1. **Homological Mirror Symmetry (cntd.)**

Last time, we studied homological mirror symmetry on $T^2$ (with area form $\lambda$) on the one hand and $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, \tau = i\lambda$ on the other. Lagrangians of slope $(p,q)$ with a $U(1)$ flat connection correspond to vector bundles of rank $p$ and degree $-q$ (for $(p,q) = (0, -1)$, this gives skyscraper sheaves). We showed that $m_2$ corresponds to theta functions and to sections and products.

1.1. **Massey Products.** We consider these in the special case of a triangulated category $\mathcal{D}$, and consider objects and morphisms $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} X_4$ where $g \circ f = 0, h \circ g = 0$. Assume that $\text{hom}(X_1, X_3[-1]) = \text{hom}(X_2, X_4[-1]) = 0$. Then $m_3(h, g, f) \in \text{hom}(X_1, X_4[-1])$. Let $K$ be s.t. $K \xrightarrow{\overline{f}} X_2 \xrightarrow{\overline{g}} X_3 \xrightarrow{[1]} K[1]$ is a distinguished triangle (i.e. $K[1] = \text{Cone}(g)$). Then $g \circ f = 0 \implies f$ factors through $X_1 \xrightarrow{\overline{f}} K \xrightarrow{} X_2$, where $\overline{f} \in \text{hom}(X_1, K)$ comes from

(1) $\text{hom}(X_1, X_3[-1]) \to \text{hom}(X_1, K) \to \text{hom}(X_1, X_2) \xrightarrow{g} \text{hom}(X_1, X_3)$

Similarly, $h \circ g = 0 \implies h$ factors through $X_3 \to K[1] \xrightarrow{\overline{h}} X_4$, and we define

(2) $m_3(h, g, f) := \overline{h}[-1] \circ \overline{f} : X_1 \xrightarrow{\overline{f}} K \xrightarrow{\overline{h}[-1]} X_4[-1]$

Now, let’s say that we had $f, g, h$ in the $A_\infty$ category of twisted complexes, $K = \{X_2 \xrightarrow{g} X_3[-1]\}$,

\[
\begin{array}{c}
X_1 \\
\downarrow f \\
X_2 \xrightarrow{g} X_3[-1] \\
\downarrow \overline{h}[-1] \\
X_4[-1]
\end{array}
\]
and \( m_2^{T^w}(\overline{h}[-1], \overline{f}) = m_3(h, g, f) \). If we add an extra step

\[
\begin{array}{c}
\begin{array}{ccc}
e = & X_2 & \xrightarrow{g} & X_3[-1] \\
\downarrow{id} & \downarrow{0} & & \downarrow{f} \\
X_2 & \xrightarrow{g} & X_3[-1] & \xrightarrow{h} & X_4[-1]
\end{array}
\end{array}
\]

(4)

then we get

(5)

\[
m_3(h, g, f) = m_3(h, m_1(e), f) = m_2(h, m_2(e, f)) + \text{other terms which vanish}
\]

Now, let \( L \to X^\vee \) be a nontrivial degree 0 holomorphic line bundle over an elliptic curve, \( p, q \) generic points. Then the pairwise compositions in

(6)

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{f} & \mathcal{O}_p \\
\mathcal{L}[1] & \xrightarrow{h} & \mathcal{O}_q[1]
\end{array}
\end{array}
\]

vanish, and we have

(7)

\[
\begin{align*}
\text{hom}(\mathcal{O}_p, \mathcal{L}[1]) & = \text{Ext}^1(\mathcal{O}_p, \mathcal{L}) \cong \text{Hom}(\mathcal{L}, \mathcal{O}_p)^\vee \\
\text{hom}(\mathcal{O}, \mathcal{L}[1]) & = \text{Ext}^1(\mathcal{O}, \mathcal{L}) \cong H^1(\mathcal{L}) = 0
\end{align*}
\]

Then \( K \cong \mathcal{L} \otimes \mathcal{O}(p) \) is a degree 1 line bundle, neither \( \mathcal{O}(p) \) nor \( \mathcal{O}(q) \): note that \( \mathcal{O}(p) \) is a degree 1 line bundle with a section \( s_p, s_p^{-1}(0) = \{p\} \). Then we have a long exact sequence

(8)

\[
0 \to \mathcal{L} \xrightarrow{s_p} \mathcal{L} \otimes \mathcal{O}(p) \to \mathcal{O}_p \to 0
\]

giving us an exact triangle in the derived category

(9)

\[
K = \mathcal{L} \otimes \mathcal{O}(p) \to \mathcal{O}_p \xrightarrow{s_p} \mathcal{L}[1] \xrightarrow{1} K[1]
\]

via our extension class. \( f \) should factor as a map from \( \mathcal{O} \) to \( K \), and does via a nontrivial section \( \overline{f} \) of \( K = \mathcal{L} \otimes \mathcal{O}(p) \). Moreover, for \( \overline{h}[-1] \) nontrivial in \( \text{hom}(K, \mathcal{O}_q), \overline{h}[-1] \circ \overline{f} \neq 0 \).

This matches with the calculation of \( m_3 \) for the relevant Lagrangians in the Fukaya category of \( T^2 \): two horizontal lines and two vertical lines, bounding an infinite series of rectangles. See notes for a visual description of this.
2. Strominger-Yau-Zaslow (SYZ) Conjecture

Motivating question: how does one build a mirror $X^\vee$ of a given Calabi-Yau $X$? Observe that homological mirror symmetry (1994) says that $D^b\text{Coh}(X^\vee) \cong D^s\text{Fuk}(X)$. Points $p \in X^\vee$ correspond to skyscraper sheaves $\mathcal{O}_p \in D^b\text{Coh}(X^\vee)$ and $\mathcal{L}_p \in D^s\text{Fuk}(X)$. That is, we can regard $X^\vee$ as the moduli space of skyscraper sheaves in $D^b\text{Coh}(X^\vee)$ as well as a moduli space of certain objects of $D^s\text{Fuk}(X)$. The question reduces to understanding exactly what are these certain objects. Four lectures ago, we computed $\text{Ext}^k(\mathcal{O}_p, \mathcal{O}_p) \cong \bigwedge^k V$ for $V$ the tangent space at $p$. As a graded vector space, $\text{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \cong H^*(T^n; \mathbb{C})$. Four lectures before that, we showed that $HF^*(L, L)$ is in good cases isomorphic to $H^*(L)$, but if $L$ bounds disks, these are only related by a spectral sequence.

Remark. Warning: recall that in general we are dealing with $\Lambda$-coefficients. In good cases, we can set $T = e^{-2\pi}$ and hope that we have convergence. If convergence fails, we only get a formal family near LSCL.

If (optimistically) we assume $\mathcal{L}_p$ is an actual Lagrangian, then it should be a Lagrangian torus. There are not enough of these: given $T^n \cong L \subset X, V(L) \cong T^*L$, one has that Lagrangian deformations of $L$ are graphs of closed 1-forms, while Hamiltonian isotopies are graphs of exact 1-forms. Furthermore, for $T^n$, $\text{Def}_L \cong H^1(L, \mathbb{R}) \cong \mathbb{R}^n$.

Now, recall the twisted Floer homology for pairs $(L, \nabla)$, with $\nabla$ a flat $U(1)$ connection on $\mathbb{C} \to L$: $\nabla = d + A, A \in \Omega^1(L, i\mathbb{R})$. Taking this modulo gauge transformations and exact 1-forms, we obtain $H^1(L; i\mathbb{R})$. One can hope that generic points of $X^\vee$ parameterize isomorphism classes of $(L, \nabla)$, $L \subset X$ a Lagrangian torus and $U(1)$ a flat connection.