathcal{O}(-1) \otimes \pi_2^* \mathcal{O}(1) \). Recall that \( T \mathbb{P}^n \) is spanned by the vector fields \( x_i \frac{\partial}{\partial x_i} \) on \( \mathbb{C}^{n+1} \) under the relation \( \sum_{i=0}^n x_i \frac{\partial}{\partial x_i} = 0 \). Taking the Koszul resolution
\[
0 \to E^* = \Omega^{1}(1) \boxtimes \mathcal{O}(-1) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0
\]
in \( D^b \text{Coh}(\mathbb{P}^1 \times \mathbb{P}^1) \). On the other hand, \( E \in D^b \text{Coh}(X \times Y) \) gives \( \phi^E : D^b(\text{Coh}(X)) \to D^b(\text{Coh}(Y), F \mapsto R\pi_2^*(L\pi_1^* F \otimes E)) \). Exactness implies that \( \phi^O_\Delta(F) \sim F \) sits in an exact triangle with
\[
\phi^{O_\boxtimes \mathcal{O}(-1)}(F) \cong R\Gamma(F \otimes \Omega^{1}(1)) \boxtimes_\mathbb{C} \mathcal{O}(-1)
\]
which completes the proof.

The algebra of the exceptional collection \( \langle \mathcal{O}(-1), \mathcal{O} \rangle \) is given by
\[
\mathcal{A} = \text{End}^{*}(\mathcal{O}(-1) \oplus \mathcal{O})
\]
and \( D^B \text{Coh}(\mathbb{CP}^1) \) is isomorphic to the derived category of finitely-generated \( \mathcal{A} \)-modules.

Finally, the Fukaya category of \( (\mathbb{C}^*, W = z + \frac{e^{-\Lambda}}{z}) \) is the category whose objects are admissible Lagrangians with flat connections, i.e. \( L \) is a (possibly noncompact) Lagrangian submanifold with \( W|_L \) proper, \( W|_L \in \mathbb{R}_+ \) outside a compact subset. We can perturb such \( L \): for \( a \in \mathbb{R} \), let \( L^{(a)} \) be Hamiltonian isotopic to \( L, W(L^{(a)}) \in \mathbb{R}_+ + ia \) near \( \infty \). In good cases, it will be the Hamiltonian flow of \( X_{\text{Re}(W)} = \nabla \text{Im}(W) \). Then \( \text{Hom}(L, L') = CF^*(L^{(a)}, L'^{(a')}) \) for \( a > a' \) (the Floer differential is well-defined), and we obtain \( m_k, k \geq 2 \) similarly, perturbing the Lagrangians so they are in decreasing order of \( \text{Im}(W) \).

**Example.** Consider \( L_0 = \mathbb{R}_+, L_{-1} = \) an arc joining \( 0 \) to \( +\infty \) and rotating once clockwise around the origin. Then \( e^{-\Lambda/2} \in L_0, -e^{-\Lambda/2} \in L_{-1} \), so under \( W = z + \frac{e^{-\Lambda}}{z} \), we have \( W(L_0) \) being the interval \( [2e^{-\Lambda/2}, +\infty) \) on the positive real axis, while \( W(L_{-1}) \) is an arc that joins \(-2e^{-\Lambda/2} \) to \( +\infty \) in the lower half plane. Furthermore, \( \text{hom}(L_0, L_0) \cong \mathbb{C} \cdot e, e = \text{id}_{L_0} \), and same for \( L_{-1} \), while \( \text{hom}(L_0, L_{-1}) = 0 \) and \( \text{hom}(L_{-1}, L_0) = V \) has dimension \( 2 \). Then \( \text{Fuk}(\mathbb{C}^*, W) \) is generated by \( L_{-1}, L_0 \) (Seidel)

Similarly, one can obtain homological mirror symmetry for toric Fano manifolds: see M. Abouzaid.