1. Deformations of Complex Structures

An (almost) complex structure \((X, J)\) splits the complexified tangent and (wedge powers of) cotangent bundles as

\[
TX \otimes \mathbb{C} = TX^{1,0} \oplus TX^{0,1}, v^{0,1} = \frac{1}{2}(v + iJv)
\]

\[
T^*X \otimes \mathbb{C} = T^*X^{1,0} \oplus T^*X^{0,1}, T^*X^{1,0} = \text{Span}(dz_i), T^*X^{0,1} = \text{Span}(d\bar{z}_i)
\]

\[
\bigwedge^k T^*X \otimes \mathbb{C} = \bigoplus_{p+q=k} \bigwedge^p T^*X = \Omega^{p,q}(X)
\]

If \(J\) is almost complex, these are \(\mathbb{C}\)-vector bundles. \(J\) is integrable (i.e. a complex structure)

\[
[T^{1,0}, T^{1,0}] \subset T^{1,0} \iff d = \partial + \bar{\partial} \text{ maps } \Omega^{p,q} \rightarrow \Omega^{p+1,q} \oplus \Omega^{p,q+1}
\]

\[
\iff \bar{\partial}^2 = 0 \text{ on diff. forms}
\]

We obtain a Dolbeault cohomology for holomorphic vector bundles \(E\):

\[
C^q_{\bar{\partial}}(X, E) = \{C^\infty(X, E) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X, E) \xrightarrow{\bar{\partial}} \Omega^{0,2}(X, E) \rightarrow \cdots \}
\]

\[
H^q_{\bar{\partial}}(X, E) = \ker \bar{\partial}/\text{im} \bar{\partial}
\]

Deforming \(J\) to a “nearby” \(J'\) gives

\[
\Omega^{1,0}_{J'} \subset T^*X = \Omega^{1,0}_J \oplus \Omega^{0,1}_J
\]

is a graph of a linear map \((-s) : \Omega^{1,0}_J \rightarrow \Omega^{0,1}_J\). \(J'\) is determined by \(\Omega^{1,0}_{J'}\) (acted on by \(i\)) and \(\Omega^{0,1}_{J'}\) (acted on by \(i'\)). \(s\) is a section of \((\Omega^{1,0}_J)^* \otimes \Omega^{0,1}_J = T^{1,0}_J \otimes \Omega^{0,1}_J\)

i.e. a \((0, 1)_J\)-form with values in \(T^{1,0}_J X\). If \(z_1, \ldots, z_n\) are local holomorphic
coordinates for $J$, then $s = \sum s_{ij} \frac{\partial}{\partial z_i} \otimes \overline{d z_j}$. A basis of $(1, 0)$-forms for $J'$ is given by $d z_i - \sum_{j} s_{ij} d z_j$ and $(0, 1)$-vectors for $J'$ by $\frac{\partial}{\partial \overline{z_k}} + \sum_{\ell} s_{\ell k} \frac{\partial}{\partial z_{\ell}}$.

We can use this to test the integrability of $J'$. The Dolbeault complex $(\bigoplus_{q} \Omega^0,q \otimes TX^{1,0}, \overline{\partial})$ ($\overline{\partial}$ acts “on forms”) carries a Lie bracket

\[ [\alpha \otimes v, \alpha' \otimes v'] = (\alpha \wedge \alpha') \otimes [v, v'] \]

giving it the structure of a differential graded Lie algebra.

**Proposition 1.** $J'$ is integrable $\iff \overline{\partial}s + \frac{1}{2}[s, s] = 0$.

**Proof.** We want to check that the bracket of two 0, 1 tangent vectors is still 0, 1, i.e. that

\[ \left[ \frac{\partial}{\partial \overline{z_k}} + \sum_{\ell} s_{\ell k} \frac{\partial}{\partial z_{\ell}}, \frac{\partial}{\partial z_k} \right] \in TX^{0,1} \]

Evaluating this bracket gives

\[ \sum_{\ell} \left( \frac{\partial s_{\ell j}}{\partial z_i} - \frac{\partial s_{\ell i}}{\partial z_j} \right) \frac{\partial}{\partial z_{\ell}} + \sum_{k, \ell} \left( s_{k i} \frac{\partial s_{\ell j}}{\partial \overline{z_k}} - s_{k j} \frac{\partial s_{\ell i}}{\partial \overline{z_k}} \right) \frac{\partial}{\partial z_{\ell}} \]

We want this to be 0, i.e. for all $i, j, \ell$,

\[ 0 = \frac{\partial s_{\ell j}}{\partial z_i} - \frac{\partial s_{\ell i}}{\partial z_j} + \sum_{k} \left( s_{k i} \frac{\partial s_{\ell j}}{\partial \overline{z_k}} - s_{k j} \frac{\partial s_{\ell i}}{\partial \overline{z_k}} \right) \]

(coefficients of $\frac{\partial}{\partial \overline{z} \otimes (d z_i \wedge d z_j)}$ in $[\overline{\partial}s]$)

We leave the rest as an exercise. $\square$

We would now like to use this to understand the moduli space of complex structures. Define

\[ \mathcal{M}_{CX}(X) = \{ J \text{ integrable complex structures on } X \}/\text{Diff}(X) \]

(or, assuming that Aut($X, J$) is discrete, we want that near $J$, $\exists$ a universal family $X' \to U \subset \mathcal{M}_{CX}$ (complex manifolds, holomorphic fibers $\cong X$) s.t. any family of integrable complex structures $X' \to S$ induces a map $S \to U$ s.t. $X'$ pulls back to $X$). We have an action of the diffeomorphisms of $X$: for $\phi \in \text{Diff}(X)$ close to id,

\[ d\phi : TX \otimes \mathbb{C} \simeq \phi^*TX \otimes \mathbb{C} \]

\[ \partial\phi : TX^{1,0} \to \phi^*TX^{1,0} \]

\[ \overline{\partial}\phi : TX^{0,1} \to \phi^*TX^{1,0} \]
so
\[
\phi^* dz_i = dz_i \circ d\phi = dz_i \circ \partial \phi + dz_i \circ \overline{\partial} \phi
\]
\[(11)\]

Deformation by \( s \in \Omega^{0,1}(X, TX^{1,0}) \) gives \( \Omega_{J,0}^1 = \{ \alpha - s(\alpha) | \alpha \in \Omega^{1,0} \} \) (the graph of \(-s\)): taking \( s = -(\partial \phi)^{-1} \cdot \overline{\partial} \phi : TX^{0,1} \to \phi^* TX^{1,0} \to TX^{1,0} \) gives the desired element of \( \Omega^{0,1}(TX^{1,0}) \).

1.1. First-order infinitesimal deformations. Given a family \( J(t), J(0) = J \) gives \( s(t) \in \Omega^{0,1}(X, TX^{1,0}), s(0) = 0 \). By the above, this should satisfy
\[
\partial s(t) + \frac{1}{2} [s(t), s(t)] = 0
\]
\[(12)\]

In particular, \( s_1 = \frac{ds}{dt}|_{t=0} \) solves \( \overline{\partial} s_1 = 0 \). We obtain an infinitesimal action of \( \text{Diff}(X) \): for \( \phi_t, \phi_0 = \text{id}, \frac{d\phi}{dt}|_{t=0} = v \) a vector field,
\[
\frac{d}{dt}|_{t=0}(-\partial \phi_t)^{-1} \circ \overline{\partial} \phi_t = -\frac{d}{dt}|_{t=0}(\overline{\partial} \phi_t) = -\overline{\partial} v
\]
\[(13)\]

This implies that first-order deformations are given as
\[
\text{Def}^1(X, J) = \text{Ker} (\overline{\partial} : \Omega^{0,1}(TX^{1,0}) \to \Omega^{2,0}(TX^{1,0})) \quad \text{Im}(\overline{\partial} : C^\infty(TX^{1,0}) \to \Omega^{0,1}(TX^{1,0}))
\]
\[(14)\]

We can write this more compactly using Dolbeault cohomology, namely \( H^1_G(X, TX^{1,0}) \).

Furthermore, given a family
\[
X \quad \xrightarrow{S} \quad X
\]
\[(15)\]

of deformations of \((X, J)\) parameterized by \( S \), we get a map \( T_* S \to H^1(X, TX^{1,0}) \) called the \textit{Kodaira Spencer map}.

\textit{Remark.} A complex manifold \((X, J)\) is a union of complex charts \( U_i \) with biholomorphisms \( \phi_{ij} : U_{ij} \sim U_{ji} \) s.t. \( \phi_{ij} = \phi_{ji}^{-1} \) and \( \phi_{ij} \phi_{jk} = \phi_{ik} \) on \( U_{ijk} \). Deformations of \((X, J)\) come from deforming the gluing maps \( \phi_{ij} \) among the space of holomorphic maps. To first order, this is given by holomorphic vector fields \( v_{ij} \) on \( U_i \cap U_j \) s.t. \( v_{ij} = -v_{ji} \) and \( v_{ij} + v_{jk} = v_{ik} \) on \( U_{ijk} \). This is precisely the \( \check{\text{C}}\text{ech} \) 1-cocycle conditions in the sheaf of holomorphic tangent vector fields. Modding out by holomorphic functions \( \psi_i : U_i \sim U_i \) (which act by \( \phi_{ij} \mapsto \psi_j \phi_{ij} \phi_{ij}^{-1} \)) is precisely modding by the \( \check{\text{C}}\text{ech} \) coboundaries. Thus, \( \text{Def}^1(X, J) = \check{H}^1(X, TX^{1,0}) \).
1.2. Obstructions to Deformation. Given a first-order deformation $s_1$, one can ask if one can find an actual deformation $s(t) = s_1t + O(t^2)$ (or even a formal deformation, i.e. non-convergent power series). Expand
\begin{equation}
(16) \quad s(t) = s_1t + s_2t^2 + \cdots \in \Omega^{0,1}(X, TX^{1,0})
\end{equation}
Then the condition $\overline{\partial}s(t) + \frac{1}{2}[s(t), s(t)] = 0$ implies that $\overline{\partial}s_1 = 0$, $\overline{\partial}s_2 + \frac{1}{2}[s_1, s_1] = 0$, $\overline{\partial}s_3 + [s_1, s_2] = 0$, \ldots. Now, we need $[s_1, s_1] \in \text{im} (\overline{\partial}) \subset \Omega^{0,2}(TX^{1,0})$. We know that $[s_1, s_1] \in \text{Ker} (\partial)$. Thus, the primary obstruction to deforming is the class of $[s_1, s_1]$ in $H^2(X, TX^{1,0})$. If it is zero, then there is a $s_2$ s.t. $\overline{\partial}s_2 + \frac{1}{2}[s_1, s_1] = 0$, and the next obstruction is the class of $[s_1, s_2] \in H^2(X, TX^{1,0})$. We are basically attempting to apply by brute force the implicit function theorem.

If it happens that $H^2(X, TX) = 0$, then the deformations are unobstructed and the moduli space of complex structures is locally a smooth orbifold (not a manifold, because we may have to quotient by automorphisms) with tangent space $H^1(X, TX^{1,0})$. For Calabi-Yau manifolds, this will not be true: however, we still have

**Theorem 1** (Bogomolov-Tian-Todorov). For $X$ a compact Calabi-Yau ($\Omega_X^{n,0} \cong \mathcal{O}_X$) with $H^0(X, TX) = 0$ (automorphisms are discrete), deformations of $X$ are unobstructed and, assuming $\text{Aut}(X, J) = \{1\}$, $\mathcal{M}_{CX}$ is locally a smooth manifold with $T\mathcal{M}_{CX} = H^1(X, TX)$.

**Theorem 2** (Griffiths Transversality). For a family $(X_t, J_t)$, $\alpha_t \in \Omega^{p,q}(X_t, J_t) \implies \frac{d}{dt} |_{t=0} \alpha_t \in \Omega^{p,q} + \Omega^{p+1,q-1} + \Omega^{p-1,q+1}$.

**Proof.** $J_t$ is given by $s(t) \in \Omega^{0,1}(TX^{1,0})$, $s(0) = 0$. In local coordinates, we have $T^*X^{1,0}_t = \text{Span}\{dz_i^{(t)} = dz_i - \sum s_{ij}(t)dz_j\}$
\begin{equation}
(17) \quad \alpha_t = \sum_{I, J|I|=p, |J|=q} \alpha_{IJ}(t)dz_{i_1}^{(t)} \wedge \cdots \wedge dz_{i_p}^{(t)} \wedge dz_{j_1}^{(t)} \wedge \cdots \wedge dz_{j_q}^{(t)}
\end{equation}
Taking $\frac{d}{dt} |_{t=0}$, the result follows from the product rule. We mostly get $(p, q)$ terms and a few $(p + 1, q - 1), (p - 1, q + 1)$ forms (the latter from $\frac{d}{dt} |_{t=0}(dz_{i_k}^{(t)})$. \qed